Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 911 – 923
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doi:10.3906/mat-1611-113

Reflexivity of vector-valued Köthe–Orlicz sequence spaces

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Received: 28.11.2016 • **Accepted/Published Online:** 25.08.2017 • **Final Version:** 08.05.2018

Abstract: Let E be a Banach space, λ a perfect sequence space, and M an Orlicz function. Denote by $\lambda(E,M)_r$ the space of all weakly (M,λ)-summable sequences from E that are the limit of their finite sections. In this paper, we describe the continuous linear functionals on $\lambda(E,M)_r$ in terms of strongly (N,λ^*)-summable sequences in the dual E^* of E, and then we give a characterization of the reflexivity of $\lambda(E,M)$ in terms of that of λ and of E and the AK-property.

Key words: Banach spaces, vector-valued sequence spaces, Orlicz function, duality

1. Introduction

In connection with the nuclearity of a locally convex space E, Pietsch [13] introduced the spaces $\ell_p(E)$ and $\ell_p\{E\}$ respectively of weakly ℓ_p-summable and absolutely ℓ_p-summable sequences in E. This allowed him also to introduce and study absolutely p-summing operators. He introduced and studied also the spaces $\lambda\{E\}$ and $\lambda(E)$ of λ-summable and weakly λ-summable sequences in E, λ being a perfect sequence space in the sense of Köthe endowed with its normal topology.

Later, Rosier considered in [14] the general case where λ is no longer equipped with the normal topology, but with a general polar one. He obtained many results, among them a complete description of the dual space of $\lambda\{E\}$. Florencio and Paúl [3] and [4] considered a general polar topology on λ and obtained interesting results on $\lambda(E)$. In particular, using the AK property, they represent the elements of the completion $\lambda\widetilde{\otimes}_\epsilon E$ of the injective tensor product $\lambda\otimes_\epsilon E$ as weakly $\lambda-$ summable sequences in E.

In [10], the authors extend to the locally convex setting the definition of the strong summability introduced first by Cohen [1] in the case when E is a normed space. They made use of this notion to describe the continuous dual space of $\lambda(E)$. Many other results on $\lambda(E)$ have been obtained in [11], [9], and [12].

Ghosh and Srivastava in [5] deal with an Orlicz function M to extend the notion of absolute $\lambda-$summability. They introduce and study the space $F(E,M)$ of those sequences $(x_n)_n$ in a Banach space E for which $(M(\|x_n\|_E/\rho))_n\in F$, for some $\rho>0$, where F is a normal sequence space. In this paper, we introduce the space $\lambda(E,M)$ of all weakly (M,λ)-summable sequences $(x_n)_n$ from a Banach space E ; that is $(\alpha_n a(x_n))_n\in\ell_M$, for all $(\alpha_n)\in\lambda^*$ and $a\in E^*$, where ℓ_M is the Orlicz sequence space associated with the Orlicz function M. For $M(t)=t$ the spaces $\lambda(E,M)$ and $\lambda(E)$ coincide.

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2010 *AMS Mathematics Subject Classification:* 46A17, 46B35, 46A45

2. Definitions and preliminaries

In the sequel, if V is a normed space then V^*, $\|\cdot\|_{V^*}$, and B_{V^*} will denote respectively the topological dual, the norm, and the closed unit ball of V.

Let ω denote the vector space of all real or complex sequences for the usual coordinate operations. For all $k \in \mathbb{N}$, e_k will denote the k-th unit vector of ω. We mean by a sequence space any linear subspace of ω. If λ is a sequence space, we denote by λ^* its α-dual defined by

$$\lambda^* = \left\{ (\beta_n) \in \omega : \sum_{n=1}^{\infty} |\alpha_n \beta_n| \text{ converges, for all } (\alpha_n)_n \in \lambda \right\}.$$

We see that $\lambda \subset \lambda^{**} = (\lambda^*)^*$, and λ is said to be perfect if $\lambda = \lambda^{**}$. Throughout this paper, λ stands for a Banach perfect sequence space whose norm $\|\cdot\|_\lambda$ satisfies

(1) for all α and β in λ, if $\alpha \leq \beta$ then $\|\alpha\|_\lambda \leq \|\beta\|_\lambda$.

(2) λ is an AK-space. This means that every $(\alpha_n)_n \in \lambda$ is the $\|\cdot\|_\lambda-$limit of its finite sections $(\alpha_1, \ldots, \alpha_n, 0, \ldots)$, $n \in \mathbb{N}$.

In this case the topological dual of λ coincides with its $\alpha-$dual. The norm of λ^* is then defined by

$$\|\beta\|_{\lambda^*} = \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n \beta_n|, \ \alpha \in \lambda \text{ and } \|\alpha\|_\lambda \leq 1 \right\}.$$

We assume moreover that $(\lambda^*, \|\cdot\|_{\lambda^*})$ is also an AK-space. In particular, λ is a reflexive Banach space.

An Orlicz function is a continuous, convex, nondecreasing function M defined from $[0, \infty)$ to itself such that $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \mu(t)dt.$$

Define, for $s \geq 0$,

$$\nu(s) = \sup\{t : \mu(t) \leq s\}.$$

Then ν possesses the same properties as μ and the function N defined by

$$N(x) = \int_0^x \nu(t)dt$$

is an Orlicz function. The functions M and N are called mutually complementary Orlicz functions and satisfy the Young inequality,

$$uv \leq M(u) + N(v), \text{ for } u, v \geq 0. \tag{2.1}$$

The Orlicz sequence space ℓ_M, introduced in [8], is defined by

$$\ell_M = \left\{ (\alpha_n)_n \in \omega,\ \sum_{n=1}^{\infty} M\left(\frac{|\alpha_n|}{\rho}\right) < \infty,\ \text{for some } \rho > 0 \right\}.$$

ℓ_M is a Banach space with respect to the norm

$$\|(\alpha_n)_n\|_M = \inf \left\{ \rho > 0,\ \sum_{n=1}^{\infty} M\left(\frac{|\alpha_n|}{\rho}\right) \leq 1 \right\}.$$

For $M(t) = t^p,\ 1 \leq p < \infty$, the space ℓ_M coincides with the classical sequence space ℓ_p.
An Orlicz function M satisfies the condition that $M(\eta x) \leq \eta M(x)$, if $0 \leq \eta \leq 1$. It is said to satisfy Δ_2-condition if there exists a constant $K > 0$, such that, for every $x \geq 0,\ M(2x) \leq KM(x)$. In this case, $\ell_M^* = \ell_N$ (see e.g. [6], Corollary 4.2).

