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Continuous dependence of solutions to the strongly damped nonlinear Klein–Gordon equation

Şevket GÜR, Mesude Elif UYSAL*
Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya, Turkey

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Abstract: This article is devoted to the study of the initial-boundary value problem for the strongly damped nonlinear Klein–Gordon equation. It is proved that the solution depends continuously on changes in the damping terms, diffusion, mass, and nonlinearity effect term in the H^1 norm.

Key words: Structural stability, nonlinear Klein–Gordon equation, continuous dependence

1. Introduction

For a reasonable model it is expected that some controls over its structural stability should exist. One of those controls is to examine the dependency on the coefficients of the solutions of the governing model. Recently, many important works have been done on deriving stability estimates. In these calculations changes in coefficients are permitted or even the model itself can be changed. Such works were examined in books [1,4] and articles [7,10] and the references therein.

In this article, the question of structural stability for the following initial-boundary value problem (IBVP) for the strongly damped Klein–Gordon equation is investigated:

$$u_{tt} - \alpha\Delta u_t + \beta u_t - \sigma\Delta u + m^2u + \lambda|u|^{p-1}u = 0 \quad in \quad \Omega \times (0,\infty), \tag{1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad in \quad \Omega, \tag{2}$$

$$u = 0 \quad on \quad \partial\Omega \times (0,\infty). \tag{3}$$

Here α, β, σ, $m > 0$, $\lambda \in \mathbb{R}$ are physical constants that represent the first of two gradients of damping, diffusion, mass, and nonlinearity effects; Δ is a Laplacian; and $p > 0$ is a source. $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth enough boundary $\partial\Omega$, and $1 < p \leq \infty$ if $n = 1,2$ and $1 < p \leq \frac{n}{n-2}$ if $n \geq 3$.

In 1926, Oskar Klein and Walter Gordon independently proposed one of the nonlinear wave equations emerging from the relativistic motion of electrons. Since then, this equation has been known as the Klein–Gordon equation.

Equations with no damping terms ($\alpha = \beta = 0$) have been considered by many authors; see [3,6,9,12,13,15, 18] and the references therein. For these undamped equations there exists adequate knowledge about the local

*Correspondence: elifuysal13@gmail.com

solution in time for the initial value problem (1) [6,13,18]. Furthermore, for small enough initial data it is known that the global solutions of this equation exist in time; see [6,9,12,17] and the references therein. In 1985, Cezaneve [5] proved that all global solutions must remain uniformly (in time) bounded in the energy phase space.

For an equation with weak damping $(\alpha = 0, \beta > 0)$, the existence and uniqueness of a time periodic solution was proved by Gao and Guo [8]. In this, the Galerkin method and Leray–Schauder fixed point theorem were employed. Nakao [14] obtained energy decay estimates for the global solutions of equation (1). Moreover, the existence and uniqueness of solutions were analyzed by Ha and Park [11]. In this analysis, the Faedo–Galerkin method in a noncylindrical domain was used. By this, the exponential decay rate of the global solutions was proved. Polat and Taskesen [16] also investigated the existence of solutions globally for equation (1), where $\alpha = 0, \quad \beta = 1$ by using the potential well method. Moreover, asymptotic behavior of global solutions was obtained by Xu [19].

For equations with strong damping $(\alpha > 0, \beta > 0)$, much less is known about solutions. The reader referred to the work by Avrin [2] in 1987, who studied the equation (1) in $\mathbb{R}^3$ and demonstrated a global weak solution v with $\alpha = 0$ for $p > 3$. By the application of the global strong solutions for each $\alpha > 0$, a global weak solution can be approximated closely. Furthermore, Xu and Ding in [20] investigated the existence of solutions globally and asymptotic behavior of the corresponding solutions for the IBVP of equation (1).

However, a number of unsolved problems like the structural stability question for the Klein–Gordon equation (1) exist. Therefore, in this article, our main goal is to know whether small changes in coefficients α, β, σ, m, λ separately will lead to a dramatic change in the behavior of the corresponding solution.

The inequalities that can be considered as fundamental tools in the analysis here are listed below.

- *Cauchy inequality with* ϵ*:*

  For any $a, b \geq 0$ and any $\epsilon > 0$ we have the inequality

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

- *Sobolev embedding theorem:*

  Suppose that $1 \leq p \leq n, p^* = \frac{np}{n-p}$, and $u \in W^{1,p}(\mathbb{R}^n)$. Then $u \in L^{p^*}(\mathbb{R}^n)$, and there exist $C \geq 0$ such that

$$||u||_{L^{p^*}} \leq C||\nabla u||_{L^p}.$$

2. A priori estimates

In this section, a priori estimates on solutions of (1) are derived. This will be used to prove the continuous dependency for the parameters.

