Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 949 – 966
© TÜBİTAK
doi:10.3906/mat-1606-11

Variational problem involving operator curl associated with p-curl system

Junichi ARAMAKI*
Division of Science, Faculty of Science and Engineering, Tokyo Denki University, Hatoyama-machi, Saitama, Japan

Received: 05.06.2017 • **Accepted/Published Online:** 06.09.2017 • **Final Version:** 08.05.2018

Abstract: We shall study the problem of minimizing a functional involving curl of vector fields in a three-dimensional, bounded multiconnected domain with the prescribed tangent component of a given vector field on the boundary. It will be seen that the minimizers are weak solutions of the p-curl type system. We shall prove the existence and the estimate of minimizers of a more general functional that contains the L^p norm of the curl of vector fields. We shall also give the continuity with respect to the given data.

Key words: Variational problem, p-curl system, multiconnected domain

1. Introduction

In this paper we shall consider the variational problem

$$\inf\left\{\frac{1}{2}\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{u}|^2)dx-\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx\right\},$$

where $S(x,t)$ satisfies some structure condition, $\boldsymbol{f}$ is a given vector field, and the minimization is taken in an appropriate space with tangent trace on the boundary being prescribed. The structure condition contains $S(x,t)=t^{p/2}$ $(1<p<\infty)$ as a typical example. In this case, if $\boldsymbol{f}=\boldsymbol{0}$, the problem

$$\inf\int_\Omega |\mathrm{curl}\,\boldsymbol{u}|^p dx$$

was proposed by Pan [12, p. 9].

This problem is related to the mathematical theory of liquid crystals, of superconductivity, and of electromagnetic fields. See, for example, Bates and Pan [5], Pan and Qi [13], and Miranda et al. [11].

When $p=2$, $\boldsymbol{f}=\boldsymbol{0}$, $S(x,t)=t$, and Ω is a simply connected domain without holes, the authors of [5] showed the existence of a minimizer. For the multiconnected domain, the author of [12] obtained the existence of a minimizer to minimization problem (1.4) below in this case.

More precisely, let $S(x,t)$ be a Carathéodory function on $\Omega\times[0,\infty)$ and $S(x,t^2)$ be a convex function with respect to t. Moreover, assume that for a.e. $x\in\Omega$, $S(x,t)\in C^1((0,\infty))$ and there exist $1<p<\infty$ and $\lambda,\Lambda>0$ such that for a.e. $x\in\Omega$ and all $t>0$,

$$\lambda t^{(p-2)/2}\le S_t(x,t):=\frac{\partial}{\partial t}S(x,t)\le\Lambda t^{(p-2)/2}. \tag{1.1}$$

*Correspondence: aramaki@mail.dendai.ac.jp
2010 *AMS Mathematics Subject Classification:* 49J20, 58Axx, 82D30

Without loss of generality, we may assume that $S(x,0)=0$. We furthermore assume the following structure condition:

$$(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})>0 \tag{1.2}$$

for any $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$ with $\boldsymbol{a}\neq\boldsymbol{b}$. Here, for any vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$, $\boldsymbol{a}\cdot\boldsymbol{b}$ denotes the Euclidean inner product. Under (1.1) with $S(x,0)=0$, we have

$$\frac{2}{p}\lambda t^{p/2}\le S(x,t)\le\frac{2}{p}\Lambda t^{p/2}. \tag{1.3}$$

For example, the function $S(x,t)=\nu(x)t^{p/2}$ where $\nu(x)$ is a measurable function satisfying $0<\nu_*\le\nu(x)\le\nu^*<\infty$ for a.e. $x\in\Omega$ satisfies (1.1)–(1.2).

Let Ω be a bounded domain in $\mathbb{R}^3$ with C^r boundary $\partial\Omega$ $(r\ge 2)$. Let $\mathcal{H}$ be a given vector field on $\partial\Omega$ and $\mathcal{H}_T$ be the tangential component of $\mathcal{H}$. Let $W^{1,p}(\Omega,\mathbb{R}^3)$ be the standard Sobolev space of vector fields. From now on, we denote the tangential component of a vector field $\boldsymbol{u}$ by $\boldsymbol{u}_T$; that is, $\boldsymbol{u}_T=\boldsymbol{u}-(\boldsymbol{u}\cdot\boldsymbol{\nu})\boldsymbol{\nu}$ where $\boldsymbol{\nu}$ is the outer normal unit vector to the boundary $\partial\Omega$. For any given vector field

$$\mathcal{H}_T\in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3),$$

define a space of vector fields

$$W_t^{1,p}(\Omega,\mathbb{R}^3,\mathcal{H}_T)=\{\boldsymbol{u}\in W^{1,p}(\Omega,\mathbb{R}^3);\boldsymbol{u}_T=\mathcal{H}_T\text{ on }\partial\Omega\}.$$

Then it is clear that $W_t^{1,p}(\Omega,\mathbb{R}^3,\mathcal{H}_T)$ is a closed convex set in $W^{1,p}(\Omega,\mathbb{R}^3)$. We consider the minimization problem

$$R_t^p(\mathcal{H}_T,\boldsymbol{f})=\inf_{\boldsymbol{u}\in W_t^{1,p}(\Omega,\mathbb{R}^3,\mathcal{H}_T)}\left\{\frac{1}{2}\int_\Omega S(x,|\operatorname{curl}\boldsymbol{u}|^2)dx-\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}dx\right\}, \tag{1.4}$$

where $\boldsymbol{f}\in L^{p'}(\Omega,\mathbb{R}^3)$ is given. Here p' is the conjugate exponent of p; that is, $(1/p)+(1/p')=1$. When $p=2$, $S(x,t)=t$, $\boldsymbol{f}=\boldsymbol{0}$, and Ω is a simply connected domain without holes, the authors of [5] showed that (1.4) is achieved, and then in the same case and when Ω is a bounded multiconnected domain, the author of [12] succeeded to show the existence of a minimizer of (1.4) and got an estimate of the minimizer.

Since we allow Ω to be a multiconnected domain in $\mathbb{R}^3$, throughout this paper, we assume that the domain Ω satisfies the following (O1) and (O2) (cf. Dautray and Lions [6] and Amrouche and Seloula [2]).

(O1) Ω is a bounded domain in $\mathbb{R}^3$ with C^r $(r\ge 2)$ boundary $\partial\Omega$. Ω is locally situated on one side of $\partial\Omega$, $\partial\Omega$ has a finite number of connected components $\Gamma_1,\dots,\Gamma_{m+1}$ $(m\ge 0)$, and Γ_{m+1} denotes the boundary of the infinite connected component of $\mathbb{R}^3\setminus\overline{\Omega}$.

(O2) There exist n manifolds of dimension 2 and of class C^r denoted by $\Sigma_1,\dots,\Sigma_n$ $(n\ge 0)$ such that $\Sigma_i\cap\Sigma_j=\emptyset$ $(i\neq j)$ and they are nontangential to $\partial\Omega$ and such that $\Omega\setminus(\cup_{i=1}^n\Sigma_i)$ is simply connected and pseudo $C^{1,1}$.

