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On a new identity for the H-function with applications to the summation of hypergeometric series

Arjun Kumar RATHIE[1], **Luan Carlos de Sena Monteiro OZELIM**[2,*] , **Pushpa Narayan RATHIE**[3]

[1] Department of Mathematics, Vedant College of Engineering and Technology, Rajasthan Technical University, BUNDI, Rajasthan State, India

[2]Department of Civil and Environmental Engineering, University of Brasilia, Brasilia, Brazil

[3]Department of Statistics and Applied Mathematics, Federal University of Ceara, Fortaleza, Brazil

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Abstract: Using generalized hypergeometric functions to perform symbolic manipulations of equations is of great importance to pure and applied scientists. There are in the literature a great number of identities for the Meijer-G function. On the other hand, when more complex expressions arise, that function is not capable of representing them. The H-function is an alternative to overcome this issue, as it is a generalization of the Meijer-G function. In the present paper, a new identity for the H-function is derived. In short, this result enables one to split a particular H-function into the sum of two other H-functions. The new relation in addition to an old result are applied to the summation of hypergeometric series. Finally, some relations between H-functions and elementary functions are built.

Key words: H-function, hypergeometric sum, identity

1. Introduction

Special functions have proven to be essential tools while dealing with the formal mathematical manipulation of equations. In fact, most of the computational software programs that perform symbolic operations consider generalized hypergeometric functions to do so.

Generalized hypergeometric functions of the type ${}_pF_q$ have been extensively studied. For example, in the works of [1, 3, 8, 9], a series of identities have been derived for this function.

These functions, on the other hand, are able to represent just a small share of the mathematical relations commonly considered in science. Thus, more general hypergeometric functions must be considered. This is the case of the Meijer-G function [4].

Mathematica software, for example, vastly relies on the Meijer-G function to perform integration, differentiation, and algebraic manipulation of standard and special functions. This comes from the fact that most of the functions that are used in science are representable in terms of this special function.

When more complex expressions arise, Meijer-G functions are not capable of representing the functional relations that show up. Thus, a more general function is needed for this task. This is where the H-function, which is a generalization of the Meijer-G function, can be used. The H-function is a powerful hypergeometric function whose importance in pure and applied sciences has been considerably discussed [4, 7].

*Correspondence: luanoz@gmail.com

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Useful identities for this special function were presented in [4, 6]. In the present paper, a new identity for the H-function has been derived. It is shown how this identitiy can be used to provide closed form representations for hypergeometric summations. New relations between H-functions and elementary functions are also presented.

In order to better familiarize the reader with this special function, the next section presents some basic concepts regarding the H-function.

2. H-function

The H-function (see [4]) is defined as a contour complex integral that contains gamma functions in its integrands, by

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,A), & \ldots, & (a_n,A_n), & (a_{n+1},A_{n+1}), & \ldots, & (a_p,A_p)\\(b_1,B_1), & \ldots, & (b_m,B_m), & (b_{m+1},B_{m+1}), & \ldots, & (b_q,B_q)\end{matrix}\right]$$
$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}z^{-s}ds, \tag{1}$$

where A_j and B_j are assumed to be positive quantities and all the a_j and b_j may be complex. The contour L runs from $c-i\infty$ to $c+i\infty$ such that the poles of $\Gamma(b_j+B_js)$, $j=1,\ldots,m$ lie to the left of L and the poles of $\Gamma(1-a_j-A_js)$, $j=1,\ldots,n$ lie to the right of L.

By performing the variable change $s\to -r$ and adjusting the contour L to L^*, where the integral runs from $c^*-i\infty$ to $c^*+i\infty$, the H-function can be alternatively defined as:

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,A), & \ldots, & (a_n,A_n), & (a_{n+1},A_{n+1}), & \ldots, & (a_p,A_p)\\(b_1,B_1), & \ldots, & (b_m,B_m), & (b_{m+1},B_{m+1}), & \ldots, & (b_q,B_q)\end{matrix}\right]$$
$$=\frac{1}{2\pi i}\int_{L^*} \frac{\prod_{j=1}^{m}\Gamma(b_j-B_jr)\prod_{j=1}^{n}\Gamma(1-a_j+A_jr)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_jr)\prod_{j=n+1}^{p}\Gamma(a_j-A_jr)}z^{r}dr, \tag{2}$$

for which the same parameter domain restrictions apply.

By considering the definition in (2), the H-function can be expressed in computable form as follows [4]:

When the poles of $\prod_{j=1}^{m}\Gamma(b_j-B_jr)$ are simple, we have:

$$H_{p\ q}^{m\ n}(z)=\sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\frac{\prod_{j=1\neq h}^{m}\Gamma\left(b_j-B_j\dfrac{b_h+\nu}{B_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+B_j\dfrac{b_h+\nu}{B_h}\right)}$$

$$\times \frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+A_j\frac{b_h+\nu}{B_j}\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-A_j\frac{b_h+\nu}{B_h}\right)}\frac{(-1)^{\nu}z^{(b_h+\nu)/B_h}}{\nu!B_h} \tag{3}$$

for $z \neq 0$ if $\delta > 0$ and for $0 < |z| < D^{-1}$ if $\delta = 0$, where $\delta = \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j$ and $D = \prod_{j=1}^{p} A_j^{A_j} / \prod_{j=1}^{q} B_j^{B_j}$.