Theorem 2.1 *For any* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$*, there exists a solution* $u \in H_0^1(\Omega)$ *of the problem* (1)–(3)*. Moreover, here the following estimates are held:*

$$||u_t||^2 \leq D_1, \ ||\nabla u||^2 \leq D_2, \ ||u||^2 \leq D_3 \tag{4}$$

and

$$\int_0^t ||\nabla u_s(x,s)||^2 ds \leq D_4, \tag{5}$$

where $D_1, D_2, D_3, D_4 > 0$ *are constants that depend on the initial data and the parameters of* (1)

Proof Multiplying (1) by u_t in $L^2(\Omega)$, we get

$$\frac{d}{dt}\Big[\frac{1}{2}||u_t||^2 + \frac{\sigma}{2}||\nabla u||^2 + \frac{m^2}{2}||u||^2 + \frac{\lambda}{p+1}||u||_{p+1}^{p+1}\Big] + \alpha||\nabla u_t||^2 + \beta||u_t||^2 = 0. \tag{6}$$

It follows from (6) that

$$E_u(t) = \frac{1}{2}||u_t||^2 + \frac{\sigma}{2}||\nabla u||^2 + \frac{m^2}{2}||u||^2 + \frac{\lambda}{p+1}||u||_{p+1}^{p+1} \leq E_u(0). \tag{7}$$

Hence, (4) follows from (6). From (6) it is also known that

$$\frac{d}{dt}E_u(t) + \alpha||\nabla u_t||^2 \leq 0$$

and if we integrate this over [0,t] then we find (2.2) since $E_u(t) > 0$. □

3. Continuous dependence on coefficients

In this part, it will be shown that the solution of the problem (1)–(3) depends continuously on coefficients α, m, and λ.

Continuous dependence on the damping term α*:*

Suppose that u is the solution of (1)–(3) and v is the solution of

$$v_{tt} - (\alpha + \mathrm{a})\Delta v_t + \beta v_t - \sigma\Delta v + m^2 v + \lambda|v|^{p-1}v = 0 \; in \; \Omega \times (0,\infty),$$

$$v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x) \quad in \quad \Omega,$$

$$v = 0 \quad on \quad \partial\Omega \times (0,\infty).$$

The difference $w = u - v$ of the solutions of these problems is the solution of the following IBVP:

$$w_{tt} - \alpha\Delta w_t + a\Delta v_t + \beta w_t - \sigma\Delta w + m^2 w + \lambda(|u|^{p-1}u - |v|^{p-1}v) = 0 \; in \; \Omega \times (0,\infty), \tag{8}$$

$$w(x,0) = 0, \qquad w_t(x,0) = 0 \qquad in \; \Omega, \tag{9}$$

$$w = 0 \quad on \quad \partial\Omega \times (0,\infty). \tag{10}$$

Theorem 3.1 *The solution* w *of problem* (8)–(10) *satisfies the inequality*

$$\frac{1}{2}||w_t||^2 + \frac{\sigma}{2}||\nabla w||^2 + \frac{m^2}{2}||w||^2 \leq \frac{e^{M_1 t}D_4}{\alpha}a^2 \qquad \forall t > 0, \tag{11}$$

where $D_4 > 0$, $M_1 > 0$ *are constants that depend on the parameters and initial data of* (1).

Proof Multiplying (8) by w_t in $L^2(\Omega)$, we obtain

$$\frac{d}{dt}\Big[\frac{1}{2}||w_t||^2+\frac{\sigma}{2}||\nabla w||^2+\frac{m^2}{2}||w||^2\Big]+\alpha||\nabla w_t||^2-a(\nabla v_t,\nabla w_t)+\beta||w_t||^2$$
$$+\lambda\int_\Omega(|u|^{p-1}u-|v|^{p-1}v)w_t dx=0, \tag{12}$$

$$\frac{d}{dt}\Big[\frac{1}{2}||w_t||^2+\frac{\sigma}{2}||\nabla w||^2+\frac{m^2}{2}||w||^2\Big]+\alpha||\nabla w_t||^2+\beta||w_t||^2\leq a|(\nabla v_t,\nabla w_t)|+$$
$$\lambda\Big|\int_\Omega\Big(|u|^{p-1}u-|v|^{p-1}v\Big)w_t dx\Big|. \tag{13}$$