The number n is called the first Betti number and m the second Betti number of Ω. We say that Ω is simply connected if $n=0$, and Ω has no holes if $m=0$. If we define the spaces

$$\mathbb{K}_N^p(\Omega)=\{\boldsymbol{u}\in L^p(\Omega,\mathbb{R}^3);\operatorname{curl}\boldsymbol{u}=\boldsymbol{0},\operatorname{div}\boldsymbol{u}=0\text{ in }\Omega,\boldsymbol{\nu}\cdot\boldsymbol{u}=0\text{ on }\partial\Omega\}$$

and

$$\mathbb{K}_T^p(\Omega) = \{\boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3); \operatorname{curl} \boldsymbol{u} = \boldsymbol{0}, \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u}_T = \boldsymbol{0} \text{ on } \partial\Omega\},$$

then it is well known that $\dim \mathbb{K}_N^p(\Omega) = n$ and $\dim \mathbb{K}_T^p(\Omega) = m$. We note that $\mathbb{K}_N^p(\Omega)$ and $\mathbb{K}_T^p(\Omega)$ are contained in $W^{1,p}(\Omega, \mathbb{R}^3)$; moreover, $\mathbb{K}_N^p(\Omega)$ and $\mathbb{K}_T^p(\Omega)$ are closed subspaces of $W^{1,p}(\Omega, \mathbb{R}^3)$. It will be shown in Lemma 2.3 below that $\mathbb{K}_N^p(\Omega)$ and $\mathbb{K}_T^p(\Omega)$ are also closed subspaces of $L^p(\Omega, \mathbb{R}^3)$. Thus, since $\mathbb{K}_T^p(\Omega)$ is a finite dimensional closed subspace of $L^p(\Omega, \mathbb{R}^3)$, $\mathbb{K}_T^p(\Omega)$ has a complement $\mathbb{L}^p$ in $L^p(\Omega, \mathbb{R}^3)$; that is, $\mathbb{L}^p$ is a closed subspace of $L^p(\Omega, \mathbb{R}^3)$, $\mathbb{L}^p \cap \mathbb{K}_T^p(\Omega) = \{\boldsymbol{0}\}$, and $L^p(\Omega, \mathbb{R}^3) = \mathbb{L}^p \oplus \mathbb{K}_T^p(\Omega)$ (the direct sum). Therefore, for any $\boldsymbol{w} \in L^p(\Omega, \mathbb{R}^3)$, there exist uniquely $\boldsymbol{v} \in \mathbb{L}^p$ and $\boldsymbol{u} \in \mathbb{K}_T^p(\Omega)$ such that $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{u}$. We denote the projection $P : L^p(\Omega, \mathbb{R}^3) \to \mathbb{L}^p$ by $P\boldsymbol{w} = \boldsymbol{v}$.

Define

$$\begin{aligned} H^p(\Omega, \operatorname{curl}, \operatorname{div} 0) &= \{\boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3); \operatorname{curl} \boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3), \\ &\qquad \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}, \\ H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T) &= \{\boldsymbol{u} \in H^p(\Omega, \operatorname{curl}, \operatorname{div} 0); \boldsymbol{u}_T = \mathcal{H}_T \text{ on } \partial\Omega\}. \end{aligned}$$

Note that if $\boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3)$ and $\operatorname{curl} \boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3)$, then the tangent trace $\boldsymbol{u}_T$ is well defined as an element of $W^{-1/p,p}(\partial\Omega, \mathbb{R}^3)$ (cf. [2, p. 45]), and

$$H^p(\Omega, \operatorname{curl}, \operatorname{div} 0) \cap W^{1,p}(\Omega, \mathbb{R}^3) = \{\boldsymbol{u} \in W^{1,p}(\Omega, \mathbb{R}^3); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}.$$

Moreover, we note that if $\mathcal{H}_T \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$, then

$$H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T) \subset W_t^{1,p}(\Omega, \mathbb{R}^3, \mathcal{H}_T)$$

(cf. Amrouche and Seloula [1, Theorem 2.3]). We will see, in Lemma 2.1 of Section 2, that

$$R_t^p(\mathcal{H}_T, \boldsymbol{f}) = \inf_{\boldsymbol{v} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)} \left\{ \frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2) dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx \right\}. \tag{1.5}$$

We are in a position to state the main theorem.

Theorem 1.1 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain satisfying* (O1) *and* (O2) *with* $r \geq 2$*, and let* $\mathcal{H}_T \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ *and* $\boldsymbol{f} \in L^{p'}(\Omega, \mathbb{R}^3)$ *satisfying* $\operatorname{div} \boldsymbol{f} = 0$ *and* $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{z} dx = 0$ *for all* $\boldsymbol{z} \in \mathbb{K}_T^p(\Omega)$*. Then* $R_t^p(\mathcal{H}_T, \boldsymbol{f})$ *is achieved, and the minimizers* $\boldsymbol{A}$ *of* $R_t^p(\mathcal{H}_T, \boldsymbol{f})$ *in the space* $H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$ *satisfy the following estimate. There exists a constant* $C = C(\Omega) > 0$ *independent of* $\mathcal{H}_T$ *such that*

$$\|P\boldsymbol{A}\|_{W^{1,p}(\Omega)} \leq C(\|\mathcal{H}_T\|_{W^{1-1/p,p}(\partial\Omega)} + \|\boldsymbol{f}\|_{L^{p'}(\Omega)}).$$

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we consider the continuous dependence on the data of the minimizers.

2. Preliminaries

In this section, we shall give some lemmas as preliminaries.

Lemma 2.1 *Let* $\mathcal{H}_T \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ *and* $\boldsymbol{f} \in L^{p'}(\Omega, \mathbb{R}^3)$ *satisfying* $\operatorname{div} \boldsymbol{f} = 0$ *in* Ω*. Then* $R_t^p(\mathcal{H}_T, \boldsymbol{f})$ *defined by* (1.4) *satisfies* (1.5)*; that is to say, we have*

$$R_t^p(\mathcal{H}_T, \boldsymbol{f}) = \inf_{\boldsymbol{v} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)} \left\{ \frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2) dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx \right\}.$$

Proof Put

$$\begin{aligned} \alpha &= \inf_{\boldsymbol{u} \in W_t^{1,p}(\Omega, \mathbb{R}^3, \mathcal{H}_T)} \left\{ \frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{u}|^2) dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx \right\}, \\ \beta &= \inf_{\boldsymbol{v} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)} \left\{ \frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2) dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx \right\}. \end{aligned}$$

Since $H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T) \subset W_t^{1,p}(\Omega, \mathbb{R}^3, \mathcal{H}_T)$, it is trivial that $\alpha \le \beta$. For any $\boldsymbol{u} \in W^{1,p}(\Omega, \mathbb{R}^3, \mathcal{H}_T)$, the following Dirichlet problem

$$\begin{cases} \Delta\varphi = \operatorname{div} \boldsymbol{u} & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $\varphi \in W^{2,p}(\Omega)$ (cf. Girault and Raviart [10, Theorem 1.8]). If we define $\boldsymbol{v} = \boldsymbol{u} - \nabla\varphi \in W^{1,p}(\Omega, \mathbb{R}^3)$, then $\operatorname{curl} \boldsymbol{v} = \operatorname{curl} \boldsymbol{u}, \operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{u} - \Delta\varphi = 0$ in Ω and $\boldsymbol{v}_T = \boldsymbol{u}_T - (\nabla\varphi)_T = \boldsymbol{u}_T = \mathcal{H}_T$. Thus, $\boldsymbol{v} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$. Moreover, since $\operatorname{div} \boldsymbol{f} = 0$ in Ω and $\varphi = 0$ on $\partial\Omega$, we have

$$\begin{aligned} \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx + \int_\Omega \boldsymbol{f} \cdot \nabla\varphi dx \\ &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx - \int_\Omega (\operatorname{div} \boldsymbol{f})\varphi dx + \int_{\partial\Omega} (\boldsymbol{f} \cdot \boldsymbol{\nu})\varphi dS \\ &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx. \end{aligned}$$

Therefore, we have

$$\frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{u}|^2) dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx = \frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2) dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx \ge \beta.$$

Thus, we have $\alpha \ge \beta$. □

By Lemma 2.1, the minimization problem (1.4) reduces to the following problem.