When the poles of $\prod_{j=1}^{n}\Gamma(1-a_j+A_j r)$ are simple, we have

$$H_{p\,q}^{m\,n}(z) = \sum_{h=1}^{n}\sum_{\nu=0}^{\infty}\frac{\prod_{j=1\neq h}^{n}\Gamma\left(1-a_j-A_j\frac{1-a_h+\nu}{A_h}\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j+A_j\frac{1-a_h+\nu}{A_h}\right)}\times$$

$$\times \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j\frac{1-a_h+\nu}{A_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-B_j\frac{1-a_h+\nu}{A_h}\right)}\frac{(-1)^{\nu}(1/z)^{(1-a_h+\nu)/A_h}}{\nu!A_h} \tag{4}$$

for $z \neq 0$ if $\delta < 0$ and for $|z| > D^{-1}$ if $\delta = 0$.

Both representations above apply when the poles of the gamma function in the numerator of the quotients are simple. When this simplification does not hold, the residue theorem has to be applied. For details about this theorem, one may refer to [7].

Another hypergeometric function that is of interest in the present paper is ${}_pF_q$, defined as:

$${}_pF_q\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};z\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{x^n}{n!}, \tag{5}$$

where the symbols follow the same constraints as in the case of the H-function. Also, $(a)_n$ denotes the Pochhammer symbol, which can be defined in terms of the gamma function as:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{6}$$

Both the H-function and the ${}_pF_q$ function may be related by the following formula:

$${}_pF_q\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};z\right] = \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{p}\Gamma(a_k)}H_{p,q+1}^{1,p}\left[-z\left|\begin{array}{c} (1-a_1,1),\ldots,(1-a_p,1) \\ (0,1),(1-b_1,1),\ldots,(1-b_q,1) \end{array}\right.\right]. \tag{7}$$

3. Identities presented in the literature

In the present section, a few identities presented in the literature are shown in order to better familiarize the reader with the mathematics behind the proofs [2, 4, 5].

$$\prod_{j=0}^{k-1} \Gamma\left(z+\frac{j}{k}\right) = \Gamma(kz)(2\pi)^{\frac{k-1}{2}} k^{\frac{1}{2}-kz}, \tag{8}$$

where k is a positive integer; $kz \in \mathbb{C}\backslash Z_0^-$.

$$cos(\pi z) = \frac{\pi}{\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)} \tag{9}$$

$$cos(\pi z) = \frac{e^{i\pi z}+e^{-i\pi z}}{2} \tag{10}$$

$$H_{p,q}^{m,n}\left[z| \begin{array}{c} (a_p, A_p) \\ (b_q, B_q) \end{array}\right] = kH_{p,q}^{m,n}\left[z^k| \begin{array}{c} (a_p, kA_p) \\ (b_q, kB_q) \end{array}\right], \quad k>0. \tag{11}$$

$$H_{0,1}^{1,0}\left[z \left| \begin{array}{c} - \\ (0,1) \end{array}\right.\right] = e^{-z} \tag{12}$$

$$H_{1,2}^{1,1}\left[-z \left| \begin{array}{c} (0,1) \\ (0,1),(-1,1) \end{array}\right.\right] = \frac{e^z-1}{z} \tag{13}$$

$$H_{2,3}^{1,2}\left[-z \left| \begin{array}{c} (0,1),(-1,1) \\ (0,1),(-1,1),(-2,1) \end{array}\right.\right] = \frac{e^z-1-z}{z^2} \tag{14}$$

In [6], an interesting relation was derived to split an H-function into the sum of two other H-functions. This relation can be expressed as follows [6]:

Let $z \in \mathbb{C}$, and then:

$$H_{p+1,q+1}^{m,n}\left[z \left| \begin{array}{c} (a_1,A_1),\ldots,(a_p,A_p),(\alpha,\lambda) \\ (b_1,B_1),\ldots,(b_q,B_q),(\alpha,\lambda) \end{array}\right.\right] = \frac{1}{2\pi i}\Bigg(e^{i\pi\alpha} H_{p,q}^{m,n}\left[e^{-i\pi\lambda}z \left| \begin{array}{c} (a_1,A_1),\ldots,(a_p,A_p) \\ (b_1,B_1),\ldots,(b_q,B_q) \end{array}\right.\right] \\ -e^{-i\pi\alpha} H_{p,q}^{m,n}\left[e^{i\pi\lambda}z \left| \begin{array}{c} (a_1,A_1),\ldots,(a_p,A_p) \\ (b_1,B_1),\ldots,(b_q,B_q) \end{array}\right.\right]\Bigg). \tag{15}$$

In the present paper, an alternative splitting relation is derived, as shall be seen in the next section.

