

Legendrian torus knots in lens spaces

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Abstract: In this study, we first classify all topological torus knots lying on the Heegaard torus in lens spaces, and then we study Legendrian representatives of these knots. We classify oriented positive Legendrian torus knots in the universally tight contact structures on the lens spaces up to contactomorphism.

Key words: Legendrian knots, contact structures, lens spaces

1. Introduction

A *Legendrian knot* in a contact 3-manifold is a knot that is everywhere tangent to the contact planes. Legendrian knots are natural objects in contact 3-manifolds and they are used to distinguish contact structures [16], to detect topological properties of knots [19], and to detect overtwistedness of contact structures [9].

There has been some recent progress in the classification of Legendrian knots in the standard tight contact structure on S^3 after the classification of Legendrian unknots made by Eliashberg and Fraser [6] and the classification of Legendrian torus knots and the figure eight knot made by Etnyre and Honda [8]. Legendrian knots in a cabled knot type in S^3 were studied in [10] and a complete classification was given in [20]. Recently, Legendrian twists knots were classified in [11]. Legendrian knots in 3-manifolds other than the 3-sphere S^3 are also studied. For example, in [14], Legendrian linear curves in the 3-torus T^3 were classified and in [1, 12], Legendrian rational unknots in lens spaces were classified.

In this paper, we apply the classification scheme of Etnyre and Honda in [8] to Legendrian knots that are rationally null-homologous. We focus on Legendrian torus knots in lens spaces. For relatively prime integers a, b , an oriented simple closed curve that wraps a times in the meridional direction and b times in the longitudinal direction on the Heegaard torus is called an (a, b) -torus knot in the lens space. We call (a, b) -torus knots with $a, b > 0$ *positive* torus knots. In Section 2, we study the topological properties of all torus knots in lens spaces. We find a constraint on when a torus knot is null-homologous. We compute the group of a torus knot that is defined as the fundamental group of its complement. By studying the diffeotopy group of lens spaces, we completely classify all torus knots up to isotopy. Lastly, we construct a rational Seifert surface for a torus knot and we calculate its Euler characteristic.

In section 3, we give a review of the basic concepts in convex surface theory and we fix the notation. In Section 4, by using convex surface theory tools, we study Legendrian representatives of positive torus knots in the universally tight contact structures on the lens spaces. We define the rational Legendrian knot invariants

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[1, 3, 12, 18]. We compute the rational Thurston–Bennequin invariants and the rational rotation numbers of oriented positive Legendrian torus knots by using the rational Seifert surface that we constructed for torus knots. By following the strategy outlined in [8], we first classify oriented positive Legendrian torus knots with maximal rational Thurston–Bennequin invariants. Then we show that all oriented positive Legendrian torus knots with nonmaximal rational Thurston–Bennequin invariants destabilize. We prove:

Theorem 4.4. *Two oriented positive Legendrian torus knots in a universally tight contact structure on a lens space are uniquely realized up to contactomorphism if and only if their oriented knot types, rational Thurston–Bennequin invariants, and rational rotation numbers agree.*

Legendrian knots may be classified up to contact isotopy or up to global contactomorphism. By the work of Eliashberg we know that the group of coorientation preserving contactomorphisms of the standard tight S^3 is connected [5, Theorem 2.4.2]. Therefore, for Legendrian knots in the standard tight S^3 , these two classifications are equivalent. However, for arbitrary tight contact closed 3-manifolds, the group of coorientation-preserving contactomorphisms is not well understood. In particular, nothing is known for tight contact lens spaces in general.

Question 1. *Is the group of coorientation-preserving contactomorphisms of universally tight contact structures on lens spaces connected?*

We want to remark that a positive answer to Question 1 together with Theorem 4.4 provides us the classification of Legendrian torus knots up to Legendrian isotopy in the universally tight contact structures on the lens spaces.

A *transverse knot* in a contact 3-manifold is a knot that is everywhere transverse to the contact planes. There are two types of classical invariants for null-homologous transverse knots: the knot type and the self-linking number. The self-linking number may be generalized for rationally null-homologous transverse knots [1]. By [8, Theorem 2.10], two transverse knots in a contact 3-manifold are transversely isotopic if and only if their Legendrian push-offs are Legendrian isotopic after each has been negatively stabilized some number of times. As a consequence of Theorem 4.4, we have the following:

Theorem 1.1. *Two positive transverse torus knots in a universally tight contact structure on a lens space are uniquely realized up to contactomorphism if and only if their knot types and rational self-linking numbers agree.*

2. Topological torus knots in lens spaces

For fixed relatively prime integers $p > q > 0$, let (V_1, V_2) be the genus 1 Heegaard splitting of a lens space $L(p, q)$, which is described as

$$L(p, q) = V_1 \cup_{\varphi} V_2,$$

where V_1 and V_2 are both $D^2 \times S^1$. Let μ_i and λ_i be a meridian and longitude pair for V_i , $i = 1, 2$. The gluing map $\varphi : \partial V_1 \rightarrow \partial V_2$ is an orientation-reversing map given in standard longitude-meridian coordinates on the torus by the matrix

$$\begin{pmatrix} -q & q' \\ p & p' \end{pmatrix}$$

with $pq' + qp' = 1$, $p', q' \in \mathbb{Z}$. In particular, the image of the meridian μ_1 of ∂V_1 is the curve $-q\mu_2 + p\lambda_2$ in ∂V_2 . Note that $H_1(L(p, q), \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ is generated by $[\lambda_2]$. Therefore, any oriented knot K in $L(p, q)$

represents $b[\lambda_2]$ for some integer b . Any knot in a lens space $L(p, q)$ is rationally null-homologous and has an order. The order r of K is defined to be the order of $[K]$, and hence $r = p/gcd(p, b)$.