Find the minimizer $\boldsymbol{u} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$ such that

$$R_t^p(\mathcal{H}_T, \boldsymbol{f}) = \inf_{\boldsymbol{v} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)} \left\{ \frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2) dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx \right\}. \tag{2.1}$$

In the sequel, we frequently use the following lemma.

Lemma 2.2 (i) *If* $\boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3)$*,* $\operatorname{curl} \boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3)$*,* $\operatorname{div} \boldsymbol{u} \in L^p(\Omega)$*, and* $\boldsymbol{u} \cdot \boldsymbol{\nu} \in W^{1-1/p,p}(\partial\Omega)$*, then* $\boldsymbol{u} \in W^{1,p}(\Omega, \mathbb{R}^3)$*, and there exists a constant* $c_1(\Omega) > 0$ *such that*

$$\|\boldsymbol{u}\|_{W^{1,p}(\Omega)} \le c_1(\Omega)(\|\boldsymbol{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \boldsymbol{u}\|_{L^p(\Omega)} + \|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)} + \|\boldsymbol{u} \cdot \boldsymbol{\nu}\|_{W^{1-1/p,p}(\partial\Omega)}). \tag{2.2}$$

Here we note that if furthermore Ω is simply connected, we can delete the first term $\|\boldsymbol{u}\|_{L^p(\Omega)}$ in the right-hand side of (2.2)*.*

(ii) *If $\boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3)$, $\operatorname{curl} \boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3)$, $\operatorname{div} \boldsymbol{u} \in L^p(\Omega)$, and $\boldsymbol{u}_T \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$, then $\boldsymbol{u} \in W^{1,p}(\Omega, \mathbb{R}^3)$, and there exists a constant $c_2(\Omega) > 0$ such that*

$$\|\boldsymbol{u}\|_{W^{1,p}(\Omega)} \leq c_2(\Omega)(\|\boldsymbol{u}\|_{L^p(\Omega)} + \|\operatorname{curl} \boldsymbol{u}\|_{L^p(\Omega)} + \|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)} + \|\boldsymbol{u}_T\|_{W^{1-1/p,p}(\partial\Omega)}). \tag{2.3}$$

We note that if furthermore Ω has no holes, we can delete the first term $\|\boldsymbol{u}\|_{L^p(\Omega)}$ in the right-hand side of (2.3)*.*

For the proof of (2.2) and (2.3), see [2, Theorem 3.4 and Corollary 5.2]. If Ω is simply connected or has no holes, we can see the deletion of $\|\boldsymbol{u}\|_{L^p(\Omega)}$ from (2.3) or (2.4) in Aramaki's work [4, Lemma 2.2].

Lemma 2.3 *The space $\mathbb{K}_T^p(\Omega)$ is a closed subspace of $L^p(\Omega, \mathbb{R}^3)$.*

Proof Let $\mathbb{K}_T^p(\Omega) \ni \boldsymbol{u}_j \to \boldsymbol{u}$ in $L^p(\Omega, \mathbb{R}^3)$. Then from (2.3) we have

$$\|\boldsymbol{u}_j - \boldsymbol{u}_k\|_{W^{1,p}(\Omega)} \leq c_2(\Omega)\|\boldsymbol{u}_j - \boldsymbol{u}_k\|_{L^p(\Omega)}.$$

Therefore, $\{\boldsymbol{u}_j\}$ is a Cauchy sequence in $W^{1,p}(\Omega, \mathbb{R}^3)$. Hence, there exists $\boldsymbol{u}_0 \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that $\boldsymbol{u}_j \to \boldsymbol{u}_0$ in $W^{1,p}(\Omega, \mathbb{R}^3)$, so we have $\boldsymbol{u} = \boldsymbol{u}_0$ and $\boldsymbol{u}_j \to \boldsymbol{u}$ in $W^{1,p}(\Omega, \mathbb{R}^3)$ as $j \to \infty$. It is clear that $\operatorname{curl} \boldsymbol{u} = \boldsymbol{0}, \operatorname{div} \boldsymbol{u} = 0$ in Ω, and $\boldsymbol{u}_T = \boldsymbol{0}$ on $\partial\Omega$. This implies that $\boldsymbol{u} \in \mathbb{K}_T^p(\Omega)$. □

3. Proof of the main Theorem 1.1

In this section, we give a proof of Theorem 1.1. The proof consists of some lemmas and propositions.

Lemma 3.1 *Let $\boldsymbol{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$. Then the minimizing problem*

$$\gamma = \inf_{\boldsymbol{u} \in \mathbb{K}_T^p(\Omega)} \|\boldsymbol{A} - \boldsymbol{u}\|_{L^p(\Omega)} \tag{3.1}$$

has a unique minimizer.

Proof From Lemma 2.3, we know that $\mathbb{K}_T^p(\Omega)$ is a closed subspace of $L^p(\Omega, \mathbb{R}^3)$. Thus, it is well known that (3.1) has a minimizer. For the uniqueness of the minimizer, it suffices to show that the unit sphere $B = \{\boldsymbol{u} \in L^p(\Omega, \mathbb{R}^3); \|\boldsymbol{u}\|_{L^p(\Omega)} = 1\}$ does not contain any line segment $[\boldsymbol{u}, \boldsymbol{v}] = \{\lambda\boldsymbol{u} + (1-\lambda)\boldsymbol{v}; 0 \leq \lambda \leq 1\}$ for $\boldsymbol{u}, \boldsymbol{v} \in B$ and $\boldsymbol{u} \neq \boldsymbol{v}$ (cf. Fujita et al. [9, p. 306 and the remark]). However, this is clear because the functional

$$f(\boldsymbol{u}) = \int_\Omega |\boldsymbol{u}|^p dx$$

is strictly convex. □

For $\boldsymbol{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$, let $\boldsymbol{u} \in \mathbb{K}_T^p(\Omega)$ be a unique minimizer of (3.1), and define $\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{u}$. Then, since for any $\boldsymbol{z} \in \mathbb{K}_T^p(\Omega)$ and $\theta \in \mathbb{R}$, $\|\boldsymbol{B}\|_{L^p(\Omega)}^p \leq \|\boldsymbol{B} + \theta\boldsymbol{z}\|_{L^p(\Omega)}^p$, we have

$$0 = \frac{d}{d\theta}\bigg|_{\theta=0} \int_\Omega |\boldsymbol{B} + \theta\boldsymbol{z}|^p dx = p\int_\Omega |\boldsymbol{B}|^{p-2}\boldsymbol{B} \cdot \boldsymbol{z} dx.$$

If we define a space

$$B(\Omega, \mathcal{H}_T) = \{\boldsymbol{B} \in L^p(\Omega, \mathbb{R}^3); \operatorname{curl} \boldsymbol{B} \in L^p(\Omega, \mathbb{R}^3), \operatorname{div} \boldsymbol{B} = 0 \text{ in } \Omega,$$
$$\boldsymbol{B}_T = \mathcal{H}_T \text{ on } \partial\Omega \text{ and } \int_\Omega |\boldsymbol{B}|^{p-2}\boldsymbol{B} \cdot \boldsymbol{z} dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}_T^p(\Omega)\},$$

then we see that $\boldsymbol{B} \in B(\Omega, \mathcal{H}_T)$. Then we have the following.