Using the Cauchy–Schwarz inequality and the Cauchy inequality with ϵ, the following is obtained:

$$a||\nabla v_t||||\nabla w_t||\leq\epsilon||\nabla w_t||^2+\frac{a^2||\nabla v_t||^2}{4\epsilon}. \tag{14}$$

Notice that, after using the mean value theorem and Hölder and Sobolev inequalities, respectively, the following is derived:

$$\begin{aligned}\lambda\Big|\int_\Omega\Big(|u|^{p-1}u-|v|^{p-1}v\Big)w_t dx\Big| &\leq \lambda p\int_\Omega|w||w_t|\Big(|u|^{p-1}+|v|^{p-1}\Big)dx\\ &\leq \lambda p||w_t||||w||_{\frac{2n}{n-2}}\Big(||u||^{p-1}_{(p-1)n}+||v||^{p-1}_{(p-1)n}\Big)\\ &\leq \lambda p||w_t||C_1||\nabla w||C_2\Big(||\nabla u||^{p-1}+||\nabla v||^{p-1}\Big).\end{aligned} \tag{15}$$

Putting all of these estimates into inequality (13), we obtain

$$\begin{aligned}\frac{d}{dt}E_w(t)+\alpha||\nabla w_t||^2+\beta||w_t||^2 &\leq \epsilon||\nabla w_t||^2+\frac{a^2}{4\epsilon}||\nabla v_t||^2+\lambda pC_1||w_t||||\nabla w||C_2 2D_2^{\frac{p-1}{2}}\\ &\leq \epsilon||\nabla w_t||^2+\frac{a^2}{4\epsilon}||\nabla v_t||^2+\frac{C_3}{2}||w_t||^2+\frac{C_3}{2}||\nabla w||^2\\ &\quad+\frac{m^2}{2}||w||^2,\end{aligned} \tag{16}$$

where $\epsilon=\frac{\alpha}{4}$, $C_3=2\lambda pC_1C_2D_2^{\frac{p-1}{2}}$, and

$$E_w(t)=\frac{1}{2}||w_t||^2+\frac{\sigma}{2}||\nabla w||^2+\frac{m^2}{2}||w||^2. \tag{17}$$

Inequality (3.9) implies

$$\frac{d}{dt}E_w(t)\leq M_1E_w(t)+\frac{a^2}{\alpha}||\nabla v_t||^2, \tag{18}$$

where $M_1=max\left\{1,\frac{C_3}{\sigma},C_3\right\}$.

Using Gronwall's inequality, the desired result is found:

$$E_w(t) \le \frac{e^{M_1 t} D_4}{\alpha} a^2. \tag{19}$$

□

Continuous dependence on the coefficient m:

Let u be the solution of (1) and v be the solution of the following IVBP:

$$v_{tt} - \alpha \Delta v_t + \beta v_t - \sigma \Delta v + (m^2 + \mu) v + \lambda |v|^{p-1} v = 0 \ in \ \Omega \times (0, \infty),$$

$$v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x) \quad in \ \Omega,$$

$$v = 0 \ on \ \partial\Omega \times (0, \infty).$$

Hence, $w = u - v$ is a solution of the following IBVP:

$$w_{tt} - \alpha \Delta w_t + \beta w_t - \sigma \Delta w + m^2 w - \mu v + \lambda (|u|^{p-1} u - |v|^{p-1} v) = 0 \ in \ \Omega \times (0, \infty), \tag{20}$$

$$w(x,0) = 0, \quad w_t(x,0) = 0 \quad in \ \Omega, \tag{21}$$

$$w = 0 \ on \ \partial\Omega \times (0, \infty). \tag{22}$$

Theorem 3.2 *Let w be the solution of the problem (20)–(22). Then w satisfies the inequality*

$$\frac{1}{2}||w_t||^2 + \frac{\sigma}{2}||\nabla w||^2 + \frac{m^2}{2}||w||^2 \le \frac{e^{M_2 t} D_3 t}{2} \mu^2, \qquad \forall t > 0, \tag{23}$$

where $D_3 > 0$ and $M_2 > 0$ are constants that depend on the initial data and the parameters of (1).