3. The space $\lambda(E, M)$

Let E stand for a Banach space and $\omega(E)$ denote the linear space of all E-valued sequences. Define the space $\lambda(E, M)$ of weakly $(M, \lambda)-$ summable sequences of E by

$$\lambda(E, M) = \{x = (x_n)_n \subset E : \text{for all } a \in E^*, (\alpha_n)_n \in \lambda^*, (\alpha_n a(x_n)) \in \ell_M\}.$$

We have

Theorem 3.1 *With the usual coordinate operations, $\lambda(E, M)$ is a vector space on which*

$$\begin{aligned} \|x\|_{\lambda(E,M)} &= \sup\{\|(\alpha_n a(x_n))\|_M : a \in B_{E^*}, \alpha \in B_{\lambda^*}\} \\ &= \sup_{a \in B_{E^*}, \alpha \in B_{\lambda^*}} \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} M(|\alpha_n a(x_n)|/\rho) \leq 1 \right\}, \end{aligned}$$

for $x = (x_n)_n \in \lambda(E, M)$, defines a norm.

Proof For all $a \in E^*$ and $(\alpha_n)_n \in \lambda^*$, define $\varphi_{a,\alpha} : \omega(E) \to \omega$ by $\varphi_{a,\alpha}(x) = (\alpha_n a(x_n))$, for all $x = (x_n) \in \lambda(E, M)$.
It is clear that $\varphi_{a,\alpha}$ is linear and that

$$\lambda(E, M) = \bigcap_{(a,\alpha) \in E^* \times \lambda^*} \varphi_{a,\alpha}^{-1}(\ell_M).$$

Thus, $\lambda(E, M)$ is a vector space.

For the second part of the theorem, we prove only that $\|x\|_{\lambda(E,M)}$ is finite. Fix $x = (x_n)_n \in \lambda(E, M)$ and define the family of linear mappings $f_\alpha : E^* \to \ell_M$ by $f_\alpha(a) = (\alpha_n a(x_n))$, for $\alpha = (\alpha_n) \in B_{\lambda^*}$. Then f_α is linear and continuous by the closed graph theorem. The same argument shows that $g_a(\alpha) = f_\alpha(a) : \lambda^* \to \ell_M$ is continuous for every $a \in E^*$. On the other hand, for every $a \in E^*$,

$$\sup_{\alpha \in B_{\lambda^*}} \|f_\alpha(a)\|_M = \sup_{\alpha \in B_{\lambda^*}} \|g_a(\alpha)\|_M = \|g_a\|_{\mathcal{L}(\lambda^*, \ell_M)} < \infty.$$

By the uniform boundedness principle,

$$\sup\{\|(\alpha_n a(x_n))\|_M : a \in B_{E^*}, \alpha \in B_{\lambda^*}\} = \sup_{\alpha\in B_{\lambda^*}} \|f_\alpha\|_{\mathcal{L}(E^*,\ell_M)} < \infty,$$

and $\|x\|_{\lambda(E,M)}$ is finite. □

We establish now the continuity of the projections.

Lemma 3.2 *For $k \in \mathbb{N}$, let π_k denote the projection from $\lambda(E,M)$ on E defined by*

$$\pi_k(x) = x_k, \text{for all } x = (x_n) \in \lambda(E,M).$$

Then π_k is linear and continuous.

Proof Fix $k \in \mathbb{N}$, $a \in B_{E^*}$, and $(\alpha_n)_n \in B_{\lambda^*}$ with $\alpha_k > 0$. Let $\kappa = 1/(\alpha_k\|e_k\|_M)$. For all $x = (x_n) \in \lambda(E,M)$, we have

$$\begin{aligned}\alpha_k|a(x_k)|\|e_k\|_M = \|\alpha_k a(x_k)e_k\|_M &\leq \|(a(\alpha_n x_n))_n\|_M \\ &\leq \|(x_n)_n\|_{\lambda(E,M)}.\end{aligned}$$

Thus, $\|x_k\|_E \leq \kappa\|x\|_{\lambda(E,M)}$ for all $x = (x_n) \in \lambda(E,M)$ and π_k is continuous. □

Theorem 3.3 *The normed space $\lambda(E,M)$ is complete and E is isomorphic to a closed linear subspace of it.*

Proof Consider a nonzero $\alpha = (\alpha_n)_n \in \lambda$. We will show that

$$\|(\alpha_n t)_n\|_{\lambda(E,M)} \leq M(1)\|\alpha\|_\lambda\|t\|_E, \text{ for all } \alpha = (\alpha_n)_n \in \lambda \text{ and } t \in E. \tag{3.1}$$

The inequality is obvious if $t = 0$. Suppose that $t \neq 0$ and set $\rho_0 = M(1)\|\alpha\|_\lambda\|t\|_E$. If $\beta = (\beta_n)_n \in \lambda^*$ with $\|\beta\|_{\lambda^*} \leq 1$ and $a \in E^*$ with $\|a\|_{E^*} \leq 1$, then by the convexity of M,

$$\sum_{n=1}^{\infty} M\left(\frac{|\alpha_n\beta_n a(t)|}{\rho_0}\right) \leq \sum_{n=1}^{\infty} \frac{|\alpha_n\beta_n||a(t)|}{\rho_0} M(1) \leq 1.$$