Lemma 3.2 *We can see that for any* $\boldsymbol{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$, *there exist uniquely* $\boldsymbol{B} \in B(\Omega, \mathcal{H}_T)$ *and* $\boldsymbol{u} \in \mathbb{K}_T^p(\Omega)$ *such that*

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{u}.$$

Proof For any $\boldsymbol{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$, as in the above we can write

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{u} \text{ where } \boldsymbol{B} \in B(\Omega, \mathcal{H}_T) \text{ and } \boldsymbol{u} \in \mathbb{K}_T^p(\Omega).$$

We show the uniqueness of the above decomposition. If we can write

$$\boldsymbol{A} = \boldsymbol{B}_1 + \boldsymbol{u}_1 = \boldsymbol{B}_2 + \boldsymbol{u}_2$$

where $\boldsymbol{B}_1, \boldsymbol{B}_2 \in B(\Omega, \mathcal{H}_T), \boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{K}_T^p(\Omega)$, then $\boldsymbol{B}_1 - \boldsymbol{B}_2 = \boldsymbol{u}_2 - \boldsymbol{u}_1 \in \mathbb{K}_T^p(\Omega)$. Therefore, we have

$$\int_\Omega |\boldsymbol{B}_1|^{p-2}\boldsymbol{B}_1 \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) dx = 0, \int_\Omega |\boldsymbol{B}_2|^{p-2}\boldsymbol{B}_2 \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) dx = 0.$$

Hence,

$$\int_\Omega (|\boldsymbol{B}_1|^{p-2}\boldsymbol{B}_1 - |\boldsymbol{B}_2|^{p-2}\boldsymbol{B}_2) \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) dx = 0. \tag{3.2}$$

Here we use the following inequality. There exists a constant $c > 0$ such that

$$(|\boldsymbol{a}|^{p-2}\boldsymbol{a} - |\boldsymbol{b}|^{p-2}\boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) \geq \begin{cases} c|\boldsymbol{a} - \boldsymbol{b}|^p & \text{if } p \geq 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2}|\boldsymbol{a} - \boldsymbol{b}|^2 & \text{if } 1 < p < 2 \end{cases} \tag{3.3}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. For the proof of this inequality, see DiBenedetto [7, Lemma 4.4] for $p \geq 2$, and see [11, (7C')] for $1 < p < 2$. Applying (3.3) with $\boldsymbol{a} = \boldsymbol{B}_1, \boldsymbol{b} = \boldsymbol{B}_2$ to (3.2), we have

$$\int_\Omega |\boldsymbol{B}_1 - \boldsymbol{B}_2|^p dx = 0 \text{ for } p \geq 2$$

and

$$\int_\Omega (|\boldsymbol{B}_1| + |\boldsymbol{B}_2|)^{p-2}|\boldsymbol{B}_1 - \boldsymbol{B}_2|^2 dx = 0 \text{ for } 1 < p < 2.$$

From these equalities, we have $\boldsymbol{B}_1 = \boldsymbol{B}_2$, so $\boldsymbol{u}_1 = \boldsymbol{u}_2$. □

Now we state a refinement of Fatou's lemma (cf. Evans [8, pp. 11–12]).

Lemma 3.3 *Assume $1 < p < \infty$. Let $\boldsymbol{B}_j \to \boldsymbol{B}$ weakly in $L^p(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Then we have*

$$\lim_{j\to\infty}\int_\Omega \big(|\boldsymbol{B}_j|^p - \big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\big)dx = \int_\Omega |\boldsymbol{B}|^p dx. \tag{3.4}$$

If furthermore

$$\lim_{j\to\infty}\int_\Omega |\boldsymbol{B}_j|^p dx = \int_\Omega |\boldsymbol{B}|^p dx,$$

then

$$|\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j \to |\boldsymbol{B}|^{p-2}\boldsymbol{B} \text{ strongly in } L^{p'}(\Omega, \mathbb{R}^3) \tag{3.5}$$

where p' denotes the conjugate exponent of p, i.e. $(1/p) + (1/p') = 1$. In particular, if $\boldsymbol{B}_j \to \boldsymbol{B}$ strongly in $L^p(\Omega, \mathbb{R}^3)$ and a.e. in Ω, then (3.5) holds.

Proof We use an elementary estimate. Let $1 \le q < \infty$. Then for any fixed $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, q) > 0$ such that

$$\big||\boldsymbol{a}+\boldsymbol{b}|^q - |\boldsymbol{a}|^q\big| \le \varepsilon|\boldsymbol{a}|^q + C|\boldsymbol{b}|^q \tag{3.6}$$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ (cf. [8, (1.13)]). Define

$$g_j^\varepsilon = \Big[\Big|\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j\big|^{p'} - \big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'} - \big||\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\Big| - \varepsilon\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\Big]^+,$$

where $[a]^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. Then we have

$$\begin{aligned}
g_j^\varepsilon &\le \Big[\Big|\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j\big|^{p'} - \big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\Big| + \big||\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'} \\
&\quad -\varepsilon\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\Big]^+ \\
&= \Big[\Big|\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B} + |\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'} \\
&\quad -\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\Big| \\
&\quad +\big||\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'} - \varepsilon\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\Big]^+.
\end{aligned}$$

If we apply (3.6) with $\boldsymbol{a} = |\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j - |\boldsymbol{B}|^{p-2}\boldsymbol{B}$, $\boldsymbol{b} = |\boldsymbol{B}|^{p-2}\boldsymbol{B}$ and $q = p'$, we have

$$g_j^\varepsilon \le (C+1)\big||\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'} = (C+1)|\boldsymbol{B}|^p.$$

We note that the right-hand side is integrable. By the hypothesis, we can see $g_j^\varepsilon \to 0$ a.e. in Ω. Therefore, by the Lebesgue dominated theorem, we have

$$\lim_{j\to\infty}\int_\Omega g_j^\varepsilon dx = 0.$$

Therefore, we have

$$\begin{aligned}
&\limsup_{j\to\infty}\int_\Omega\Big|\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j\big|^{p'}-\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j-|\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}-\big||\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\Big|dx\\
&\le\varepsilon\limsup_{j\to\infty}\int_\Omega\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j-|\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}dx\\
&\le\varepsilon 2^{p'}\limsup_{j\to\infty}\int_\Omega\big(\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j\big|^{p'}+\big||\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\big)dx\\
&=\varepsilon 2^{p'}\limsup_{j\to\infty}\int_\Omega\big(|\boldsymbol{B}_j|^p+|\boldsymbol{B}|^p\big)dx.
\end{aligned}$$

Since $\boldsymbol{B}_j\to\boldsymbol{B}$ weakly in $L^p(\Omega,\mathbb{R}^3)$, $\|\boldsymbol{B}_j\|_{L^p(\Omega)}$ is bounded. Since ε is arbitrary, we have

$$\lim_{j\to\infty}\int_\Omega\big(|\boldsymbol{B}_j|^p-\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j-|\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}\big)dx=\int_\Omega|\boldsymbol{B}|^p dx.$$

If furthermore

$$\lim_{j\to\infty}\int_\Omega|\boldsymbol{B}_j|^p dx=\int_\Omega|\boldsymbol{B}|^p dx,$$

then we have

$$\lim_{j\to\infty}\int_\Omega\big||\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j-|\boldsymbol{B}|^{p-2}\boldsymbol{B}\big|^{p'}dx=0.$$

This completes the proof. □

Lemma 3.4 *$B(\Omega,\mathcal{H}_T)$ is a weakly closed set in $W^{1,p}(\Omega,\mathbb{R}^3)$.*

Proof Let $\boldsymbol{B}_j\in B(\Omega,\mathcal{H}_T),\boldsymbol{B}_j\to\boldsymbol{B}$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$. Then we have $\operatorname{curl}\boldsymbol{B}\in L^p(\Omega,\mathbb{R}^3),\operatorname{div}\boldsymbol{B}=0$ in Ω, $\boldsymbol{B}_T=\mathcal{H}_T$ on $\partial\Omega$, and

$$\int_\Omega|\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j\cdot\boldsymbol{z}dx=0\text{ for all }\boldsymbol{z}\in\mathbb{K}^p_T(\Omega).$$