4. Results

First, one identity for the H-function is presented. Then applications of the new identity derived are shown together with (15).

Theorem 1 *Let* $z, \alpha, \lambda \in \mathbb{C}$, *and then:*

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{l}(\alpha,\lambda),(a_2,A_2),\ldots,(a_p,A_p)\\(\alpha,\lambda),(b_2,B_2),\ldots,(b_q,B_q)\end{array}\right] = e^{i\pi\alpha} H_{p,q}^{m,n}\left[e^{-i\pi\lambda}z \,\middle|\, \begin{array}{l}(2\alpha,2\lambda),(a_2,A_2),\ldots,(a_p,A_p)\\(2\alpha,2\lambda),(b_2,B_2),\ldots,(b_q,B_q)\end{array}\right]$$
$$+e^{-i\pi\alpha} H_{p,q}^{m,n}\left[e^{i\pi\lambda}z \,\middle|\, \begin{array}{l}(2\alpha,2\lambda),(a_2,A_2),\ldots,(a_p,A_p)\\(2\alpha,2\lambda),(b_2,B_2),\ldots,(b_q,B_q)\end{array}\right]. \tag{16}$$

Proof First, by using (1), one shall consider the contour integral representation of the H-function in (16), given as:

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{l}(\alpha,\lambda),\ldots,(a_p,A_p)\\(\alpha,\lambda),\ldots,(b_q,B_q)\end{array}\right] =$$
$$= \frac{1}{2\pi i}\int_L \frac{\Gamma(\alpha+\lambda s)\Gamma(1-\alpha-\lambda s)\prod_{j=2}^{m}\Gamma(b_j+B_js)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)} z^{-s}ds. \tag{17}$$

By considering (8), it is easy to see that:

$$\Gamma(\alpha+\lambda s)\Gamma\left(\frac{1}{2}+\alpha+\lambda s\right) = \Gamma(2\alpha+2\lambda s)(2\pi)^{\frac{1}{2}}2^{\frac{1}{2}-2\alpha-2\lambda s}$$
$$\Gamma\left(\frac{1}{2}-\alpha-\lambda s\right)\Gamma\left(1-\alpha-\lambda s\right) = \Gamma(1-2\alpha-2\lambda s)(2\pi)^{\frac{1}{2}}2^{\frac{1}{2}-1+2\alpha+2\lambda s}. \tag{18}$$

Also, (18) implies that:

$$\Gamma(\alpha+\lambda s)\Gamma\left(1-\alpha-\lambda s\right) = 2\pi\frac{\Gamma(2\alpha+2\lambda s)\Gamma(1-2\alpha-2\lambda s)}{\Gamma\left(\frac{1}{2}+\alpha+\lambda s\right)\Gamma\left(\frac{1}{2}-\alpha-\lambda s\right)}. \tag{19}$$

Equation (19) can be further simplified by using (9) and (10), resulting in:

$$\Gamma(\alpha+\lambda s)\Gamma\left(1-\alpha-\lambda s\right) = \Gamma(2\alpha+2\lambda s)\Gamma(1-2\alpha-2\lambda s)(e^{i\pi(\alpha+\lambda s)}+e^{-i\pi(\alpha+\lambda s)}). \tag{20}$$

Finally, by inserting (20) into (17), (16) is retrieved.

□

The following two forms of (15) and (16) are easily derived:

- By taking $(a_1, A_1) = (b_1, B_1) = (\alpha, \lambda)$ in (15), one gets:

$$H_{p-1,q-1}^{m-1,n-1}\left[z \,\middle|\, \begin{array}{l}(a_2,A_2),\ldots,(a_p,A_p)\\(b_2,B_2),\ldots,(b_q,B_q)\end{array}\right] = \frac{1}{2\pi i}\left(e^{i\pi\alpha} H_{p,q}^{m,n}\left[e^{-i\pi\lambda}z \,\middle|\, \begin{array}{l}(\alpha,\lambda),(a_2,A_2),\ldots,(a_p,A_p)\\(\alpha,\lambda),(b_2,B_2),\ldots,(b_q,B_q)\end{array}\right]\right.$$
$$\left.-e^{-i\pi\alpha} H_{p,q}^{m,n}\left[e^{i\pi\lambda}z \,\middle|\, \begin{array}{l}(\alpha,\lambda),(a_2,A_2),\ldots,(a_p,A_p)\\(\alpha,\lambda),(b_2,B_2),\ldots,(b_q,B_q)\end{array}\right]\right), \tag{21}$$

for $p \geq n \geq 1$ and $q \geq m \geq 1$.