Thus, $\|(\beta_n\alpha_n a(t))_n\|_M \leq \rho_0$. However,

$$\begin{aligned}\|(\alpha_n t)_n\|_{\lambda(E,M)} &= \sup_{a\in B_{E^*},\alpha\in B_{\lambda^*}} \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} M(|\alpha_n a(x_n)|/\rho) \leq 1\right\} \\ &\leq \rho_0 = M(1)\|\alpha\|_\lambda\|t\|_E.\end{aligned}$$

For a fixed $\gamma = (\gamma_n)_n \in \lambda$, with $\gamma \neq 0$ the mapping $t \in E \to (\gamma_n t)_n \in \lambda(E,M)$ is well defined, injective, and continuous by (3.1). Let $(t_k)_k$ be a sequence in E such that $(\gamma t_k)_k$ converges in $\lambda(E,M)$ to $x = (x_n)_n$. For every $m \in \mathbb{N}$ with $\gamma_m \neq 0$, the sequence $(t_k)_k$ converges to $\frac{1}{\gamma_m}x_m$, by Lemma 3.2. If t denotes the limit of $(t_k)_k$ then $x_n = t$ if $\gamma_n \neq 0$ and $x_n = 0$ otherwise, and so $x = \gamma t$, and the range of E is closed in $\lambda(E,M)$.

Let $x^k = (x_n^k), k = 1, 2, \ldots,$ be a Cauchy sequence in $\lambda(E,M)$. For a fixed $n \in \mathbb{N}$, by Lemma 3.2, $x_n^k, k = 1, 2, \ldots,$ is a Cauchy sequence in E; let $x_n \in E$ be its limit. We will prove that $x = (x_n)_n \in \lambda(E,M)$

and that $(x^k)_k$ converges to x. Fix $\alpha = (\alpha_n) \in \lambda^*$ and $a \in E^*$. It is clear that the mapping $\varphi_{\alpha,a} : y = (y_n) \in \lambda(E, M) \to (\alpha_n a(y_n)) \in \ell_M$ is linear and continuous. Thus, $\varphi_{\alpha,a}(x^k) = (\alpha_n a(x^k_n)), k = 1, 2, \ldots,$ is a Cauchy sequence in the Banach space ℓ_M. Let $\beta = (\beta_n)$ be its limit in ℓ_M. For every $n \in \mathbb{N}$, we have

$$\begin{aligned} \alpha_n a(x_n) &= \alpha_n a\left(\lim_{k\to\infty} x^k_n\right) \\ &= \lim_{k\to\infty} \alpha_n a(x^k_n) = \beta_n. \end{aligned}$$

Hence, $(\alpha_n a(x_n)) = \beta \in \ell_M$. Thus, $x \in \lambda(E, M)$. It remains to show that $(x^k)_k$ converges to x.
For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $q \geq p \geq N$, $\alpha = (\alpha_n) \in B_{\lambda^*}$ and $a \in B_{E^*}$, there exists $0 < \rho < \varepsilon$ that satisfies

$$\sup_{K\in\mathbb{N}} \sum_{n=1}^{K} M(|\alpha_n a(x^p_n - x^q_n)|/\rho) = \sum_{n=1}^{\infty} M(|\alpha_n a(x^p_n - x^q_n)|/\rho) \leq 1.$$

Since M is continuous, letting $q \to \infty$, we get $\sum_{n=1}^{K} M(|\alpha_n a(x^p_n - x_n)|/\varepsilon) \leq 1$ for $K \geq N$; and then

$$\|x^p - x\|_{\lambda(E,M)} = \sup_{a\in B_{E^*},\alpha\in B^*_\lambda} \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} M(|\alpha_n a(x^p_n - x_n)|/\rho) \leq 1.\right\} \leq \varepsilon,$$

for every $p \geq N$. This completes the proof. □

4. The space $\lambda\langle E, M\rangle$

A sequence $(x_n)_n$ is said to be strongly (M, λ)-summable in E, if for every $(a_n)_n \in \lambda^*(E^*, N)$, one has $(a_n(x_n))_n \in \ell_1$. The space of these sequences will be denoted $\lambda\langle E, M\rangle$.

That is

$$\lambda\langle E, M\rangle = \{x = (x_n)_n \subset E : \text{for all } a = (a_n)_n \in \lambda^*(E^*, N), (a_n(x_n))_n \in \ell_1\}.$$

If we endow $\lambda\langle E, M\rangle$ with the standard coordinate operations $\lambda\langle E, M\rangle$ is a vector space over $\mathbb{K}$ that contains the finite sequences of E. Indeed, if $a = (a_n)_n \in \lambda^*(E^*, N)$, the map φ_a from $\omega(E)$ into ω defined by $\varphi_a(x) = (a_n(x_n))$, for all $x = (x_n) \in \lambda\langle E, M\rangle$ is linear such that

$$\lambda\langle E, M\rangle = \bigcap_{a\in\lambda^*(E^*,N),} \varphi_a^{-1}(\ell_1).$$

Although many properties of the spaces $\lambda(E, M)$ and $\lambda\langle E, M\rangle$ are similar, the techniques of their proofs are different.

Next, we define a norm on $\lambda\langle E, M\rangle$.

Theorem 4.1 *For $x = (x_n)_n \in \lambda\langle E, M\rangle$ set*

$$\|x\|_{\lambda\langle E,M\rangle} = \sup\left\{\sum_{n=1}^{\infty} |a_n(x_n)| : a = (a_n)_n \in \lambda^*(E^*, N),\ \|a\|_{\lambda^*(E^*,N)} \leq 1\right\}.$$

Then $\|x\|_{\lambda\langle E,M\rangle}$ defines a norm on $\lambda\langle E, M\rangle$.

Proof Fix $x = (x_n)_n \in \lambda\langle E, M\rangle$ and define the family of linear mappings $\varphi_x : \lambda^*(E^*, N) \to \ell_1$ by $\varphi_x(a) = (a_n(x_n))$, for all $a = (a_n) \in \lambda^*(E^*, N)$.
Then φ_x is linear and it is easy to check that the graph of φ_x is closed. As $\lambda^*(E^*, N)$ is a Banach space by Theorem 3.3, φ_x is continuous. This proves that $\|x\|_{\lambda\langle E,M\rangle}$ is finite.