Passing to a subsequence, we may assume that $\boldsymbol{B}_j\to\boldsymbol{B}$ strongly in $L^p(\Omega,\mathbb{R}^3)$ and a.e. in Ω. Thus, from Lemma 3.3, we have $|\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j\to|\boldsymbol{B}|^{p-2}\boldsymbol{B}$ in $L^{p'}(\Omega,\mathbb{R}^3)$. Therefore, we have

$$\int_\Omega|\boldsymbol{B}|^{p-2}\boldsymbol{B}\cdot\boldsymbol{z}dx=0\text{ for all }\boldsymbol{z}\in\mathbb{K}^p_T(\Omega).$$

This implies that $\boldsymbol{B}\in B(\Omega,\mathcal{H}_T)$. □

Lemma 3.5 *There exists a constant $c(\Omega)>0$ such that for all $\boldsymbol{B}\in W^{1,p}(\Omega,\mathbb{R}^3)$ satisfying $\operatorname{div}\boldsymbol{B}=0$ in Ω and*

$$\int_\Omega|\boldsymbol{B}|^{p-2}\boldsymbol{B}\cdot\boldsymbol{z}dx=0\textit{ for all }\boldsymbol{z}\in\mathbb{K}^p_T(\Omega),$$

we have

$$\|\boldsymbol{B}\|_{W^{1,p}(\Omega)}\le c(\Omega)(\|\operatorname{curl}\boldsymbol{B}\|_{L^p(\Omega)}+\|\boldsymbol{B}_T\|_{W^{1-1/p,p}(\partial\Omega)}).\tag{3.7}$$

Proof If the conclusion (3.7) is false, there exists a sequence $\{\boldsymbol{B}_j\} \subset W^{1,p}(\Omega, \mathbb{R}^3)$ satisfying $\operatorname{div} \boldsymbol{B}_j = 0$ in Ω and

$$\int_\Omega |\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j \cdot \boldsymbol{z} dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}^p_T(\Omega),$$

such that $\|\boldsymbol{B}_j\|_{W^{1,p}(\Omega)} = 1, \|\operatorname{curl} \boldsymbol{B}_j\|_{L^p(\Omega)} \to 0, \|\boldsymbol{B}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)} \to 0$ as $j \to \infty$. After passing to a subsequence, we may assume that $\boldsymbol{B}_j \to \boldsymbol{B}_0$ weakly in $W^{1,p}(\Omega, \mathbb{R}^3)$, strongly in $L^p(\Omega, \mathbb{R}^3)$, and a.e. in Ω. Therefore, we have $\operatorname{div} \boldsymbol{B}_0 = 0, \operatorname{curl} \boldsymbol{B}_0 = \boldsymbol{0}$ in Ω, and $\boldsymbol{B}_{0,T} = \boldsymbol{0}$ on $\partial\Omega$, so $\boldsymbol{B}_0 \in \mathbb{K}^p_T(\Omega)$. From Lemma 3.3,

$$\int_\Omega |\boldsymbol{B}_0|^p dx = \int_\Omega |\boldsymbol{B}_0|^{p-2}\boldsymbol{B}_0 \cdot \boldsymbol{B}_0 dx = \lim_{j\to\infty} \int_\Omega |\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j \cdot \boldsymbol{B}_0 dx = 0.$$

Thus, we have $\boldsymbol{B}_0 = \boldsymbol{0}$. Hence, $\boldsymbol{B}_j \to \boldsymbol{0}$ strongly in $L^p(\Omega, \mathbb{R}^3)$. From (2.3), we see that

$$\|\boldsymbol{B}_j\|_{W^{1,p}(\Omega)} \le c_2(\Omega)(\|\boldsymbol{B}_j\|_{L^p(\Omega)} + \|\operatorname{curl} \boldsymbol{B}_j\|_{L^p(\Omega)} + \|\boldsymbol{B}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)}) \to 0$$

as $j \to \infty$. This contradicts $\|\boldsymbol{B}_j\|_{W^{1,p}(\Omega)} = 1$. □

Proposition 3.6 *Let* $\mathcal{H}_T \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ *and* $\boldsymbol{f} \in L^{p'}(\Omega, \mathbb{R}^3)$ *satisfying*

$$\operatorname{div} \boldsymbol{f} = 0 \textit{ in } \Omega \textit{ and } \int_\Omega \boldsymbol{f} \cdot \boldsymbol{z} dx = 0 \textit{ for all } \boldsymbol{z} \in \mathbb{K}^p_T(\Omega). \tag{3.8}$$

Then the minimizing problem

$$\inf_{\boldsymbol{B}\in B(\Omega,\mathcal{H}_T)} \left\{ \frac{1}{2}\int_\Omega S(x, |\operatorname{curl} \boldsymbol{B}|^2)dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{B} dx \right\}$$

is achieved and

$$R^p_t(\mathcal{H}_T, \boldsymbol{f}) = \inf_{\boldsymbol{B}\in B(\Omega,\mathcal{H}_T)} \left\{ \frac{1}{2}\int_\Omega S(x, |\operatorname{curl} \boldsymbol{B}|^2)dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{B} dx \right\}. \tag{3.9}$$

Proof By Lemma 2.1, we can see that

$$R^p_t(\mathcal{H}_T, \boldsymbol{f}) = \inf_{\boldsymbol{A}\in H^p_t(\Omega,\operatorname{curl},\operatorname{div} 0,\mathcal{H}_T)} \left\{ \frac{1}{2}\int_\Omega S(x, |\operatorname{curl} \boldsymbol{A}|^2)dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{A} dx \right\}.$$

Since $B(\Omega, \mathcal{H}_T) \subset H^p_t(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$, it is clear that

$$R^p_t(\mathcal{H}_T, \boldsymbol{f}) \le \inf_{\boldsymbol{B}\in B(\Omega,\mathcal{H}_T)} \left\{ \frac{1}{2}\int_\Omega S(x, |\operatorname{curl} \boldsymbol{B}|^2)dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{B} dx \right\}.$$

On the other hand, for any $\boldsymbol{A} \in H^p_t(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$, we can write $\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{u}$ where $\boldsymbol{B} \in B(\Omega, \mathcal{H}_T), \boldsymbol{u} \in \mathbb{K}^p_T(\Omega)$. Since $\operatorname{curl} \boldsymbol{A} = \operatorname{curl} \boldsymbol{B}$ and

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{A} dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{B} dx + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{B} dx,$$

we have

$$\frac{1}{2}\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{A}|^2)dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{A}dx = \frac{1}{2}\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{B}|^2)dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{B}dx$$
$$\geq \inf_{\boldsymbol{B}\in B(\Omega,\mathcal{H}_T)}\left\{\frac{1}{2}\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{B}|^2)dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{B}dx\right\}.$$

Thus, (3.9) holds. We show that the right-hand side of (3.9) has a minimizer. Let $\{\boldsymbol{B}_j\}\subset B(\Omega,\mathcal{H}_T)$ be a minimizing sequence. Then

$$\frac{1}{2}\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{B}_j|^2)dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{B}_j dx = R_t^p(\mathcal{H}_T,\boldsymbol{f}) + o(1) \text{ as } j\to\infty.$$