- Similarly, taking $(a_p, A_p) = (b_q, B_q) = (\alpha, \lambda)$ in (16), one obtains:

$$H_{p-2,q-2}^{m-1,n-1}\left[z \;\middle|\; \begin{matrix}(a_2, A_2), \ldots, (a_{p-1}, A_{p-1})\\(b_2, B_2), \ldots, (b_{q-1}, B_{q-1})\end{matrix}\right] = e^{i\pi\alpha} H_{p,q}^{m,n}\left[e^{-i\pi\lambda} z \;\middle|\; \begin{matrix}(2\alpha, 2\lambda), (a_2, A_2), \ldots, (a_{p-1}, A_{p-1}), (\alpha, \lambda)\\(2\alpha, 2\lambda), (b_2, B_2), \ldots, (b_{q-1}, B_{q-1}), (\alpha, \lambda)\end{matrix}\right]$$
$$+e^{-i\pi\alpha} H_{p,q}^{m,n}\left[e^{i\pi\lambda} z \;\middle|\; \begin{matrix}(2\alpha, 2\lambda), (a_2, A_2), \ldots, (a_{p-1}, A_{p-1}), (\alpha, \lambda)\\(2\alpha, 2\lambda), (b_2, B_2), \ldots, (b_{q-1}, B_{q-1}), (\alpha, \lambda)\end{matrix}\right], \quad (22)$$

for $p-1 \geq n \geq 1$ and $q-1 \geq m \geq 1$.

Equations (15) and (16) may be applied to split H-functions into the sum of the H-function. In the present paper, an interesting application to the summation of hypergeometric series is described. First, a general summation may be studied. Then special cases are discussed case by case.

Theorem 2 *Let $x \in \mathbb{R}$, and then:*

$$\sum_{n=0}^{\infty} \frac{x^{\alpha n}\beta^n}{\Gamma(\gamma n + \delta + 1)} = \frac{1}{\Gamma(\delta+1)}\, {}_1F_\gamma\left[\begin{matrix}1\\ \frac{\delta+1}{\gamma}, \frac{\delta+2}{\gamma}, \ldots, \frac{\delta+\gamma}{\gamma}\end{matrix} ; \beta\frac{x^\alpha}{\gamma^\gamma}\right]. \quad (23)$$

The left-hand side of (23) is a generalization of the Mittag–Leffler function [2, 4].

Proof The summation in (23) can be expressed in terms of the hypergeometric function ${}_pF_q$ by noticing that:

$$\Gamma(\gamma n + \delta + 1) = \frac{\prod_{k=0}^{\gamma-1} \Gamma\left(n + \frac{\delta+1+k}{\gamma}\right)}{(2\pi)^{\frac{\gamma-1}{2}} \gamma^{-\frac{1}{2}-\gamma n-\delta}}. \quad (24)$$

By combining the left-hand side of equation (23) and equation (24), the former may be rewritten as:

$$\frac{1}{\Gamma(\delta+1)} \sum_{n=0}^{\infty} \frac{(\beta x^\alpha \gamma^{-\gamma})^n (1)_n}{\prod_{k=0}^{\gamma-1}\left(\frac{\delta+1+k}{\gamma}\right)_n n!}. \quad (25)$$

By the representation in (5), equation (25) implies (23). □

Corollary 3 *The summation in (23) can be expressed in terms of the H-function as:*

$$\sum_{n=0}^{\infty} \frac{x^{\alpha n}\beta^n}{\Gamma(\gamma n + \delta + 1)} = H_{1,2}^{1,1}\left[-\beta x^\alpha \middle| \begin{matrix}(0,1)\\(0,1), (-\delta, \gamma)\end{matrix}\right]. \quad (26)$$

Proof By means of (7), (23) is expressible in terms of the H-function as:

$$\sum_{n=0}^{\infty} \frac{x^{\alpha n}\beta^n}{\Gamma(\gamma n + \delta + 1)} = \frac{1}{\Gamma(\delta+1)} \prod_{k=1}^{\gamma} \Gamma\left(\frac{\delta+k}{\gamma}\right) H_{1,\gamma+1}^{1,1}\left[-\beta\frac{x^\alpha}{\gamma^\gamma} \middle| \begin{matrix}(0,1)\\(0,1), (1-\frac{\delta+1}{\gamma}, 1), \ldots, (1-\frac{\delta+\gamma}{\gamma}, 1)\end{matrix}\right]. \quad (27)$$

On the other hand, by using the contour integral representation (1) of (27) and the multiplication theorem for the gamma function (8), (26) follows from (27).

□

Based on the general results presented, one shall proceed to the applications sections.

5. Applications

In the present section, results obtained for the summation of hypergeometric series are presented. These results are important as they present interesting relations between generalized hypergeometric functions and elementary functions.

5.1. When $\gamma = 2$ and $\beta = -1$ in (23)

For the cases where $\beta = -1$, the results may be obtained by performing the transformation $x^{\alpha/2} \to ix^{\alpha/2}$. This provides the results in terms of standard trigonometrical functions. Also, these are the cases treated in [8] for $\Omega_i = 1$, $\forall i \in \mathbb{N}$, where the parameters considered are $\gamma = 2$ ($\delta = 0$ and 1) and $\gamma = 3$ ($\delta = 0, 1$ and 2).