The other properties of the norm derive from that of $\|\cdot\|_M$ and the supremum. □

Next, we establish the continuity of the projections.

Lemma 4.2 *For $k \in \mathbb{N}$, let π_k denote the projection from $\lambda\langle E, M\rangle$ on E defined by*

$$\pi_k(x) = x_k, \text{for all } x = (x_n) \in \lambda\langle E, M\rangle.$$

Then π_k is linear and continuous.

Proof Fix $k \in \mathbb{N}$. Let $x^* \in E^*$ such that $\|x^*\|_{E^*} \leq 1$. Set

$$\delta_k = \sup\left\{\|\alpha_k e_k\|_N : \alpha = (\alpha_n)_n \in \lambda \text{ and } \|\alpha\|_\lambda \leq 1\right\}.$$

Define $a = 1/\delta_k x^* e_k$. It is easy to check that $a \in \lambda^*(E^*, N)$ with $\|a\|_{\lambda^*(E^*,N)} \leq 1$ and $|x^*(x_k)| \leq \delta_k \|x\|_{\lambda\langle E,M\rangle}$, for every $x = (x_n) \in \lambda\langle E, M\rangle$. Since this is satisfied for any $x^* \in E^*$ such that $\|x^*\|_{E^*} \leq 1$, we have

$$\|x_k\|_E \leq \delta_k \|x\|_{\lambda\langle E,M\rangle}, \text{ for every } x = (x_n) \in \lambda\langle E, M\rangle.$$

This shows the continuity of π_k. □

Theorem 4.3 *The normed space $\lambda\langle E, M\rangle$ is complete and E is isomorphic to a closed linear subspace of it.*

Proof Fix $p \in \mathbb{N}$ and define $\theta_p : E \to \lambda\langle E, M\rangle$ by $\theta_p(t) = te_p$ for every $t \in E$. It is clear that θ is linear and injective. Suppose that $\|t\|_E < 1$ and choose $\alpha = (\alpha_n)_n \in \lambda$ with $\|\alpha\|_\lambda \leq 1$ and $\alpha_p > 0$. Let $\kappa = 1/\|\alpha_p e_p\|_N$ and $a \in \lambda^*(E^*, N)$ with $\|a\|_{\lambda^*(E^*,N)} \leq 1$. Then we have

$$|a_p(t)|\|\alpha_p e_p\|_N = \|\alpha_p a_p(t) e_p\|_N \leq 1.$$

However, $\|te_p\|_{\lambda\langle E,M\rangle} = \sup\left\{|a_p(t)| : \|a\|_{\lambda^*(E^*,N)} \leq 1\right\} \leq \kappa$. This means that $\|te_p\|_{\lambda\langle E,M\rangle} \leq \kappa\|t\|_E$ for every $t \in E$ and then θ is continuous. On the other hand, it is easy to check that $\|te_p\|_{\lambda\langle E,M\rangle} \geq \|t\|_E \|e_p\|_\lambda$ for every $t \in E$ and θ is open.

For the completeness of $\lambda\langle E, M\rangle$, let $x^k = (x_n^k), k = 1, 2, \ldots$ be a Cauchy sequence in $\lambda\langle E, M\rangle$. For a fixed $n \in \mathbb{N}$, by Lemma 4.2, the sequence $x_n^k, k = 1, 2, \ldots$ is Cauchy in E and then converges to an $x_n \in E$. Set $x = (x_n)$. We will prove that $x \in \lambda\langle E, M\rangle$ and that $(x^k)_k$ converges to x in $\lambda\langle E, M\rangle$. Let X denote the unit ball of $\lambda^*(E^*, N)$. For every $k \in \mathbb{N}$, let $f_k : X \to \ell_1$ be defined by $f_k(a) = (a_n(x_n^k))_n$ for all $a = (a_n)_n \in X$. Since $(f_k)_k$ is a uniformly Cauchy sequence and ℓ_1 is a Banach space, $(f_k)_k$ must converge uniformly on X to a function $f : X \to \ell_1$. Let $a = (a_n)_n \in X$ and $\alpha = (\alpha_n)_n = f(a)$. Then $\alpha \in \ell_1$. On the other hand, for every $n \in \mathbb{N}$, the sequence $a_n(x_n^k)$, $k = 1, 2, \ldots$ converges to $a_n(x_n)$. However, $f_k(a) = (a_n(x_n^k))_n$ converges to $f(a) = \alpha$. This gives $(a_n(x_n))_n \in \ell_1$ and then $x \in \lambda\langle E, M\rangle$. Since $(f_k)_k$ converges uniformly on X to f, the sequence $(x^k)_k$ converges in $\lambda\langle E, M\rangle$ to x. □

5. Dual space of $\lambda(E,M)$

If $x=(x_n)\in\omega(E)$ then we denote by $x^{(k)}=(x_1,x_2,\ldots,x_k,0\ldots)$ the sequence of the finite sections of x. If $x\in\lambda(E,M)$, then $x^{(k)}\in\lambda(E,M)$ for all $k\in\mathbb{N}$. Using the notation $x^{(k)}=\sum_{n=1}^{k}x_ne_n$, we see that if x is the limit of its finite sections, then

$$x=\lim_{k\to\infty}x^{(k)}=\sum_{n=1}^{\infty}x_ne_n. \tag{5.1}$$

If $\lambda(E,M)_r$ denotes the subspace of $\lambda(E,M)$ of the sequences of $\lambda(E,M)$, which are the limit of their finite sections, then $\lambda(E,M)$ is said to have the AK-property if $\lambda(E,M)=\lambda(E,M)_r$.