By (1.3), we have

$$\frac{2}{p}\lambda\int_\Omega |\mathrm{curl}\,\boldsymbol{B}_j|^p dx - \|\boldsymbol{f}\|_{L^{p'}(\Omega)}\|\boldsymbol{B}_j\|_{L^p(\Omega)} \leq R_t^p(\mathcal{H}_T,\boldsymbol{f}) + o(1).$$

Using Lemma 3.5, for any $\varepsilon>0$, there exists $C(\varepsilon)>0$ such that

$$\|\boldsymbol{f}\|_{L^{p'}(\Omega)}\|\boldsymbol{B}_j\|_{L^p(\Omega)} \leq \varepsilon\|\boldsymbol{B}_j\|^p_{L^p(\Omega)} + C(\varepsilon)\|\boldsymbol{f}\|^{p'}_{L^{p'}(\Omega)}$$
$$\leq C(\Omega)\varepsilon(\|\mathrm{curl}\,\boldsymbol{B}_j\|^p_{L^p(\Omega)} + \|\mathcal{H}_T\|^p_{W^{1-1/p,p}(\partial\Omega)}) + C(\varepsilon)\|\boldsymbol{f}\|^{p'}_{L^{p'}(\Omega)}.$$

If we choose $\varepsilon>0$ so that $C(\Omega)\varepsilon<2\lambda/p$, we can see that

$$\int_\Omega |\mathrm{curl}\,\boldsymbol{B}_j|^p dx \leq R_t^p(\mathcal{H}_T,\boldsymbol{f}) + C(\|\mathcal{H}_T\|^p_{W^{1-1/p,p}(\partial\Omega)} + \|\boldsymbol{f}\|^{p'}_{L^{p'}(\Omega)}) + o(1).$$

Then it follows from Lemma 3.5 that $\{\boldsymbol{B}_j\}$ is bounded in $W^{1,p}(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{B}_j\to\boldsymbol{B}_0$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$, strongly in $L^p(\Omega,\mathbb{R}^3)$, and a.e. in Ω. Therefore, we have $\mathrm{div}\,\boldsymbol{B}_0=0$ in Ω and $\boldsymbol{B}_{0,T}=\mathcal{H}_T$ on $\partial\Omega$. Since

$$\int_\Omega |\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j\cdot\boldsymbol{z}dx = 0 \text{ for all } \boldsymbol{z}\in\mathbb{K}_T^p(\Omega),$$

it follows from Lemma 3.3 that

$$\int_\Omega |\boldsymbol{B}_0|^{p-2}\boldsymbol{B}_0\cdot\boldsymbol{z}dx = 0 \text{ for all } \boldsymbol{z}\in\mathbb{K}_T^p(\Omega).$$

Therefore, $\boldsymbol{B}_0\in B(\Omega,\mathcal{H}_T)$. It suffices to prove that

$$\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{B}_0|^2)dx \leq \liminf_{j\to\infty}\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{B}_j|^2)dx.$$

In fact, we can choose a subsequence $\{\mathrm{curl}\,\boldsymbol{B}_{j_k}\}$ of $\{\mathrm{curl}\,\boldsymbol{B}_j\}$ so that

$$\lim_{k\to\infty}\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{B}_{j_k}|^2)dx = \liminf_{j\to\infty}\int_\Omega S(x,|\mathrm{curl}\,\boldsymbol{B}_j|^2)dx.$$

Since $\operatorname{curl} \boldsymbol{B}_{j_k} \to \operatorname{curl} \boldsymbol{B}_0$ weakly in $L^p(\Omega, \mathbb{R}^3)$, it follows from the Mazur theorem that there exist $\boldsymbol{g}_l \in L^p(\Omega, \mathbb{R}^3)$ such that $\boldsymbol{g}_l \in$ convex hull of $\{\operatorname{curl} \boldsymbol{B}_{j_k}; k \geq l\}$ and $\boldsymbol{g}_l \to \operatorname{curl} \boldsymbol{B}_0$ strongly in $L^p(\Omega, \mathbb{R}^3)$. Hence, we can choose a subsequence $\{\boldsymbol{g}_{l_m}\}$ of $\{\boldsymbol{g}_l\}$ so that $\boldsymbol{g}_{l_m} \to \operatorname{curl} \boldsymbol{B}_0$ strongly in $L^p(\Omega, \mathbb{R}^3)$ and a.e. in Ω. By the Fatou lemma, we have

$$\int_\Omega S(x, |\operatorname{curl} \boldsymbol{B}_0|^2)dx \leq \liminf_{m\to\infty} \int_\Omega S(x, |\boldsymbol{g}_{l_m}|^2)dx.$$

Since $S(x, t^2)$ is a convex function with respect to t, we have

$$\int_\Omega S(x, |\boldsymbol{g}_{l_m}|^2)dx \leq \sup \left\{ \int_\Omega S(x, |\operatorname{curl} \boldsymbol{B}_{j_k}|^2)dx; k \geq l_m \right\}.$$

Therefore, we have

$$\begin{aligned} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{B}_0|^2)dx &\leq \liminf_{m\to\infty} \int_\Omega S(x, |\boldsymbol{g}_{l_m}|^2)dx \\ &\leq \lim_{m\to\infty} \sup \left\{ \int_\Omega S(x, \operatorname{curl} \boldsymbol{B}_{j_k}|^2)dx; k \geq l_m \right\} \\ &= \lim_{k\to\infty} \int_\Omega S(x, \operatorname{curl} \boldsymbol{B}_{j_k}|^2)dx \\ &= \liminf_{j\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{B}_j|^2)dx. \end{aligned}$$

This completes the proof. □

Lemma 3.7 *Let* $\boldsymbol{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$ *be a minimizer of* $R_t^p(\mathcal{H}_T, \boldsymbol{f})$. *Then* $\boldsymbol{A}$ *is a weak solution of the following system:*

$$\begin{cases} \operatorname{curl} [S_t(x, |\operatorname{curl} \boldsymbol{A}|^2)\operatorname{curl} \boldsymbol{A}] = \boldsymbol{f}, \ \operatorname{div} \boldsymbol{A} = 0 & \text{in } \Omega, \\ \boldsymbol{A}_T = \mathcal{H}_T & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Proof If $\boldsymbol{A} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathcal{H}_T)$ is a minimizer of $R_t^p(\mathcal{H}_T, \boldsymbol{f})$, then we can see that for any $\boldsymbol{w} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathbf{0})$, we have

$$\left.\frac{d}{d\theta}\right|_{\theta=0} \left\{ \frac{1}{2} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{A} + \theta \operatorname{curl} \boldsymbol{w}|^2)dx - \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{A} + \theta \boldsymbol{w})dx \right\} = 0.$$

Thus, we have

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{A}|^2)\operatorname{curl} \boldsymbol{A} \cdot \operatorname{curl} \boldsymbol{w} dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} dx = 0 \tag{3.11}$$

for all $\boldsymbol{w} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \mathbf{0})$. For any $\boldsymbol{u} \in W_t^{1,p}(\Omega, \mathbb{R}^3, \mathbf{0})$, we choose a unique solution $\phi \in W^{2,p}(\Omega)$ of the Dirichlet problem

$$\begin{cases} \Delta\phi = \operatorname{div} \boldsymbol{u} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

and put $\boldsymbol{w} = \boldsymbol{u} - \nabla\phi$. Then $\operatorname{curl}\boldsymbol{w} = \operatorname{curl}\boldsymbol{u} \in L^p(\Omega,\mathbb{R}^3)$, $\operatorname{div}\boldsymbol{w} = \operatorname{div}\boldsymbol{u} - \Delta\phi = 0$ in Ω, and $\boldsymbol{w}_T = \boldsymbol{u}_T - (\nabla\phi)_T = \boldsymbol{u}_T = \boldsymbol{0}$. Thus, we have $\boldsymbol{w} \in H_t^p(\Omega, \operatorname{curl}, \operatorname{div} 0, \boldsymbol{0})$. Since