Another interesting result may be discussed for the case where $\gamma = 2$ and $\beta = -1$.

5.1.1. Case when $\delta = 0$

First, let us consider (23) for this case:

$$\sum_{n=0}^{\infty} \frac{x^{\alpha n}(-1)^n}{\Gamma(2n+1)} = {}_0F_1\left[\begin{matrix} - \\ \frac{1}{2} \end{matrix}; -\frac{x^\alpha}{4}\right]. \tag{28}$$

The literature shows that when $\alpha = 2$, we have [1, eq.(3.2)]:

$$e^x {}_0F_1\left[\begin{matrix} - \\ \frac{1}{2} \end{matrix}; -\frac{x^2}{4}\right] = \sum_{n=0}^{\infty} 2^{n/2} cos\left(\frac{n\pi}{4}\right)\frac{x^n}{n!}. \tag{29}$$

By considering (9), the right-hand side of (29) may be rewritten as:

$$e^x {}_0F_1\left[\begin{matrix} - \\ \frac{1}{2} \end{matrix}; -\frac{x^2}{4}\right] = \pi \sum_{n=0}^{\infty} \frac{2^{n/2}}{\Gamma\left(\frac{1}{2}+\frac{n}{4}\right)\Gamma\left(\frac{1}{2}-\frac{n}{4}\right)}\frac{x^n}{n!}. \tag{30}$$

Finally, by considering the H-function series representation given in (3), (30) provides:

$$e^x {}_0F_1\left[\begin{matrix} - \\ \frac{1}{2} \end{matrix}; -\frac{x^2}{4}\right] = \pi H_{1,2}^{1,0}\left[-\sqrt{2}x \,\middle|\, \begin{matrix} (\frac{1}{2},\frac{1}{4}) \\ (0,1),(\frac{1}{2},\frac{1}{4}) \end{matrix}\right]. \tag{31}$$

The alternative representation (26) of the left-hand side of (31) results in the following relations:

$$e^x {}_0F_1\left[\begin{matrix} - \\ \frac{1}{2} \end{matrix}; -\frac{x^2}{4}\right] = e^x H_{1,2}^{1,1}\left[x^2 \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(0,2) \end{matrix}\right] = \pi H_{1,2}^{1,0}\left[-\sqrt{2}x \,\middle|\, \begin{matrix} (\frac{1}{2},\frac{1}{4}) \\ (0,1),(\frac{1}{2},\frac{1}{4}) \end{matrix}\right]. \tag{32}$$

By considering the transformation $x \to ix$, (32) is also expressible in terms of elementary functions, as $e^x cos(x)$, upon using results recorded in [2].

5.1.2. Case when $\delta = 1$

For the case where $\alpha = 2$ and $\delta = 1$, by considering the relation in [1, Eq. (3.3)] and following a similar procedure as in the case where $\delta = 0$, the following relation is obtained:

$$e^x {}_0F_1\left[\begin{matrix} - \\ \frac{3}{2} \end{matrix}; -\frac{x^2}{4}\right] = e^x H_{1,2}^{1,1}\left[x^2 \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(-1,2) \end{matrix}\right] = \sqrt{2}\pi H_{2,3}^{1,1}\left[-\sqrt{2}x \,\middle|\, \begin{matrix} (0,1),(\frac{3}{4},\frac{1}{4}) \\ (0,1),(-1,1),(\frac{3}{4},\frac{1}{4}) \end{matrix}\right]. \tag{33}$$

On the other hand, by considering the transformation $x \to ix$, (33) can be expressed in terms of trigonometrical functions, as $x^{-1}e^x sin(x)$ upon using results recorded in[2].

5.2. When $\gamma = 3$ and $\beta = -1$ in (23)

For the next subcases, a similar procedure as in the cases where $\gamma = 2$ and $\beta = -1$ can be followed. Thus, by considering [1, Eq. (3.4)], [1, Eq. (3.5)], and [1, Eq. (3.6)] for the cases where $\delta = 0$, $\delta = 1$, and $\delta = 2$, respectively, the following relations between H-functions are retrieved:

$$e^x {}_0F_2\left[\begin{matrix} - \\ \frac{1}{3}, \frac{2}{3} \end{matrix}; -\frac{x^3}{27}\right] = e^x H_{1,2}^{1,1}\left[x^3 \left| \begin{matrix} (0,1) \\ (0,1),(0,3) \end{matrix} \right.\right] = \frac{1}{3} + \frac{2\pi}{3} H_{1,2}^{1,0}\left[-\sqrt{3}x \left| \begin{matrix} (\frac{1}{2},\frac{1}{6}) \\ (0,1),(\frac{1}{2},\frac{1}{6}) \end{matrix} \right.\right], \tag{34}$$