We have

Theorem 5.1 *$\lambda(E,M)_r$ is a closed subspace of $\lambda(E,M)$.*

Proof It is easy to check that if $x=(x_n)\in\lambda(E,M)$ then $\|x^{(k)}\|_{\lambda(E,M)}\le\|x\|_{\lambda(E,M)}$. Suppose that x is in the closure of $\lambda(E,M)_r$ and $\varepsilon>0$. There exist $y\in\lambda(E,M)_r$ and $K\in\mathbb{N}$ such that $\|x-y\|_{\lambda(E,M)}<\varepsilon/3$ and $\|y^{(k)}-y\|_{\lambda(E,M)}<\varepsilon/3$ for all $k\ge K$. Now,

$$\begin{aligned}\|x^{(k)}-x\|_{\lambda(E,M)}\le\|x^{(k)}-y^{(k)}\|_{\lambda(E,M)}+\|y-y^{(k)}\|_{\lambda(E,M)}+\|x-y\|_{\lambda(E,M)}\\ <2\|x-y\|_{\lambda(E,M)}+\varepsilon/3<\varepsilon,\end{aligned}$$

for all $k\ge K$. Then $x\in\lambda(E,M)_r$ □

The following theorem gives an analogue of a result of [10] given for $M(t)=t$, when λ and E are Banach spaces.

Theorem 5.2 *Let F be a continuous linear functional on $\lambda(E,M)$ and, for every $n\in\mathbb{N}$ and $t\in E$, $a_n(t)=F(te_n)$. Then the sequence $(a_n)_n$ is strongly (N,λ^*)-summable in E^*.*

Proof Since F is continuous, there exists $\kappa>0$ such that

$$|F(x)|\le\kappa\|x\|_{\lambda(E,M)},\ \text{for all } x=(x_n)_n\in\lambda(E,M).$$

Fix $n\in\mathbb{N}$ and $t\in E$. We have

$$|a_n(t)|=|F(te_n)|\le\kappa\|te_n\|_{\lambda(E,M)}\le\kappa M(1)\|e_n\|_\lambda\|t\|_E.$$

This means that $(a_n)_n\subset E^*$.
It remains to show that $(a_n)_n\in\lambda^*\langle E^*,N\rangle$. To this end, let $(f_n)_n\in\lambda(E^{**},M)$, $k\in\mathbb{N}$, and $\delta>0$ be given. Then, due to the principle of local reflexivity (cf. [2]), there exists a continuous operator $u_k\colon\operatorname{span}\{f_1,f_2,\ldots,f_k\}\to E$ such that $\|u_k\|_{E^{***}}\le1+\delta$ and $a_n(u_kf_n)=f_n(a_n)$ for all $n\in\{1,2,\ldots,k\}$. Since every a_n is continuous, there exist $0<\delta_n\le\dfrac{\delta}{k(1+\|e_n\|_\lambda)}$ and $x_n\in E$ such that $\|x_n-u_kf_n\|_E\le\ \delta_n$

and $|a_n(x_n - u_k f_n)| \leq \dfrac{\delta}{k(1+\|e_n\|_\lambda)}$. Now,

$$\begin{aligned}
\left|\sum_{n=1}^{k} f_n(a_n)\right| &= \left|\sum_{n=1}^{k} a_n(u_k f_n)\right| \\
&\leq \left|\sum_{n=1}^{k} a_n(x_n - u_k f_n)\right| + \left|\sum_{n=1}^{k} a_n(x_n)\right| \\
&\leq \sum_{n=1}^{k} |a_n(x_n - u_k f_n)| + \left|F\left(\sum_{n=1}^{k} x_n e_n\right)\right| \\
&\leq \delta + \kappa \left\|\sum_{n=1}^{k} x_n e_n\right\|_{\lambda(E,M)}.
\end{aligned}$$

However, for $\alpha = (\alpha_n)_n \in \lambda^*$, with $\|\alpha\|_{\lambda^*} \leq 1$ and $a \in E^*$, with $\|a\|_{E^*} \leq 1$,

$$\left\|\sum_{n=1}^{k} \alpha_n a(x_n) e_n\right\|_M \leq \left\|\sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n\right\|_M + \left\|\sum_{n=1}^{k} \alpha_n a(u_k f_n) e_n\right\|_M. \tag{5.2}$$

On one hand,

$$\begin{aligned}
\sum_{n=1}^{k} M(|\alpha_n a(x_n - u_k f_n)|/\delta) &\leq \sum_{n=1}^{k} M(|\alpha_n|/k(1+\|e_n\|_\lambda)) \\
&\leq \sum_{n=1}^{k} (|\alpha_n|/k(1+\|e_n\|_\lambda) M(1/k) \\
&\leq k\frac{|\alpha_n|}{\|e_n\|_\lambda} M(1/k) \leq kM(1/k) \leq M(1).
\end{aligned}$$

Thus, $\left\|\sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n\right\|_M \leq \delta$, if $M(1) \leq 1$, and $\left\|\sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n\right\|_M \leq M(1)\delta$, if $M(1) \geq 1$. Replacing δ by $M(1)\delta$ if necessary, we may suppose that

$$\left\|\sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n\right\|_M \leq \delta. \tag{5.3}$$

On the other hand,

$$\left\|\sum_{n=1}^{k} \alpha_n a(x_n - u_k f_n) e_n\right\|_M \leq (1+\delta) \left\|\sum_{n=1}^{k} f_n e_n\right\|_{\lambda(E^{**},M)} \leq (1+\delta) \|(f_n)_n\|_{\lambda(E^{**},M)}. \tag{5.4}$$

Combining (5.3) and (5.4) in (5.2) and taking the supremum on B_{E^*} and B_{λ^*}, we get $\left\|\sum_{n=1}^{k} x_n e_n\right\|_{\lambda(E,M)} \leq \delta + (1+\delta) \|(f_n)_n\|_{\lambda(E^{**},M)}$.