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}dx + \int_\Omega \boldsymbol{f}\cdot\nabla\phi dx \\ &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}dx - \int_\Omega (\operatorname{div}\boldsymbol{f})\phi dx + \int_{\partial\Omega}(\boldsymbol{f}\cdot\boldsymbol{\nu})\phi dS \\ &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}dx,\end{aligned}$$

it follows from (3.11) that

$$\int_\Omega S_t(x, |\operatorname{curl}\boldsymbol{A}|^2)\operatorname{curl}\boldsymbol{A}\cdot\operatorname{curl}\boldsymbol{u}dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx = 0$$

for all $\boldsymbol{u} \in W_t^{1,p}(\Omega,\mathbb{R}^3,\boldsymbol{0})$. Since $\mathcal{D}(\Omega,\mathbb{R}^3) \subset W_t^{1,p}(\Omega,\mathbb{R}^3,\boldsymbol{0})$, we can see that (3.10) holds. □

Remark 3.8 *The system* (3.10) *with* $S(x,t) = t^{p/2}$ *is the so-called* p*-curl system. When* Ω *is a bounded, simply connected domain in* $\mathbb{R}^3$ *without holes, and with* $C^{2+\alpha}$ *boundary for some* $\alpha \in (0,1)$*. If* $\mathcal{H}_T = \boldsymbol{0}$ *and* $\boldsymbol{f} \in C^\alpha(\overline{\Omega},\mathbb{R}^3)$ *satisfying* $\operatorname{div}\boldsymbol{f} = 0$ *in* Ω*, then Aramaki [4] showed that the weak solution* $\boldsymbol{A}$ *of the system* (3.10) *satisfies that* $\boldsymbol{A} \in C^{1+\beta}(\overline{\Omega},\mathbb{R}^3)$ *for some* $\beta \in (0,1)$ *and there exists a constant* C *depending only on* p,Ω *such that* $\|\boldsymbol{A}\|_{C^{1+\beta}(\overline{\Omega})} \le C$.

Lemma 3.9 *Let* $\mathcal{H}_T \in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$ *and* $\boldsymbol{f} \in L^{p'}(\Omega,\mathbb{R}^3)$ *satisfying* (3.8)*. If* $\boldsymbol{B}_0 \in B(\Omega,\mathcal{H}_T)$ *is a minimizer of* (3.9)*, then any minimizer* $\boldsymbol{A} \in H_T^p(\Omega,\operatorname{curl},\operatorname{div} 0,\mathcal{H}_T)$ *of* (2.1) *must have the form* $\boldsymbol{A} = \boldsymbol{B}_0 + \boldsymbol{u}$ *where* $\boldsymbol{u} \in \mathbb{K}_T^p(\Omega)$*. In particular, the minimizer of* (3.9) *is unique.*

Proof Since for any $\boldsymbol{u} \in \mathbb{K}_T^p(\Omega)$, we see that

$$\boldsymbol{B}_0 + \boldsymbol{u} \in H_t^p(\Omega,\operatorname{curl},\operatorname{div} 0,\mathcal{H}_T),$$

and

$$\int_\Omega |\operatorname{curl}(\boldsymbol{B}_0+\boldsymbol{u})|^p dx = \int_\Omega |\operatorname{curl}\boldsymbol{B}_0|^p dx$$

and

$$\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{B}_0+\boldsymbol{u})dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{B}_0 dx,$$

thus, $\boldsymbol{B}_0+\boldsymbol{u}$ is a minimizer of (2.1). On the other hand, for any minimizer $\boldsymbol{A} \in H_t^p(\Omega,\operatorname{curl},\operatorname{div} 0,\mathcal{H}_T)$ of (2.1), define $\boldsymbol{w} = \boldsymbol{A} - \boldsymbol{B}_0$. Then $\boldsymbol{w} \in H_t^p(\Omega,\operatorname{curl},\operatorname{div} 0,\boldsymbol{0})$. Since $\boldsymbol{A}$ and $\boldsymbol{B}_0$ are minimizers of $R_t^p(\mathcal{H}_T,\boldsymbol{f})$, it follows from Lemma 3.7 that

$$\begin{aligned}\int_\Omega S_t(x,|\operatorname{curl}\boldsymbol{A}|^2)\operatorname{curl}\boldsymbol{A}\cdot\operatorname{curl}\boldsymbol{w}dx &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}dx, \\ \int_\Omega S_t(x,|\operatorname{curl}\boldsymbol{B}_0|^2)\operatorname{curl}\boldsymbol{B}_0\cdot\operatorname{curl}\boldsymbol{w}dx &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}dx.\end{aligned}$$

Therefore,

$$\int_\Omega (S_t(x,|\operatorname{curl}\boldsymbol{A}|^2)\operatorname{curl}\boldsymbol{A} - S_t(x,|\operatorname{curl}\boldsymbol{B}_0|^2)\operatorname{curl}\boldsymbol{B}_0)\cdot(\operatorname{curl}\boldsymbol{A} - \operatorname{curl}\boldsymbol{B}_0)dx = 0.$$

By the structure condition (1.2), we have $\operatorname{curl}(\boldsymbol{A}-\boldsymbol{B}_0) = \boldsymbol{0}$ in Ω, so $\boldsymbol{A}-\boldsymbol{B}_0 \in \mathbb{K}_T^p(\Omega)$.

If $\boldsymbol{B} \in B(\Omega,\mathcal{H}_T) \subset H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathcal{H}_T)$ is a minimizer of (3.9), we can write $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{u}$ where $\boldsymbol{u} \in \mathbb{K}_T^p(\Omega)$. If follows from Lemma 3.2 that we see that $\boldsymbol{u} = \boldsymbol{0}$. Thus, the minimizer of (3.9) in $B(\Omega,\mathcal{H}_T)$ is unique. □

For $\mathcal{H}_T \in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$ and $\boldsymbol{f} \in L^{p'}(\Omega,\mathbb{R}^3)$ satisfying (3.8), let $\boldsymbol{A} = \boldsymbol{A}(\mathcal{H}_T,\boldsymbol{f}) \in H_t^p(\Omega,\operatorname{curl},\operatorname{div}0,\mathcal{H}_T)$ be a minimizer of (2.1). Then there exist uniquely $\boldsymbol{B}_0 = \boldsymbol{B}_0(\mathcal{H}_T,\boldsymbol{f}) \in B(\Omega,\mathcal{H}_T)$, which is a minimizer of (3.9), and $\boldsymbol{u} = \boldsymbol{u}(\mathcal{H}_T,\boldsymbol{f}) \in \mathbb{K}_T^p(\Omega)$, such that

$$\boldsymbol{A}(\mathcal{H}_T,\boldsymbol{f}) = \boldsymbol{B}_0(\mathcal{H}_T,\boldsymbol{f}) + \boldsymbol{u}(\mathcal{H}_T,\boldsymbol{f}). \tag{3.12}$$

Proposition 3.10 *There exists a constant $c = c(\Omega)$ independent of $\mathcal{H}_T$ and $\boldsymbol{f}$ satisfying the above such that*

$$\|\boldsymbol{B}_0(\mathcal{H}_T,\boldsymbol{f})\|_{W^{1,p}(\Omega)} \le c(\|\mathcal{H}_T\|_{W^{1-1/p,p}(\partial\Omega)} + \|\boldsymbol{f}\|_{L^{p'}(\Omega)}).$$