By Theorem 1 of [4], one can fix the Heegaard torus up to isotopy in a lens space. This allows us to define torus knots on the Heegaard torus ∂V_2 . For relatively prime integers a, b , an oriented simple closed curve $K_{(a,b)}$ that wraps a times in the meridional direction and b times in the longitudinal direction on ∂V_2 is called an (a, b) -torus knot in the lens space $L(p, q)$. For a knot $K_{(a,b)}$ of order r in $L(p, q)$, $p \mid rb$ and $K_{(a,b)}$ is null-homologous if and only if $p \mid b$.

Proposition 2.1. *Let $K_{(a,b)}$ be an (a, b) -torus knot on the Heegaard torus ∂V_2 in lens space $L(p, q)$.*

1. *The group of a torus knot $K_{(a,b)}$ can be presented as*

$$\pi_1(L(p, q) - K_{(a,b)}) = \langle u, v \mid u^b = v^{pa+qb} \rangle.$$

2. *Two torus knots $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups if and only if $|b| = |b'|$ and $|pa + qb| = |pa' + qb'|$ or $|b| = |pa' + qb'|$ and $|b'| = |pa + qb|$.*

Proof The complement of a neighborhood $\nu(K_{(a,b)})$ of the torus knot $K_{(a,b)}$ in $L(p, q)$ is the union of two solid tori glued along an annulus A where the core C of the annulus A is isotopic to the torus knot $K_{(a,b)}$. Namely, $L(p, q) \setminus \nu(K_{(a,b)}) = \tilde{V}_1 \cup \tilde{V}_2$ where $\tilde{V}_i = \overline{V_i \setminus \nu(K_{(a,b)})}$, $i = 1, 2$, are two solid tori glued along the annulus $A = (L(p, q) \setminus \nu(K_{(a,b)})) \cap \partial V_2$.

Let $\tilde{\mu}_i$ and $\tilde{\lambda}_i$ be a meridian and longitude pair for \tilde{V}_i where $\tilde{\mu}_i$ and $\tilde{\lambda}_i$ represent the trivial element and a generator of $\pi_1(\tilde{V}_i)$, respectively. Note that the homotopy class $[C] = [\tilde{\mu}_1]^{-p'a+q'b} [\tilde{\lambda}_1]^{pa+qb} = [\tilde{\lambda}_1]^{pa+qb}$ since $K_{(a,b)}$ is on ∂V_2 and

$$\begin{pmatrix} -q & q' \\ p & p' \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -p' & q' \\ p & q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -p'a + q'b \\ pa + qb \end{pmatrix}$$

for $pp' + qq' = 1$, $p', q' \in \mathbb{Z}$. Also, $[C] = [\tilde{\mu}_2]^a [\tilde{\lambda}_2]^b = [\tilde{\lambda}_2]^b$. Then, by the Seifert–van Kampen theorem,

$$\pi_1(K_{(a,b)}) = \langle u, v \mid u^b = v^{pa+qb} \rangle,$$

where $u = [\tilde{\lambda}_1]$ and $v = [\tilde{\lambda}_2]$. This proves (1).

The subgroup $\langle u^b \rangle$ is the center of the knot group $\pi_1(K_{(a,b)})$ and $\pi_1(K_{(a,b)}) / \langle u^b \rangle = \mathbb{Z}_{|b|} * \mathbb{Z}_{|pa+qb|}$. Note that u and v generate nonconjugate maximal finite cyclic subgroups of order $|b|$ and $|pa + qb|$ of $\mathbb{Z}_{|b|} * \mathbb{Z}_{|pa+qb|}$, respectively. Therefore, if $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups, then $|b| = |b'|$ and $|pa + qb| = |pa' + qb'|$ or $|b| = |pa' + qb'|$ and $|b'| = |pa + qb|$. Now if $|b| = |b'|$ and $|pa + qb| = |pa' + qb'|$ or $|b| = |pa' + qb'|$ and $|b'| = |pa + qb|$, then from (1) it is straightforward that $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups, proving (2).

Lemma 2.2. *1. If $K_{(a,b)}$ and $K_{(a',b')}$ are two null-homologous oriented torus knots in $L(p, q)$ that have isomorphic groups, then (a', b') is equal to one of the following pairs:*

- $A = (a, b), -A = (-a, -b),$
 $B = (\frac{-2qb-pa}{p}, b), -B = (\frac{2qb+pa}{p}, -b),$
 $C = (\frac{b-qp a-q^2 b}{p}, pa + qb), -C = (\frac{-b+qp a+q^2 b}{p}, -pa - qb),$
 $D = (\frac{b+qp a+q^2 b}{p}, -pa - qb), -D = (\frac{-b-qp a-q^2 b}{p}, pa + qb).$

2. If $K_{(a,b)}$ and $K_{(a',b')}$ are two rationally null-homologous but not null-homologous oriented torus knots in $L(p, q)$ that have isomorphic groups, then (a', b') is equal to one of the following pairs in the following cases:

- $A = (a, b), -A = (-a, -b)$ if $p \neq 2$ and $q^2 \not\equiv \pm 1 \pmod{p},$
- $A = (a, b), -A = (-a, -b), C = (\frac{b-qp a-q^2 b}{p}, pa + qb), -C = (\frac{-b+qp a+q^2 b}{p}, -pa - qb)$ if $p \neq 2$ and $q^2 \equiv 1 \pmod{p},$
- $A = (a, b), -A = (-a, -b), D = (\frac{b+qp a+q^2 b}{p}, -pa - qb), -D = (\frac{-b-qp a-q^2 b}{p}, pa + qb)$ if $p \neq 2$ and $q^2 \equiv -1 \pmod{p},$
- $A = (a, b), -A = (-a, -b), B = (-b-a, b), -B = (b+a, -b), C = (-a, 2a+b), -C = (a, -2a-b), D = (a+b, -2a-b), -D = (-a-b, 2a+b)$ if $p = 2.$

Proof By Proposition 2.1(2) we know that $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups if and only if $|b| = |b'|$ and $|pa+qb| = |pa'+qb'|$ or $|b| = |pa'+qb'|$ and $|b'| = |pa+qb|$. Case (1) follows from the analysis of these cases using that $p \mid b$. For Case (2), we know that if $K_{(a,b)}$ is not null-homologous then $p \nmid b$. Therefore, when $p \neq 2$ the cases $(\frac{-2qb-pa}{p}, b), (\frac{2qb+pa}{p}, -b)$ do not occur and the cases $(\frac{b-qp a-q^2 b}{p}, pa + qb), (\frac{-b+qp a+q^2 b}{p}, -pa - qb)$ occur only if $p \mid (1 - q^2)$. Similarly, the cases $(\frac{b+qp a+q^2 b}{p}, -pa - qb), (\frac{-b-qp a-q^2 b}{p}, pa + qb)$ occur only if $p \mid (1 + q^2)$. The case when $p = 2$ and hence $q = 1$ is clear.

Let us now classify all topological torus knots in lens spaces up to isotopy. Recall that two knots K_1 and K_2 in a 3-manifold M are isotopic if there is a diffeomorphism $g : M \rightarrow M$ such that $g(K_1) = K_2$ and g is isotopic to the identity map.

Theorem 2.3. *The oriented torus knot $K_{(a,b)}$ is isotopic to $K_{(a',b')}$ in $L(p, q)$ if and only if (a', b') is an element of one of the following:*

1. $\{(a, b)\}$ if $q \neq 1$ or $p - 1,$
2. $\{(a, b), (\frac{b-qp a-q^2 b}{p}, pa + qb)\}$ if $p \neq 2$ and $q = 1$ or $p - 1,$
3. $\{(a, b), (-a, -b), (-a, 2a + b), (a, -2a - b)\}$ if $p = 2.$

For the proof of Theorem 2.3 we need the following theorem:

Theorem 2.4 ([4, Theorem 3]). *The group of isotopy classes of diffeomorphisms of $L(p, q)$ for $p \geq 2$ is given by*

1. \mathbb{Z}_2 with generator τ if $q^2 \not\equiv \pm 1 \pmod{p}$,
2. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with generator τ and σ_+ if $q^2 \equiv 1$ and $q \not\equiv \pm 1 \pmod{p}$,
3. \mathbb{Z}_2 with generator τ if $q \equiv \pm 1 \pmod{p}$ and $p \neq 2$,
4. \mathbb{Z}_4 with generator σ_- if $q^2 \equiv -1 \pmod{p}$ and $p \neq 2$,
5. \mathbb{Z}_2 with generator σ_- if $p = 2$.

Let (V_1, V_2) be the genus 1 Heegaard splitting of the lens space $L(p, q)$ defined as above. In Theorem 2.4, the diffeomorphism τ preserves each of the solid tori $V_i = D^2 \times S^1$ and acts by a complex conjugation on each factor (as viewed in \mathbb{C}) of each Heegaard torus. Note that τ always exists and if $p = 2$, then τ is isotopic to the identity. In general, $L(p, q)$ does not admit a diffeomorphism that exchanges V_1 and V_2 except when $q^2 \equiv \pm 1 \pmod{p}$. If $q^2 \equiv 1 \pmod{p}$, there exists a diffeomorphism σ_+ that exchanges the Heegaard tori, namely $\sigma_+ : (u, v) \in V_1 \mapsto (u, v) \in V_2$. If $q = 1$ or $p = 1$ then σ_+ is isotopic to the identity. Similarly, when $q^2 \equiv -1 \pmod{p}$, $L(p, q)$ admits a diffeomorphism σ_- that exchanges V_1 and V_2 , $\sigma_- : (u, v) \in V_1 \mapsto (\bar{u}, v) \in V_2$ and $(u, v) \in V_2 \mapsto (u, \bar{v}) \in V_1$.

Proof [Proof of Theorem 2.3] Let us first consider the case of null-homologous knots; the case of not null-homologous knots follows from the same argument. Let $K_{(a,b)}$ and $K_{(a',b')}$ be two isotopic null-homologous oriented knots on ∂V_2 in $L(p, q)$. We always fix the Heegaard torus ∂V_2 in $L(p, q)$ up to isotopy from the very beginning by using the Theorem 1 of [4] so that the knots are homologous on the Heegaard torus too. Since $K_{(a,b)}$ and $K_{(a',b')}$ have isomorphic groups, from Lemma 2.2, we know that the candidates for (a', b') are $A, -A, B, -B, C, -C, D,$ and $-D$.