$$e^x {}_0F_2\left[\begin{matrix} - \\ \frac{2}{3}, \frac{4}{3} \end{matrix}; -\frac{x^3}{27}\right] = e^x H_{1,2}^{1,1}\left[x^3 \left| \begin{matrix} (0,1) \\ (0,1),(-1,3) \end{matrix} \right.\right] = \frac{2\pi}{\sqrt{3}} H_{2,3}^{1,1}\left[-\sqrt{3}x \left| \begin{matrix} (0,1),(\frac{2}{3},\frac{1}{6}) \\ (0,1),(-1,1),(\frac{2}{3},\frac{1}{6}) \end{matrix} \right.\right], \tag{35}$$

$$e^x {}_0F_2\left[\begin{matrix} - \\ \frac{4}{3}, \frac{5}{3} \end{matrix}; -\frac{x^3}{27}\right] = e^x H_{1,2}^{1,1}\left[x^3 \left| \begin{matrix} (0,1) \\ (0,1),(-2,3) \end{matrix} \right.\right] = 4\pi H_{3,4}^{1,2}\left[-\sqrt{3}x \left| \begin{matrix} (0,1),(-1,1),(\frac{5}{6},\frac{1}{6}) \\ (0,1),(-1,1),(-2,1),(\frac{5}{6},\frac{1}{6}) \end{matrix} \right.\right]. \tag{36}$$

Each of the equations from (34) to (36) can be expressed in terms of elementary functions. This can be accomplished by using (15) and [2].

5.2.1. Case when $\delta = 0$

By combining (34) and (15), the following is obtained:

$$\begin{aligned} 1 + \frac{2\pi}{3} H_{1,2}^{1,0}\left[-\sqrt{3}x \left| \begin{matrix} (\frac{1}{2},\frac{1}{6}) \\ (0,1),(\frac{1}{2},\frac{1}{6}) \end{matrix} \right.\right] &= 1 + \frac{1}{3i}\Big(e^{i\pi/2} H_{0,1}^{1,0}\left[-e^{-i\pi/6}\sqrt{3}x \left| \begin{matrix} - \\ (0,1) \end{matrix} \right.\right] \\ &\quad - e^{-i\pi/2} H_{0,1}^{1,0}\left[-e^{i\pi/6}\sqrt{3}x \left| \begin{matrix} - \\ (0,1) \end{matrix} \right.\right]\Big). \end{aligned} \tag{37}$$

Using (12) on (37) leads to:

$$\frac{1}{3} + \frac{2\pi}{3} H_{1,2}^{1,0}\left[-\sqrt{3}x \left| \begin{matrix} (\frac{1}{2},\frac{1}{6}) \\ (0,1),(\frac{1}{2},\frac{1}{6}) \end{matrix} \right.\right] = \frac{1}{3} + \frac{2}{3} e^{3x/2} cos\left(\frac{\sqrt{3}x}{2}\right), \tag{38}$$

which is an alternate form of (34) in terms of trigonometrical functions.

5.2.2. Case when $\delta = 1$

When $\delta = 1$, (35) and (15) provide:

$$\begin{aligned} \frac{2\pi}{\sqrt{3}} H_{2,3}^{1,1}\left[-\sqrt{3}x \left| \begin{matrix} (0,1),(\frac{2}{3},\frac{1}{6}) \\ (0,1),(-1,1),(\frac{2}{3},\frac{1}{6}) \end{matrix} \right.\right] &= \frac{1}{\sqrt{3}i}\Big(e^{2i\pi/3} H_{1,2}^{1,1}\left[-e^{-i\pi/6}\sqrt{3}x \left| \begin{matrix} (0,1) \\ (0,1),(-1,1) \end{matrix} \right.\right] \\ &\quad - e^{-2i\pi/3} H_{1,2}^{1,1}\left[-e^{i\pi/6}\sqrt{3}x \left| \begin{matrix} (0,1) \\ (0,1),(-1,1) \end{matrix} \right.\right]\Big). \end{aligned} \tag{39}$$

By combining (13) and (39), the following result is obtained:

$$\frac{2\pi}{\sqrt{3}} H_{2,3}^{1,1}\left[-\sqrt{3}x \middle| \begin{array}{c} (0,1),(\frac{2}{3},\frac{1}{6}) \\ (0,1),(-1,1),(\frac{2}{3},\frac{1}{6}) \end{array}\right] = \frac{2}{3x}\left(-\frac{1}{2}+e^{\frac{3x}{2}} sin\left(\frac{5\pi}{6}-\frac{\sqrt{3}x}{2}\right)\right), \quad (40)$$

which is an alternate form for (35).