Hence

$$\left|\sum_{n=1}^{k} f_n(a_n)\right| \le \delta(\kappa+1) + \kappa(1+\delta)\left\|(f_n)_n\right\|_{\lambda(E^{**},M)},\ (f_n)_n \in \lambda(E^{**},M),\ \delta>0,\ \text{and } k\in\mathbb{N}.$$

Further, let $(\epsilon_n)_n$ be such that $|f_n(a_n)| = \epsilon_n f_n(a_n)$, $n\in\mathbb{N}$. Then $(\epsilon_n f_n)_n \in \lambda(E^{**},M)$ and

$$\sum_{n=1}^{k} |f_n(a_n)| = \sum_{n=1}^{k} \epsilon_n f_n(a_n) \le \delta(\kappa+1) + \kappa(1+\delta)\left\|(f_n)_n\right\|_{\lambda(E^{**},M)}.$$

It follows that $(f_n(a_n))_n \in \ell_1$ and $(a_n)_n \in \lambda^*\langle E^*, N\rangle$. □

Remark 5.3 *From the preceding proof, since δ is arbitrary, one gets*

$$\sum_{n=1}^{\infty} |f_n(a_n)| \le \kappa \left\|(f_n)_n\right\|_{\lambda(E^{**},M)},\ \text{for all } (f_n)_n \in \lambda(E^{**},M). \tag{5.5}$$

Therefore, $\|(a_n)_n\|_{\lambda^*\langle E^*,N\rangle} \le \|F\|_{\lambda(E,M)^*}$.

In order to establish the converse of the last result we need the following characterization of weakly (M,λ)-summable sequences in E^*.

Lemma 5.4

$$\lambda(E^*,M) = \{(a_n)_n \subset E^* : (a_n(\alpha_n x))_n \in \ell_M, \text{for all } x\in E, (\alpha_n)\in\lambda^*\}$$

Proof Let $a=(a_n)_n \in \lambda(E^*,M)$. For all $x\in E$, the evaluation $\delta_x(u)=u(x)$ can be regarded as an element of E^{**}. Then, for every $(\alpha_n)_n\in\lambda^*$, $(\alpha_n\delta_x(a_n))_n = (a_n(\alpha_n x))_n \in \ell_M$. Conversely, assume that for all $x\in E$, $(\alpha_n)_n\in\lambda^*$, $(\alpha_n a_n(x))_n\in\ell_M$ and let $f\in E^{**}$. We shall use the fact that ℓ_M is perfect, since M is supposed to satisfy $\Delta 2$ condition. Let $(\gamma_n)_n \in \ell_M^*$ be given. It suffices to show that the series $\sum|\gamma_n\alpha_n f(a_n)|$ is convergent. Choose $(\epsilon_n)_n$ so that $\epsilon_n f(\gamma_n\alpha_n a_n) = |f(\gamma_n\alpha_n a_n)|$ for all n and set

$$A = \left\{\sum_{n=1}^{p} \epsilon_n\gamma_n\alpha_n a_n : p\in\mathbb{N}\right\}.$$

For all $p\in\mathbb{N}$ and all $x\in E$, one has

$$\sum_{n=1}^{p} |\epsilon_n\gamma_n\alpha_n a_n(x)| \le \sum_{n=1}^{\infty} |\gamma_n\alpha_n a_n(x)|,$$

which is finite since $(\alpha_n a_n(x))_n\in\ell_M$. The set A is then weak*-bounded in E^*, and so A is weakly bounded in E^*. Hence there exists $\rho_f>0$ such that $\sum_{n=1}^{p}\epsilon_n\gamma_n\alpha_n f(a_n)\le\rho_f$, for all $p\in\mathbb{N}$. This proves that the series $\sum|\gamma_n\alpha_n f(a_n)|$ is convergent and that $(\alpha_n f(a_n))_n\in\ell_M$. □

We establish now the converse of Theorem 5.2.

Theorem 5.5 *For every $a = (a_n)_n \in \lambda^*\langle E^*, N\rangle$, the mapping*

$$f_a : x \mapsto \sum_{n=1}^{\infty} a_n(x_n)$$

defines a continuous linear functional on $\lambda(E, M)$.

Proof Let $a = (a_n)_n \in \lambda^*\langle E^*, N\rangle$ and $x = (x_n)_n \in \lambda(E, M)$. We have $(\delta_n)_n \subset E^{**}$, where δ_n is the evaluation $u \mapsto u(x_n)$ at x_n, $u \in E^*$. Thanks to lemma 5.4, since $(\alpha_n \delta_n(u))_n \in \ell_M$, for every $(\alpha_n)_n \in \lambda^*$, we have $(\delta_n)_n \in \lambda(E^{**}, M)$. Hence $\sum |\delta_n(a_n)|$ converges and f_a is well defined.
Next consider the map φ_a defined from $\lambda(E, M)$ into ℓ_1 by $\varphi_a((f_n)_n) = (f_n(a_n))_n$. Then φ_a is well defined. Moreover, suppose that $(x^i)_{i\in\mathbb{N}} \in \lambda(E, M)$ converges to $x := (x_n)_n$ and $(\varphi_a(x^i))_i$ converges in ℓ_1 to $(\alpha_n)_n$. By the continuity of the projections (Lemma 3.2), $(x^i_n)_{i\in\mathbb{N}}$ converges to x_n for every $n \in \mathbb{N}$ and then $(a_n(x^i_n))_{i\in\mathbb{N}}$ converges to $a_n(x_n)$ as well. It follows that $(a_n(x_n))_n = (\alpha_n)_n$, showing that the graph of φ_a is closed and then that φ_a is continuous, since $\lambda(E, M)$ is a Banach space (Theorem 6.4). Then there exists $c > 0$ so that

$$\sum_{n=1}^{\infty} |a_n(x_n)| \leq c\|(x_n)_n\|_{\lambda(E,M)}, \text{ for all } (x_n)_n \in \lambda(E, M).$$

This shows that f_a is continuous on $\lambda(E, M)$. □

We now obtain the promised characterization of continuous linear functionals on $\lambda(E, M)_r$.