We are now going to identify the diffeomorphisms that send $K_{(a,b)}$ to possible $K_{(a',b')}$ s and then we are going to analyze when such diffeomorphisms are isotopic to the identity. Clearly, τ sends $K_{(a,b)}$ to $K_{(-a,-b)}$. Note that σ_+ sends $K_{(a,b)}$ on ∂V_2 to $K_{(a,b)}$ on ∂V_1 . Then, after applying the gluing map $\varphi : \partial V_1 \rightarrow \partial V_2$ with $pq' + qp' = 1$, we get

$$\begin{pmatrix} -q & q' \\ p & p' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -qa + q'b \\ pa + p'b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}.$$

Note that for $a' = -qa + q'b$ and $b' = pa + p'b$ we have $pa' + qb' = p(-qa + q'b) + q(pa + p'b) = (pq' + qp')b = b$. By Proposition 2.1(2), it follows that we are in the case when $b = pa' + qb'$ and $|b'| = |pa + qb|$. More precisely, $b = pa' + qb'$ and $b' = pa + p'b = pa + qb$ or $b = pa' + qb'$ and $b' = pa + p'b = -pa - qb$. Since $(a, b) = 1$, the latter case does not occur. If we choose p' such that $qp' \equiv 1 \pmod{p}$, then we are left with only the case $b = pa' + qb'$ and $b' = pa + qb$ and in this case $(a', b') = (\frac{b - qp'a - q^2b}{p}, pa + qb) = C$. Therefore, σ_+ sends $K_{(a,b)}$ to K_C . In homology one has $[K_{(a,b)}] = [K_C]$; that is, $b[\lambda_2] = (pa + qb)[\lambda_2] = qb[\lambda_2]$. Thus, the case C occurs only if $q = 1$. Now $\tau \circ \sigma_+$ sends $K_{(a,b)}$ to K_{-C} and by homological reasons the case $-C$ occurs only if $q = p - 1$.

By a similar argument, the diffeomorphism σ_- sends $K_{(a,b)}$ to K_D . Moreover, in homology $[K_{(a,b)}] = [K_D] = b[\lambda_2] = (-pa - qb)[\lambda_2] = -qb[\lambda_2]$. Thus, the case D occurs only if $q = p - 1$. $\tau \circ \sigma_-$ sends $K_{(a,b)}$ to K_{-D} and by similar homological reasons the case $-D$ occurs only if $q = 1$. However, by Theorem 2.4, the

diffeomorphism σ_- exists only when $q^2 \equiv -1 \pmod{p}$; for this reason, the knot $K_{(a,b)}$ is not isotopic to K_D or K_{-D} via a diffeomorphism that is isotopic to the identity.

We want to remark that there is no diffeomorphism of $L(p, q)$ sending $K_{(a,b)}$ to K_B or K_{-B} . We see that such a diffeomorphism cannot be σ_+ or σ_- . It cannot be τ either since $\tau^2 = id$ gives us a contradiction.

Now, using Theorem 2.4, we observe that when $p = 2$ the diffeomorphisms τ and σ_+ are isotopic to the identity and hence we have Case (3). In Case (2), when $p \neq 2$ and $q = 1$ or $p - 1$ the knots $K_{(a,b)}$ and K_C are isotopic since in this case only σ_+ is isotopic to the identity. In the remaining cases, only τ exists, and when $p \neq 2$, τ is not isotopic to the identity. This proves Case (1).

Lemma 2.5. *A torus knot $K_{(a,b)}$ in $L(p, q)$ has a rational Seifert surface $S_{K_{(a,b)}}$ of Euler characteristic*

$$\chi(S_{K_{(a,b)}}) = \frac{|rb| + (1 - |rb|)|rap + rbq|}{p}$$

where r is the order of $K_{(a,b)}$.

See [2, Lemma 2.2] for rational Seifert surface construction for torus knots.

Proof Let $K_{(a,b)}$ be a rationally null-homologous torus knot of order r in $L(p, q)$. Considering the corresponding meridional curves of the Heegaard splitting on ∂V_2 , for any torus knot $K_{(a,b)}$ of order r we have

$$r[K] = m[\mu_1] + l[\mu_2] = m(-q[\mu_2] + p[\lambda_2]) + l[\mu_2],$$

where $m = \frac{rb}{p}$, $l = ra + mq = ra + \frac{rb}{p}q$, and $p \mid rb$.

We may construct a rational Seifert surface $S_{K_{(a,b)}}$ for r copies of $K_{(a,b)}$ by taking $|m|$ parallel copies of the meridional disk μ_1 of ∂V_1 and $|l|$ parallel copies of the meridional disk μ_2 of ∂V_2 and then attaching a half twisted band at each intersection for a total number of $p|l||m| = |l||rb|$ bands. Then the Euler characteristic $\chi(S_{K_{(a,b)}})$ of $S_{K_{(a,b)}}$ is $\chi(S_{K_{(a,b)}}) = \#(disks) - \#(bands)$:

$$\begin{aligned} \chi(S_{K_{(a,b)}}) &= |l| + |m| - |l||rb| \\ &= \left|ra + \frac{rbq}{p}\right| + \frac{|rb|}{p} - \left|ra + \frac{rbq}{p}\right||rb| \\ &= \frac{|rb| + (1 - |rb|)|rap + rbq|}{p}. \end{aligned}$$

3. Convex surfaces

An oriented smooth surface Σ in a contact 3-manifold is called *convex* if there is a *contact vector field* v that is a vector field whose flow preserves the contact structure ξ , transverse to Σ . Given a convex surface Σ in a contact 3-manifold with a contact vector field v , the *dividing set* Γ_Σ of Σ is defined as $\Gamma_\Sigma = \{x \in \Sigma : v(x) \in \xi_x\}$. The dividing set Γ_Σ is a multicurve and possibly disconnected. The dividing set Γ_Σ is transverse to the characteristic foliation, splitting Σ into two subsurfaces $\Sigma \setminus \Gamma_\Sigma = \Sigma_+ \sqcup \Sigma_-$, and there is a vector field w that expands/contracts a volume form ω on Σ_+ / Σ_- and w points out of Σ_+ .