Proof Let us take an inner product of (20) with w_t in $L^2(\Omega)$; then we have

$$\begin{aligned} \frac{d}{dt} E_w(t) + \alpha ||\nabla w_t||^2 + \beta ||w_t||^2 &\le \mu |(v, w_t)| + \lambda \Big| \int_\Omega (|u|^{p-1} u - |v|^{p-1} v) w_t dx \Big| \\ &\le \frac{\mu^2}{2}||v||^2 + \frac{1}{2}||w_t||^2 + \frac{C_3}{2}||w_t||^2 + \frac{C_3}{2}||\nabla w||^2 \\ &+ \frac{m^2}{2}||w||^2. \end{aligned}$$

Then $\frac{d}{dt} E_w(t) \le \frac{D_3}{2}\mu^2 + M_2 E_w(t)$ where $M_2 = max\left\{1, \frac{C_3}{\sigma}, 1 + C_3, \right\}$.

That is,

$$E_w(t) \le \frac{e^{M_2 t} D_3 t}{2} \mu^2,$$

which indicates continuous dependency on m: □

Continuous dependence on the coefficient λ.

Let u be the solution of the problem (1)–(3) and v be the solution of the following IBVP:

$$v_{tt} - \alpha\Delta v_t + \beta v_t - \sigma\Delta v + m^2 v + (\lambda + L)|v|^{p-1}v = 0 \ in \ \Omega \times (0,\infty),$$

$$v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x) \quad in \ \Omega,$$

$$v = 0 \ on \ \partial\Omega \times (0,\infty).$$

Now $w = u - v$ is a solution of the following IBVP:

$$w_{tt} - \alpha\Delta w_t + \beta w_t - \sigma\Delta w + m^2 w + \lambda(|u|^{p-1}u - |v|^{p-1}v) - L|v|^{p-1}v = 0, \tag{24}$$

$$w(x,0) = 0, \quad w_t(x,0) = 0 \quad in \ \Omega, \tag{25}$$

$$w = 0 \ on \ \partial\Omega \times (0,\infty). \tag{26}$$

Theorem 3.3 *Assume that w is the solution of the problem (24)–(26). Then w satisfies*

$$\frac{1}{2}||w_t||^2 + \frac{\sigma}{2}||\nabla w||^2 + \frac{m^2}{2}||w||^2 \leq \frac{e^{M_3 t} D_2^p t}{2} L^2, \qquad \forall t > 0, \tag{27}$$

where $M_3 > 0$ and $D_2 > 0$ are constants depending on the parameters and the initial data for the equation (1).

Proof Multiplying equation (24) by w_t in the L^2 sense and employing useful inequalities that were used before, the following is obtained:

$$\begin{aligned}
\frac{d}{dt}E_w(t) + \alpha||\nabla w_t||^2 + \beta||w_t||^2 &\leq \lambda\left|\int_\Omega \Big(|u|^{p-1}u - |v|^{p-1}v\Big) w_t dx\right| + L\left|\Big(|v|^{p-1}v, w_t\Big)\right| \\
&\leq \frac{C_3}{2}||w_t||^2 + \frac{C_3}{2}||\nabla w||^2 + \frac{L^2}{2}||v||_{2p}^{2p} + \frac{1}{2}||w_t||^2 \\
&\leq \frac{C_3}{2}||w_t||^2 + \frac{C_3}{2}||\nabla w||^2 + \frac{L^2}{2}C||\nabla v||^{2p} + \frac{1}{2}||w_t||^2 \\
&\leq \frac{C_3}{2}||w_t||^2 + \frac{C_3}{2}||\nabla w||^2 + \frac{CD_2^p}{2}L^2 + \frac{1}{2}||w_t||^2 + \frac{m^2}{2}||w||^2.
\end{aligned}$$

The last inequality implies

$$\frac{d}{dt}E_w(t) \leq M_3 E_w(t) + \frac{CD_2^p}{2}L^2,$$

where $M_3 = max\left\{1, \frac{C_3}{\sigma}, 1 + C_3\right\}$. Therefore, we obtain

$$E_w(t) \leq \frac{e^{M_3 t} C D_2^p t}{2} L^2.$$

Hence, the proof is completed. □

Remark 3.1 Besides the above approach, continuous dependency on the coefficients β and σ can also be studied in the similarly proved calculations for the other coefficients.

4. Conclusion

In this paper, from the assessment of (4), it is shown by the multiplier method that the solution of the Klein–Gordon equation (1) depends continuously on its coefficients.

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