Theorem 5.6 *The following equality holds algebraically and topologically*

$$(\lambda(E, M)_r)^* = \lambda^*\langle E^*, N\rangle. \tag{5.6}$$

Proof Consider the mapping $\varphi : a \mapsto f_a$ from $\lambda^*\langle E^*, N\rangle$ to $(\lambda(E, M)_r)^*$ defined in Theorem 5.5. φ is clearly linear. Suppose that there exists $a = (a_n)_n \in \lambda^*\langle E^*, N\rangle$ such that $f_a(x) = 0$, for every $x = (x_n)_n \in \lambda(E, M)_r$. Fix $k \in \mathbb{N}$ and $t \in E$. We have $a_k(t) = f_a(te_n) = 0$, which means that $a_k = 0$. Since k was arbitrary, $a = (a_n)_n = 0$ and φ is one to one. Conversely, if $f \in (\lambda(E, M)_r)^*$ then let $a = (a_n)_n \in \lambda^*\langle E^*, N\rangle$ as defined in Theorem 5.2. If $x = (x_n)_n \in \lambda(E, M)_r$, then $x = \sum_{n=1}^{\infty} x_n e_n$ by (5.1). As f is continuous, $f(x) = \sum_{n=1}^{\infty} f(x_n e_n) = \sum_{n=1}^{\infty} a_n(x_n)$, which gives $\varphi(a) = f$ and φ is onto, and (5.6) holds algebraically. Since φ^{-1} is defined between Banach spaces (Theorems 3.3 and 4.3), and is continuous by (5.5), φ is an isomorphism by the open mapping theorem. □

6. Reflexivity of $\lambda(E, M)$

In the sequel, we denote by $\lambda\langle E, M\rangle_r$ the subspace of $\lambda\langle E, M\rangle$ formed by the sequences of $\lambda\langle E, M\rangle$, which are the limit of their finite sections.
The proof of the following theorem is along the same lines as that of Theorem 5.2; we give it for the sake of completeness.

Theorem 6.1 *Let G be a continuous linear functional on $\lambda\langle E, M\rangle$ and, for every $n \in \mathbb{N}$ and $t \in E$, $a_n(t) = G(te_n)$. Then the sequence $(a_n)_n$ is weakly (N, λ^*)-summable in E^*.*

Proof Since G is continuous, there exists $\eta > 0$ such that

$$|G(x)| \leq \eta \|x\|_{\lambda\langle E,M\rangle}, \text{ for all } x = (x_n)_n \in \lambda\langle E, M\rangle.$$

Fix $n \in \mathbb{N}$ and $t \in E$, and put

$$a_n(t) = G(te_n) = G(0, \ldots, t, 0, \ldots).$$

Then a_n is a linear functional on E with

$$G \circ \theta_n = a_n,$$

where

$$\theta_n : E \longrightarrow \lambda \langle E, M\rangle, t \longrightarrow (0, \ldots, t, 0, \ldots).$$

It follows from Theorem 4.3, that a_n is continuous and then $(a_n)_n \subset E^*$.
It remains to show that $(a_n)_n \in \lambda^*(E^*, N)$. To this end, let $(\alpha_n)_n \in B_\lambda$ and $u \in B_E$. We shall show that $(\alpha_n a_n(u))_n \in \ell_N$. Let $(\beta_n)_n \in \ell_M$ and for every $n \in \mathbb{N}$, ε_n be such that $|\beta_n \alpha_n a_n(u)| = \varepsilon_n \beta_n \alpha_n a_n(u)$. Fix $k \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{n=1}^{k} |\beta_n \alpha_n a_n(u)| = G\left(\sum_{n=1}^{k} \varepsilon_n \beta_n \alpha_n u e_n\right) &\leq \eta \|\sum_{n=1}^{k} \varepsilon_n \beta_n \alpha_n u e_n\|_{\lambda\langle E,M\rangle} \\ &= \eta \sup\left\{\sum_{n=1}^{k} |\beta_n \alpha_n b_n(u)| : \|(b)_n\|_{\lambda^*(E^*,N)} \leq 1\right\}.\end{aligned}$$

For every $\varepsilon > 0$, there exists $b = (b_n)_n \in \lambda^*(E^*, N)$ such that $\|(b_n)_n\|_{\lambda^*(E^*,N)} < 1$ with $\| \sum_{n=1}^{k} \varepsilon_n \beta_n \alpha_n u e_n\|_{\lambda\langle E,M\rangle} \leq \varepsilon + \sum_{n=1}^{k} |\beta_n \alpha_n b_n(u)|$.

However, from (2.1), we have

$$\sum_{n=1}^{k} |\beta_n \alpha_n b_n(u)| \leq \sum_{n=1}^{k} M(|\beta_n|) + \sum_{n=1}^{k} N(|\alpha_n b_n(u)|) \leq \sum_{n=1}^{\infty} M(|\beta_n|) + \sum_{n=1}^{\infty} N(|\alpha_n b_n(u)|).$$

Since $\|(b_n)_n\|_{\lambda^*(E^*,N)} < 1$, we have $\|(\alpha_n b_n(u))_n\|_N < 1$ and then there exists $0 < \rho < 1$ such that $\sum_{n=1}^{\infty} N(|\alpha_n b_n(u)|) \leq \sum_{n=1}^{\infty} N(|\alpha_n b_n(u)|/\rho) \leq 1$. Therefore, $\sum_{n=1}^{k} |\beta_n \alpha_n a_n(u)| \leq \eta(\varepsilon + \sum_{n=1}^{\infty} M(|\beta_n|) + 1)$, for every $k \in \mathbb{N}$.
Hence, the series $\sum_{n=1}^{\infty} \beta_n \alpha_n b_n(u)$ converges, and $(\alpha_n a_n(u))_n \in (\ell_M)^* = \ell_N$. That is, $(a_n)_n \in \lambda^*(E^*, N)$. □

Remark 6.2 *From the preceding proof, since* ε *is arbitrary, one gets*

$$\sum_{n=1}^{\infty} |\beta_n \alpha_n a_n(u)| \leq 2\eta, \text{ for all } (\alpha_n)_n \in B_\lambda, (\beta_n)_n \in B_{\ell_M}, u \in B_E.$$

Therefore, $\|(a_n)_n\|_{\lambda^*(E^*,N)} \leq 2\|G\|_{\lambda\langle E,M\rangle^*}$.