Theorem 3.1 (Giroux’s tightness criterion). *A convex surface Σ in a contact 3-manifold has a tight neighborhood if and only if $\Sigma \neq S^2$ and Γ_Σ has no homotopically trivial dividing curves or $\Sigma = S^2$ and Γ_Σ is connected.*

For more information and details, see [13, 15].

3.1. Legendrian knots

The *positive stabilization/negative stabilization* $S_+(L)/S_-(L)$ of a Legendrian knot L in the standard tight contact structure on \mathbb{R}^3 is obtained by modifying the front projection of L by adding a down cusp/an up cusp as in Figure 1, respectively. Since stabilizations are performed locally, by Darboux’s theorem this defines stabilizations of Legendrian knots in any contact 3-manifold.

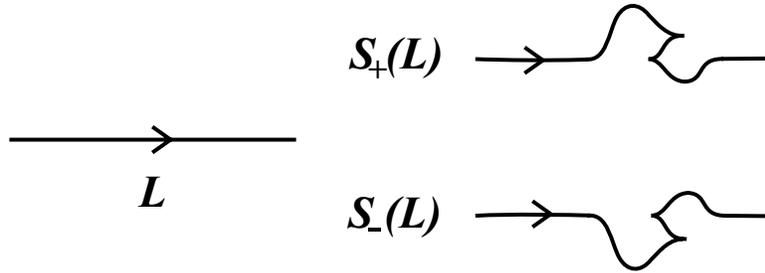


Figure 1. A positive and a negative stabilization of L .

Proposition 3.2 (Kanda [17]). *Let L be a Legendrian curve on a surface Σ and let $\text{tw}_\Sigma(L)$ denote the twisting of the contact planes along L measured with respect to the framing on L given by Σ . Then Σ may be made convex relative to L if and only if $\text{tw}_\Sigma(L) \leq 0$. If Σ is a convex surface with dividing curve Γ , then*

$$\text{tw}_\Sigma(L) = -\frac{1}{2}\#(L \cap \Gamma), \tag{1}$$

where $\#(L \cap \Gamma)$ is the unsigned count of the intersection number of L and Γ . Moreover, if Σ is a Seifert surface of a single oriented Legendrian curve L , the above formula computes the Thurston–Bennequin invariant $\text{tb}(L)$ of L and in this case the rotation number $\text{rot}(L)$ of L is

$$\text{rot}(L) = \chi(\Sigma_+) - \chi(\Sigma_-). \tag{2}$$

From Proposition 3.2 we have:

Lemma 3.3. *A surface Σ with Legendrian boundary may be made convex if and only if the twisting of contact planes along each boundary component is less than or equal to zero.*

3.2. Convex torus in standard form

For relatively prime integers a, b , the *slope* of an (a, b) -curve on a torus is $\frac{b}{a}$. A *convex torus (in standard form)* with slope s is a torus whose characteristic foliation consists of $2n$ lines of singularities with slope s , called *Legendrian divides*, and the rest of the foliation is by nonsingular lines of slope $r \neq s$, called *Legendrian rulings* where r and s are rational numbers. The $2n$ curves of the dividing set lie between the Legendrian divides. By Giroux’s flexibility theorem, [13, 15], any convex torus with slope s in a tight contact 3-manifold can be put in a standard form with any ruling slope $r \neq s$.

Theorem 3.4 (Classification of tight contact structures on a solid torus, [15]). *There are $|(r_0 + 1) \cdots (r_{k-1} + 1)(r_k)|$ tight contact structures on a solid torus $S^1 \times D^2$ with standard convex boundary having two dividing curves of slope $-\frac{p}{q}$, where $p > q > 0$ and $-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \frac{1}{\ddots - \frac{1}{r_k}}}}$ for $|r_i| < -1$. Moreover, all these contact structures are distinguished by the number of positive regions on a convex meridional disk with Legendrian boundary.*

Proposition 3.5 ([15, Proposition 4.16]). *Let ξ be a tight contact structure on $T^2 \times I$ with convex boundary having boundary slopes s_0 and s_1 on the boundary. Then, for any s between s_0 and s_1 , there is a convex torus parallel to the boundary of $T^2 \times I$ with slope s .*

3.3. Bypasses

Letting Σ be a convex surface in a contact 3-manifold, a bypass for Σ is a convex half disk D (or D with opposite orientation) with Legendrian boundary such that:

1. $\partial D = \gamma_0 \cup \gamma_1$ where γ_0, γ_1 are two arcs that intersect at their end points,
2. $D \cap \Sigma = \gamma_0$,
3. the characteristic foliation of D has three elliptic singularities along γ_0 , two positive elliptic singularities at the end points of γ_0 and one negative elliptic singularity on the interior of γ_0 , and only positive singularity along γ_1 , alternating between positive elliptic and positive hyperbolic singularities,
4. γ_0 intersects Γ_Σ exactly at the three elliptic singularities of γ_0 .

The *sign* of a bypass disk is defined to be the sign of the singularity at the center of the half disk. Figure 2 is a diagram illustrating a bypass disk.