5.2.3. Case when $\delta = 2$

In this case, the combination of (36) and (15) leads to:

$$4\pi H_{3,4}^{1,2}\left[-\sqrt{3}x \middle| \begin{array}{c} (0,1),(-1,1),(\frac{5}{6},\frac{1}{6}) \\ (0,1),(-1,1),(-2,1),(\frac{5}{6},\frac{1}{6}) \end{array}\right] = \frac{2}{i}\left(e^{5i\pi/6} H_{2,3}^{1,2}\left[-e^{-i\pi/6}\sqrt{3}x \middle| \begin{array}{c} (0,1),(-1,1) \\ (0,1),(-1,1),(-2,1) \end{array}\right]\right. \quad (41)$$

$$\left. - e^{-5i\pi/6} H_{2,3}^{1,2}\left[-e^{i\pi/6}\sqrt{3}x \middle| \begin{array}{c} (0,1),(-1,1) \\ (0,1),(-1,1),(-2,1) \end{array}\right]\right).$$

It is clear from (14) that (41) may be rewritten as:

$$4\pi H_{3,4}^{1,2}\left[-\sqrt{3}x \middle| \begin{array}{c} (0,1),(-1,1),(\frac{5}{6},\frac{1}{6}) \\ (0,1),(-1,1),(-2,1),(\frac{5}{6},\frac{1}{6}) \end{array}\right] = \frac{2}{3x^2}\left(1-2e^{\frac{3x}{2}} cos\left(\frac{\pi}{3}+\frac{\sqrt{3}x}{2}\right)\right), \quad (42)$$

giving an alternate form for (36) in terms of elementary functions.

5.3. When $\gamma = 2$ and $\beta = 1$ in (23)

The cases where $\beta = 1$, $\gamma = 3$ ($\delta = 0, 1$ and 2), and $\gamma = 4$ ($\delta = 0, 1, 2$ and 3) were explored in [9]. In that work, by taking $\Omega_i = 1, \forall i \in \mathbb{N}$, the authors provided some general results for summations similar to the ones obtained in the earlier subsections.

In this subsection, we obtain two results not recorded in [9], by taking $\gamma = 2$ and $\beta = 1$ in (23) and (26) utilizing (16).

For example, the result (2.1) for $\Omega_i = 1, \forall i \in \mathbb{N}$ taken in [9] is written in the following form:

$$\begin{aligned} e^x H_{1,2}^{1,1}\left[-x^3 \middle| \begin{array}{c} (0,1) \\ (0,1),(0,3) \end{array}\right] &= e^x {}_0F_2\left[\begin{array}{c} - \\ \frac{1}{3},\frac{2}{3} \end{array} ; \frac{x^3}{27}\right] \\ &= \frac{2}{3}\sum_{m=0}^{\infty}\frac{x^m}{m!}\left[2^{m-1}+cos\left(\frac{m\pi}{3}\right)\right] \\ &= \frac{e^{2x}}{3}+\frac{2\pi}{3}H_{1,2}^{1,0}\left[-x \middle| \begin{array}{c} (\frac{1}{2},\frac{1}{3}) \\ (0,1),(\frac{1}{2},\frac{1}{3}) \end{array}\right] \\ &= \frac{e^{2x}}{3}+\frac{2}{3}e^{\frac{x}{2}} cos\left(\frac{\sqrt{3}x}{2}\right). \end{aligned} \quad (43)$$

The first equality in (43) is easily obtained from (23), (26), and (2.1) for $\Omega_i = 1, \forall i \in \mathbb{N}$ taken in [9]. The third and fourth inequalities are obtained below:

We start with:

$$\frac{2}{3}\sum_{m=0}^{\infty}\frac{x^m}{m!}\left[2^{m-1}+cos\left(\frac{m\pi}{3}\right)\right] = \frac{1}{3}\sum_{m=0}^{\infty}\frac{(2x)^m}{m!}+\frac{2}{3}\sum_{m=0}^{\infty}\frac{x^m}{m!}cos\left(\frac{m\pi}{3}\right) \quad (44)$$

$$= \frac{e^{2x}}{3}+\frac{2\pi}{3}\sum_{m=0}^{\infty}\frac{x^m}{m!\Gamma\left(\frac{1}{2}+\frac{m}{3}\right)\Gamma\left(\frac{1}{2}-\frac{m}{3}\right)}$$

$$= \frac{e^{2x}}{3}+\frac{2\pi}{3}H_{1,2}^{1,0}\left[-x\left|\begin{array}{c}(\frac{1}{2},\frac{1}{3})\\(0,1),(\frac{1}{2},\frac{1}{3})\end{array}\right.\right],$$

using the series representation in (3). On the other hand,

$$\frac{e^{2x}}{3}+\frac{2\pi}{3}H_{1,2}^{1,0}\left[-x\left|\begin{array}{c}(\frac{1}{2},\frac{1}{3})\\(0,1),(\frac{1}{2},\frac{1}{3})\end{array}\right.\right]=\frac{e^{2x}}{3}+\frac{2\pi}{3}\frac{e^{xcos(\pi/3)}}{\pi}sin\left(\frac{\pi}{2}-xsin\left(\frac{\pi}{3}\right)\right) \quad (45)$$

by using [6] as done in §5.1.1. Finally:

$$\frac{e^{2x}}{3}+\frac{2\pi}{3}H_{1,2}^{1,0}\left[-x\left|\begin{array}{c}(\frac{1}{2},\frac{1}{3})\\(0,1),(\frac{1}{2},\frac{1}{3})\end{array}\right.\right]=\frac{e^{2x}}{3}+\frac{2}{3}e^{\frac{x}{2}}cos\left(\frac{\sqrt{3}x}{2}\right). \quad (46)$$

The other six results given in [9] with $\Omega_i = 1, \forall i \in \mathbb{N}$ can be written in terms of H-functions by using (3) or (4) and in elementary functions utilizing [2, 6]. These results are too complicated to be included here.