We establish now the converse of Theorem 6.1.

Theorem 6.3 *For every $a = (a_n)_n \in \lambda^*(E^*, N)$, the mapping*

$$g_a : x \mapsto \sum_{n=1}^{\infty} a_n(x_n)$$

defines a continuous linear functional on $\lambda\langle E, M\rangle$.

Proof Let $a = (a_n)_n \in \lambda^*(E^*, N)$. Then, for every $x \in \lambda\langle E, M\rangle$, $(a_n(x_n))_n \in \ell_1$, by the definition of $\lambda\langle E, M\rangle$. Therefore g_a is well defined. Suppose that $(x^i)_{i\in\mathbb{N}} \in \lambda\langle E, M\rangle$ converges to $x := (x_n)_n$ and $(\varphi_a(x^i))_i$ converges in ℓ_1 to $(\alpha_n)_n$. By the continuity of the projections (Lemma 4.2), $(x^i_n)_{i\in\mathbb{N}}$ converges to x_n for every $n \in \mathbb{N}$ and then $(a_n(x^i_n))_{i\in\mathbb{N}}$ converges to $a_n(x_n)$ as well. It follows that $(a_n(x_n))_n = (\alpha_n)_n$ and that the graph of φ_a is closed. This shows that φ_a is continuous. Hence g_a is continuous on $\lambda\langle E, M\rangle$. □

We now state the characterization of continuous linear functionals on $\lambda\langle E, M\rangle_r$.

Theorem 6.4 *The following equality holds algebraically and topologically*

$$(\lambda\langle E, M\rangle_r)^* = \lambda^*(E^*, N). \tag{6.1}$$

Proof Consider the mapping $\psi : a \mapsto g_a$ from $\lambda^*(E^*, N)$ to $(\lambda\langle E, M\rangle_r)^*$ defined in Theorem 6.3. It is clear that ψ is linear. Suppose that there exists $a = (a_n)_n \in \lambda^*(E^*, N)$ such that $g_a(x) = 0$, for every $x = (x_n)_n \in \lambda\langle E, M\rangle_r$. Fix $k \in \mathbb{N}$ and $t \in E$. We have $a_k(t) = g_a(te_n) = 0$, which means that $a_k = 0$. Since k was arbitrary, $a = (a_n)_n = 0$ and ψ is one to one.

Conversely, let $g \in (\lambda\langle E, M\rangle_r)^*$ and $a = (a_n)_n \in \lambda^*(E^*, N)$ as defined in Theorem 6.1. If $x = (x_n)_n \in \lambda\langle E, M\rangle_r$, then $x = \sum_{n=1}^{\infty} x_n e_n$. As g is continuous, $g(x) = \sum_{n=1}^{\infty} g(x_n e_n) = \sum_{n=1}^{\infty} a_n(x_n)$, and $\psi(a) = g$. Thus ψ is onto. The equality (6.1) holds algebraically.
However, according to Remark 6.2, $\|(a_n)_n\|_{\lambda^*(E^*,N)} \leq 2\|g_a\|_{\lambda\langle E,M\rangle^*}$ and then ψ is open. Since ψ is bijective between Banach spaces (Theorems 3.3, 4.3), ψ is continuous by the open mapping theorem. This finishes the proof. □

We give our main result in the following

Theorem 6.5 *If M and N possess the $\Delta 2$-condition, then $\lambda(E, M)$ is reflexive if and only if the following assertions hold:*

(i) E is reflexive,

(ii) $\lambda(E, M)$ is an AK-space,

(iii) $\lambda^*\langle E^*, N\rangle$ is an AK-space.

Proof If $\lambda(E, M)$ is reflexive, then E is reflexive as a closed subspace of $\lambda(E, M)$, by Theorem 3.3. Hence, (i) holds.
By [7, 23.5(10)] and our Theorem 5.1, $\lambda(E, M)_r$ is also reflexive as a closed subspace of $\lambda(E, M)$. It is then weakly quasi-complete by [7, 23.5(2)]. Thus, $\lambda(E, M)_r$ is weakly sequentially complete.
Let $x = (x_n)_n \in \lambda(E, M)$. Then the sequence $(x^{(k)})_{k\in\mathbb{N}}$ consisting of the finite sections of x is contained in

$\lambda(E,M)_r$ and is weakly Cauchy in it. In fact, let a be in $(\lambda(E,M)_r)^*$. By Theorem 5.5, the series $\sum a_n(x_n)$ converges, and $(\langle x^{(k)},a\rangle)_k = (\sum_{n=1}^k a_n(x_n))_k$ is then a Cauchy sequence; hence $(x^{(k)})_{k\in\mathbb{N}}$ converges weakly to a limit $y=(y_n)_n \in \lambda(E,M)_r$ and it is obvious that $x=y$ so that (ii) holds.
Now, since $\lambda(E,M)_r$ is reflexive, the same holds for its dual $\lambda^*\langle E^*,N\rangle$ and the argumentation above still works to infer that (iii) holds.
Conversely, assume that (i), (ii), and (iii) are satisfied. Then, since λ and E are reflexive, an application of Theorems 5.6 and 6.4 gives, algebraically and topologically,

$$\begin{aligned}(\lambda(E,M))^{**} &= (\lambda(E,M)_r)^{**}, && \text{(by (ii))}\\ &= (\lambda^*\langle E^*,N\rangle)^* = (\lambda^*\langle E^*,N\rangle_r)^*, && \text{(by (iii))}\\ &= \lambda^{**}(E^{**},M), && \text{(by Theorem 6.4)}\\ &= \lambda(E,M), && \text{(by (i)).}\end{aligned}$$

Then $\lambda(E,M)$ is reflexive. □

Acknowledgment

The author thanks the Deanship of Scientific Research at Al Imam Mohammad Ibn Saud Islamic University in the Kingdom of Saudi Arabia to have supported this project number 331217, in 2013. We are also indebted to the referee for relevant suggestions that have improved the quality of the paper.

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