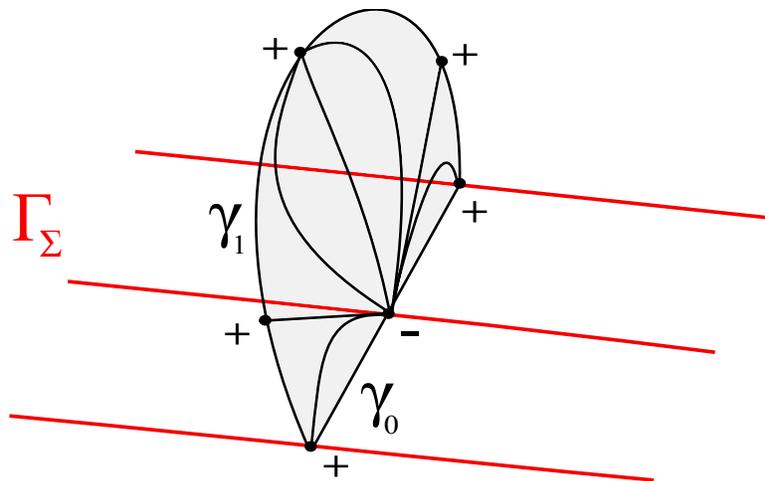


Figure 2. A bypass disk D .

A dividing curve $\gamma \subset \Gamma_\Sigma$ is called *boundary parallel* if γ cuts off a half disk of Σ that contains no other component of Γ_Σ in its interior. A boundary-parallel dividing curve allows us to find bypasses.

Proposition 3.6 (Honda [15, Proposition 3.18]). *Let Σ be a convex surface with Legendrian boundary. If the dividing set Γ_Σ contains a boundary-parallel dividing curve γ , then there exists a bypass for Σ , provided that Σ is not a disk with $\text{tb}(\partial\Sigma) = -1$.*

Proposition 3.7 (Imbalance Principle, [15, Proposition 3.17]). *Let $\Sigma = S^1 \times [0, 1]$ be a convex annulus with Legendrian boundary embedded in a tight contact 3-manifold. If $tw_\Sigma(S^1 \times \{0\}) < tw_\Sigma(S^1 \times \{1\}) \leq 0$, then there exists a boundary-parallel curve and hence a bypass along $S^1 \times \{0\}$.*

4. Legendrian torus knots in lens spaces

A tight contact structure on a 3-manifold is *universally tight* if its pullback to the universal cover is tight. In this section, we classify oriented Legendrian torus knots $L_{(a,b)}$ of knot type $K_{(a,b)}$ for $a, b > 0$ in universally tight contact structures on a lens space $L(p, q)$. Legendrian torus knots $L_{(a,b)}$ with $a, b > 0$ are called *positive* Legendrian torus knots.

There are exactly two tight contact structures on $L(p, q)$ with $q \neq p - 1$ that are universally tight, and there is only one if $q = p - 1$ [15, Proposition 5.1(3)]. According to [15], we can express $L(p, q)$ as the union of two solid tori V_1 and V_2 where ∂V_1 has dividing curves of slope ∞ . Then we split V_2 into a thickened torus $T^2 \times [0, 1]$ with two dividing curves of slope $s_0 = 0$ on $T^2 \times \{0\}$ and $s_1 = -\frac{p}{q}$ on $T^2 \times \{1\}$ and a solid torus with a unique tight contact structure on it. The universally tight contact structures on $L(p, q)$ are induced from the universally tight contact structures on $T^2 \times [0, 1]$. By Proposition 5.1 (1) of [15] there are two such universally tight contact structures on $T^2 \times [0, 1]$ and they satisfy $PD(e(\xi, s)) = \pm((-q, p) - (-1, 0))$. We assume that ξ_{ut} is induced from the one with $PD(e(\xi, s)) = (-q, p) - (-1, 0)$. The results in this section similarly hold for the other case and can be easily written down.

Remark 4.1. *By Proposition 3.5, in a universally tight lens space $L(p, q)$, one can find a convex Heegaard torus T with two dividing curves of any slope in $(-\frac{p}{q}, 0)$.*

Let $L_{(a,b)}$ be a Legendrian torus knot of knot type $K_{(a,b)}$ of order r in ξ_{ut} on $L(p, q)$. We define the rational Legendrian knot invariants, which were defined and studied in [1, 3, 12, 18] for rationally null-homologous knots. By Equation (1) in Proposition 3.2, by using the set of dividing curves Γ for the Heegaard torus containing $L_{(a,b)}$, the twisting of $L_{(a,b)}$ is $-\frac{1}{2}\#(L_{(a,b)} \cap \Gamma)$. Since $L_{(a,b)}$ is rationally null-homologous, $L_{(a,b)}$ has a framing given by a rational Seifert surface. By using the rational Seifert surface constructed in Lemma 2.5, we compute the rational Seifert framing of $L_{(a,b)}$ as $\frac{1}{r}\frac{plm}{r} = \frac{1}{r}lb$ where $m = \frac{rb}{p}$, $l = ra + \frac{rb}{p}q$. The *rational Thurston–Bennequin invariant* of $L_{(a,b)}$ is defined as the twisting of $L_{(a,b)}$ with respect to the rational Seifert framing and it is denoted by $\text{tb}_\mathbb{Q}(L_{(a,b)})$.