5.3.1. Case when $\delta = 0$

The case where $\delta = 0$ in (26) implies:

$$\sum_{n=0}^{\infty}\frac{x^{\alpha n}}{(2n)!}=H_{1,2}^{1,1}\left[-x^{\alpha}\left|\begin{array}{c}(0,1)\\(0,1),(0,2)\end{array}\right.\right]. \quad (47)$$

On the other hand, the result in (16) provides:

$$H_{1,2}^{1,1}\left[-x^{\alpha}\left|\begin{array}{c}(0,1)\\(0,1),(0,2)\end{array}\right.\right] = H_{1,2}^{1,1}\left[-e^{-i\pi}x^{\alpha}\left|\begin{array}{c}(0,2)\\(0,2),(0,2)\end{array}\right.\right]$$

$$+H_{1,2}^{1,1}\left[-e^{i\pi}x^{\alpha}\left|\begin{array}{c}(0,2)\\(0,2),(0,2)\end{array}\right.\right]. \quad (48)$$

Also, by considering the identity (11) with $k = 1/2$, (48) becomes:

$$H_{1,2}^{1,1}\left[-x^{\alpha}\left|\begin{array}{c}(0,1)\\(0,1),(0,2)\end{array}\right.\right] = \frac{1}{2}H_{1,2}^{1,1}\left[x^{\alpha/2}\left|\begin{array}{c}(0,1)\\(0,1),(0,1)\end{array}\right.\right]$$

$$+\frac{1}{2}H_{1,2}^{1,1}\left[-x^{\alpha/2}\left|\begin{array}{c}(0,1)\\(0,1),(0,1)\end{array}\right.\right]. \quad (49)$$

By further manipulating the contour integral representation of the H-functions in (49), it is clear from (12) that (49) turns to [2]:

$$H_{1,2}^{1,1}\left[-x^{\alpha}\Big|\begin{array}{c}(0,1)\\(0,1),(0,2)\end{array}\right] = \frac{1}{2}e^{x^{\alpha/2}}+\frac{1}{2}e^{-x^{\alpha/2}} = cosh(x^{\alpha/2}). \quad (50)$$

5.3.2. Case when $\delta = 1$

When $\delta = 1$:

$$\sum_{n=0}^{\infty}\frac{x^{\alpha n}}{(2n+1)!} = H_{1,2}^{1,1}\left[-x^{\alpha}\Big|\begin{array}{c}(0,1)\\(0,1),(-1,2)\end{array}\right]. \quad (51)$$

On the other hand, the result in (16) provides:

$$H_{1,2}^{1,1}\left[-x^{\alpha}\Big|\begin{array}{c}(0,1)\\(0,1),(-1,2)\end{array}\right] = H_{1,2}^{1,1}\left[-e^{-i\pi}x^{\alpha}\Bigg|\begin{array}{c}(0,2)\\(0,2),(-1,2)\end{array}\right] + H_{1,2}^{1,1}\left[-e^{i\pi}x^{\alpha}\Bigg|\begin{array}{c}(0,2)\\(0,2),(-1,2)\end{array}\right]. \quad (52)$$

Also, by considering the identity (11) with $k = 1/2$, (52) becomes:

$$H_{1,2}^{1,1}\left[-x^{\alpha}\Big|\begin{array}{c}(0,1)\\(0,1),(-1,2)\end{array}\right] = \frac{1}{2}H_{1,2}^{1,1}\left[x^{\alpha/2}\Bigg|\begin{array}{c}(0,1)\\(0,1),(-1,1)\end{array}\right] + \frac{1}{2}H_{1,2}^{1,1}\left[-x^{\alpha/2}\Bigg|\begin{array}{c}(0,1)\\(0,1),(-1,1)\end{array}\right]. \quad (53)$$

By using result (13), one may see that (53) turns to [2]:

$$H_{1,2}^{1,1}\left[-x^{\alpha}\Big|\begin{array}{c}(0,1)\\(0,1),(-1,2)\end{array}\right] = \frac{(1-e^{-x^{\alpha/2}})x^{-\alpha/2}}{2}+\frac{(e^{x^{\alpha/2}}-1)x^{-\alpha/2}}{2} = x^{-\alpha/2}sinh(x^{\alpha/2}). \quad (54)$$

6. Conclusions

A new identity that enables one to split certain H-functions into the sum of two other H-functions has been derived. This new formula has been applied to simplify the summation of hypergeometric series. By using this identity, new relations between H-functions have been established.

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