Note that an arbitrary (a, b) -curve and (c, d) -curve on a torus intersect $|\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}|$ times. If the dividing curves Γ of the Heegaard torus containing $L_{(a,b)}$ have slope $-\frac{s}{t}$ for integers $s, t > 0$ and if $2n$ is the number of dividing curves, then the *rational Thurston–Bennequin invariant* of $L_{(a,b)}$ is

$$\text{tb}_\mathbb{Q}(L_{(a,b)}) = \frac{1}{r}lb - n|as + bt|.$$

Let $\mathcal{L}(K)$ denote the set of all rationally null-homologous Legendrian knots in knot type K . The *maximal rational Thurston–Bennequin invariant* $\overline{\text{tb}}_{\mathbb{Q}}(K)$ of the knot type K is defined as

$$\overline{\text{tb}}_{\mathbb{Q}}(K) = \max\{\text{tb}_{\mathbb{Q}}(L) \mid L \in \mathcal{L}(K)\}.$$

Theorem 4.2. *For $a, b > 0$ relatively prime integers, the maximal rational Thurston–Bennequin invariant $\overline{\text{tb}}_{\mathbb{Q}}(K_{(a,b)})$ is*

$$\overline{\text{tb}}_{\mathbb{Q}}(K_{(a,b)}) = ab - a - b + b^2q/p.$$

Proof By Remark 4.1, we can find a convex Heegaard torus T with two dividing curves of any slope in $(-\frac{p}{q}, 0)$. In particular, there is a convex Heegaard torus T with two dividing curves of slope -1 . For $a, b > 0$ relatively prime integers, isotope the Legendrian ruling curves on T to have slope $\frac{b}{a}$ so that the ruling curves are Legendrian torus knots $L_{(a,b)}$ of knot type $K_{(a,b)}$. Since the intersection number $\#(L_{(a,b)} \cap \Gamma)$ is minimal on this convex torus T , the rational Thurston–Bennequin invariant is maximal. The maximal rational Thurston–Bennequin invariant of the knot type $K_{(a,b)}$ is computed as

$$\overline{\text{tb}}_{\mathbb{Q}}(K_{(a,b)}) = \frac{1}{r}lb - |a + b|,$$

where $l = ra + \frac{rb}{p}q$.

The *rational rotation number* $\text{rot}_{\mathbb{Q}}(L)$ of an oriented rationally null-homologous Legendrian knot L of order r is defined as the winding number of TL after trivializing the contact structure along a rational Seifert surface for L divided by r .

Let $L_{(a,b)}$ be an oriented Legendrian torus knot of order r with maximal rational Thurston–Bennequin invariant that sits on a convex Heegaard torus T . In what follows, we will explain how to compute the rational rotation number of $L_{(a,b)}$ in a similar way as Etnyre and Honda computed for Legendrian torus knots in the standard tight S^3 [8].

Let $L(p, q) = V_1 \cup_T V_2$ where V_1 and V_2 are both $D^2 \times S^1$ with meridional curve μ_1 and μ_2 , respectively. Define an invariant f_T of homology classes of curves on a convex Heegaard torus T as follows: let v be any globally nowhere zero section of ξ_{ut} and w a nowhere zero section of $\xi_{ut}|_T$, which is tangent to the Legendrian divides and transverse to and twists with ξ along the Legendrian ruling curves. Let $f_T(\gamma)$ equal the rotation of v relative to w along a closed oriented curve γ on T . For details and the properties of the function f_T , see [7] and [8]. If L is a Legendrian ruling or a Legendrian divide of order r on T then $f_T(L) = r \text{rot}_{\mathbb{Q}}(L)$. The rational rotation number of $L_{(a,b)}$ of order r on the Heegaard torus $\partial V_2 = T$ is

$$r \text{rot}_{\mathbb{Q}}(L_{(a,b)}) = mf_T(\mu_1) + lf_T(\mu_2),$$

where $m = \frac{rb}{p}$, $l = ra + \frac{rb}{p}q$.

Theorem 4.3. *Let $L_{(a,b)}$ be an oriented Legendrian torus knot of order r with maximal rational Thurston–Bennequin invariant. If $a, b > 0$, then the range of possible rational rotation numbers $\text{rot}_{\mathbb{Q}}(L_{(a,b)})$ is*

$$\{\pm b(1 - \frac{q+1}{p})\}.$$

Proof By the proof of Theorem 4.2, a Legendrian knot $L_{(a,b)}$ with maximal rational Thurston–Bennequin invariant is on a convex Heegaard torus T with two dividing curves of slope -1 in $L(p, q) = V_1 \cup_T V_2$. To compute the rational rotation number for $L_{(a,b)}$, we need to compute $f_T(\mu_1)$ and $f_T(\mu_2)$.

$f_T(\mu_1) = \pm(p - q - 1)$: Recall that the meridional curve $\mu_1 = \partial D_{V_1}$ is a $(-q, p)$ -curve on $T = \partial V_2$. Isotope the Legendrian ruling curves on T to be $(-q, p)$ -curves so that μ_1 is a ruling curve. In this case, the twisting of the contact planes along μ_1 is $-(p - q) < 0$. Thus, by Lemma 3.3, we may isotope D_{V_1} to be convex. By the proof of Theorem 4.2, the dividing curves on T have slope -1 and hence they are $(-1, 1)$ -curves and intersect μ_1 $2(p - q)$ times. Therefore, the dividing curves on D_{V_1} intersect μ_1 $2(p - q)$ times. By following the proof of Fact 1 of [8], the dividing curves on D_{V_1} separate off disks of the same sign that contain no other dividing curves. Then, by Equation (2) of Proposition 3.2, we have $f_T(\mu_1) = (p - q) - 1$ or $f_T(\mu_1) = 1 - (p - q)$.

$f_T(\mu_2) = 0$: Isotope the Legendrian ruling curves on $T = \partial V_2$ to be meridional. Let D_{V_2} be the meridional disk of V_2 where $\mu_2 = \partial D_{V_2}$ is a ruling curve. Since the twisting of the contact planes along μ_2 is -1 , by Lemma 3.3, we can isotope D_{V_2} to be convex. We know that the dividing curves on D_{V_2} intersect μ_2 twice. Thus, we have only one possible configuration for the dividing curves of D_{V_2} . By Equation (2) of Proposition 3.2, $f_T(\mu_2) = 0$.

Theorem 4.4. *Two oriented positive Legendrian torus knots in a universally tight contact structure on a lens space are uniquely realized up to contactomorphism if and only if their oriented knot types, rational Thurston–Bennequin invariants, and rational rotation numbers agree.*

The following two lemmas provide us the proof of the Theorem 4.4.

Lemma 4.5. *Two oriented Legendrian (a, b) -torus knots, $a, b > 0$, L and L' in $(L(p, q), \xi_{ut})$ with maximal rational Thurston–Bennequin invariant are uniquely realized up to contactomorphism if and only if $\text{rot}_{\mathbb{Q}}(L) = \text{rot}_{\mathbb{Q}}(L')$.*

Proof Let T and T' be standard convex Heegaard tori on which L and L' respectively sit in $L(p, q)$. Also, let $V_1 \cup_T V_2$ and $V'_1 \cup_{T'} V'_2$ be the Heegaard splittings associated to T and T' . Since $\text{tb}_{\mathbb{Q}}(L) = \text{tb}_{\mathbb{Q}}(L') = \overline{\text{tb}}_{\mathbb{Q}}(K_{(a,b)})$, the slopes of the dividing curves on T and T' are the same. Then, by Theorem 3.4, by the classification of tight contact structures on solid tori, there is a contactomorphism $g : V_1 \rightarrow V'_1$ such that $g(L) = L'$. By Theorem 3.4 again, the contactomorphism type of a tight contact structure on V_2 or V'_2 is determined by the number of positive bypasses on meridional disks. If r is the order of L and L' in $L(p, q)$, then the number of positive bypasses on meridional disks is determined by r times the rational rotation number of the Legendrian knots L and L' , respectively. We can extend the contactomorphism g to all of $L(p, q)$ provided that L and L' have the same rational rotation number.

Lemma 4.6. *If $L_{(a,b)}$ is a positive Legendrian torus knot in $(L(p, q), \xi_{ut})$ with nonmaximal rational Thurston–Bennequin invariant then there is a Legendrian torus knot $L'_{(a,b)}$ such that $L_{(a,b)}$ is a stabilization of $L'_{(a,b)}$.*

Proof Let T be a standard convex Heegaard torus on which the positive Legendrian torus knot $L_{(a,b)}$ sits. Let $-\frac{s}{t}$ for $s, t > 0$ be the slope of the dividing curves Γ_T on T and let $2n$ be the number of dividing curves. By Theorem 4.2, Legendrian torus knots with maximal rational Thurston–Bennequin invariant sit on a convex Heegaard torus with two dividing curves of slope -1 . Since $\text{tb}_{\mathbb{Q}}(L_{(a,b)}) < \overline{\text{tb}}_{\mathbb{Q}}(K_{(a,b)})$, we have two cases for the slope and the number of dividing curves of Γ_T on T : $-\frac{s}{t} = -1$ and $n > 1$ or $-\frac{s}{t} \neq -1$ and $n \geq 1$.

We are now going to show that we can find a bypass disk in both cases. By Remark 4.1, we know that there is a convex torus T' with two dividing curves of slope -1 . In fact, by [15], we can assume that T' is a standardly embedded convex torus parallel and disjoint from T . Now take the $T^2 \times [0, 1]$ region between T and T' and take the annulus $A = L_{(a,b)} \times [0, 1]$ between $T = T^2 \times \{0\}$ and $T' = T^2 \times \{1\}$. Furthermore, isotope the ruling curves on both boundary components of $T^2 \times [0, 1]$ to have slope $\frac{b}{a}$. Then $\partial A = L_{(a,b)} \cup L'_{(a,b)}$ are Legendrian ruling curves on the boundary of $T^2 \times [0, 1]$ and the twisting of contact planes along each boundary component will be less than zero. Therefore, by Lemma 3.3, we can make A convex. The dividing curves on $T = T^2 \times \{0\}$ are $(-t, s)$ -curves and intersect A in $2n|\det \begin{pmatrix} a & -t \\ b & s \end{pmatrix}| = 2n(sa + tb)$ points and the dividing curves on $T' = T^2 \times \{1\}$ are $(-1, 1)$ -curves and intersect A in $2|\det \begin{pmatrix} a & -1 \\ b & 1 \end{pmatrix}| = 2(a + b)$ points. In both cases, when $-\frac{s}{t} = -1$ and $n > 1$ or when $-\frac{s}{t} \neq -1$ and $n \geq 1$, we have $2n(sa + tb) > 2(a + b)$. Thus, there is a boundary parallel dividing curve along $T = T^2 \times \{0\}$ and hence by Proposition 3.7 a bypass for $L_{(a,b)}$. In other words, $L_{(a,b)}$ destabilizes.

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