Proof Assume that the conclusion is false. Then there exists a sequence $\{\mathcal{H}_{j,T}\} \subset W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$ and $\boldsymbol{f}_j \in L^{p'}(\Omega,\mathbb{R}^3)$ satisfying (3.8) such that $\|\boldsymbol{B}_0(\mathcal{H}_{j,T},\boldsymbol{f}_j)\|_{W^{1,p}(\Omega)} = 1$ and

$$\|\mathcal{H}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)} \to 0 \text{ and } \|\boldsymbol{f}_j\|_{L^{p'}(\Omega)} \to 0 \text{ as } j \to \infty.$$

For brevity of notation, we write $\boldsymbol{B}_j = \boldsymbol{B}_0(\mathcal{H}_{j,T},\boldsymbol{f}_j)$. Passing to a subsequence, we may assume that $\boldsymbol{B}_j \to \boldsymbol{B}$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$, strongly in $L^p(\Omega,\mathbb{R}^3)$, and a.e. in Ω. Thus, $\operatorname{curl}\boldsymbol{B} \in L^p(\Omega,\mathbb{R}^3)$, $\operatorname{div}\boldsymbol{B} = 0$ in Ω, and $\boldsymbol{B}_T = \boldsymbol{0}$ on $\partial\Omega$. Since $\boldsymbol{B}_j$ satisfies

$$\int_\Omega |\boldsymbol{B}_j|^{p-2}\boldsymbol{B}_j\cdot\boldsymbol{z}dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}_T^p(\Omega)$$

and $\boldsymbol{B}_j \to \boldsymbol{B}$ strongly in $L^p(\Omega,\mathbb{R}^3)$ and a.e. in Ω, it follows from Lemma 3.3 that

$$\int_\Omega |\boldsymbol{B}|^{p-2}\boldsymbol{B}\cdot\boldsymbol{z}dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}_T^p(\Omega). \tag{3.13}$$

Hence, we have $\boldsymbol{B} \in B(\Omega,\boldsymbol{0})$. On the other hand, $\boldsymbol{B}_j$ is a weak solution of

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{B}_j|^2)\operatorname{curl}\boldsymbol{B}_j] = \boldsymbol{f}_j,\ \operatorname{div}\boldsymbol{B}_j = 0 & \text{in } \Omega, \\ \boldsymbol{B}_{j,T} = \mathcal{H}_{j,T} & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Since $S_t(x,|\operatorname{curl}\boldsymbol{B}_j|^2)\operatorname{curl}\boldsymbol{B}_j \in L^{p'}(\Omega,\mathbb{R}^3)$ and

$$\operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{B}_j|^2)\operatorname{curl}\boldsymbol{B}_j] = \boldsymbol{f}_j \in L^{p'}(\Omega,\mathbb{R}^3),$$

we see that

$$\boldsymbol{\nu} \times S_t(x, |\mathrm{curl}\,\boldsymbol{B}_j|^2)\mathrm{curl}\,\boldsymbol{B}_j\big|_{\partial\Omega} \in W^{-1/p',p'}(\partial\Omega, \mathbb{R}^3)$$

(cf. [2]). Since $\mathcal{H}_{j,T} \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3) = W^{1/p',p}(\partial\Omega, \mathbb{R}^3)$, it follows from (3.13) that

$$\begin{aligned}\int_\Omega \boldsymbol{f}_j \cdot \boldsymbol{B}_j dx &= \int_\Omega \mathrm{curl}\,[S_t(x, |\mathrm{curl}\,\boldsymbol{B}_j|^2)\mathrm{curl}\,\boldsymbol{B}_j] \cdot \boldsymbol{B}_j dx \\ &= \int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{B}_j|^2)\mathrm{curl}\,\boldsymbol{B}_j \cdot \mathrm{curl}\,\boldsymbol{B}_j dx \\ &\quad + \int_{\partial\Omega} \langle \boldsymbol{B}_{j,T}, \boldsymbol{\nu} \times S_t(x, |\mathrm{curl}\,\boldsymbol{B}_j|^2)\mathrm{curl}\,\boldsymbol{B}_j\rangle dS \end{aligned} \tag{3.15}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between the spaces $W^{1/p',p}(\partial\Omega, \mathbb{R}^3)$ and $W^{-1/p',p'}(\partial\Omega, \mathbb{R}^3)$. Here we note that for any $\boldsymbol{B} \in L^{p'}(\Omega, \mathbb{R}^3)$ satisfying $\mathrm{curl}\,\boldsymbol{B} \in L^{p'}(\Omega, \mathbb{R}^3)$, we have

$$\|\boldsymbol{\nu} \times \boldsymbol{B}\|_{W^{-1/p',p'}(\partial\Omega)} \le C(\|\boldsymbol{B}\|_{L^{p'}(\Omega)} + \|\mathrm{curl}\,\boldsymbol{B}\|_{L^{p'}(\Omega)}).$$

See, for example, [2, p. 45]. Therefore, we have

$$\begin{aligned}\left|\int_{\partial\Omega} \langle \mathcal{H}_{j,T}, \boldsymbol{\nu} \times S_t(x, |\mathrm{curl}\,\boldsymbol{B}_j|^2)\mathrm{curl}\,\boldsymbol{B}_j\rangle dS\right| &\le C\|\mathcal{H}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)}(\|S_t(x, |\mathrm{curl}\,\boldsymbol{B}_j|^2)\mathrm{curl}\,\boldsymbol{B}_j\|_{L^{p'}(\Omega)} + \|\boldsymbol{f}_j\|_{L^{p'}(\Omega)}) \\ &\le C\|\mathcal{H}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)}\left\{\left(\int_\Omega (\Lambda|\mathrm{curl}\,\boldsymbol{B}_j|^{p-1})^{p'} dx\right)^{1/p'} + \|\boldsymbol{f}_j\|_{L^{p'}(\Omega)}\right\} \\ &\le C_1\|\mathcal{H}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)}(\|\mathrm{curl}\,\boldsymbol{B}_j\|_{L^p(\Omega)}^{p/p'} + \|\boldsymbol{f}_j\|_{L^{p'}(\Omega)}).\end{aligned}$$

Since $\mathrm{curl}\,\boldsymbol{B}_j \to \mathrm{curl}\,\boldsymbol{B}$ weakly in $L^p(\Omega, \mathbb{R}^3)$, we see that $\|\mathrm{curl}\,\boldsymbol{B}_j\|_{L^p(\Omega)}$ is bounded. Since $\|\mathcal{H}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)} \to 0$, we have

$$\int_{\partial\Omega} \langle \boldsymbol{\nu} \times \mathcal{H}_{j,T}, S_t(x, |\mathrm{curl}\,\boldsymbol{B}_j|^2)\mathrm{curl}\,\boldsymbol{B}_j\rangle dS \to 0$$

as $j \to \infty$. By Lemma 3.5, we have

$$\|\boldsymbol{B}_j\|_{L^p(\Omega)} \le C(\Omega)(\|\mathrm{curl}\,\boldsymbol{B}_j\|_{L^p(\Omega)} + \|\mathcal{H}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)}) \le C.$$

Since $\boldsymbol{f}_j \to 0$ in $L^{p'}(\Omega, \mathbb{R}^3)$, we see that

$$\int_\Omega \boldsymbol{f}_j \cdot \boldsymbol{B}_j dx \to 0 \text{ as } j \to \infty.$$

Since $\mathrm{curl}\,\boldsymbol{B}_j \to \mathrm{curl}\,\boldsymbol{B}$ weakly in $L^p(\Omega, \mathbb{R}^3)$, using (3.15),

$$\begin{aligned}\int_\Omega |\mathrm{curl}\,\boldsymbol{B}|^2 dx &\le \liminf_{j\to\infty} \int_\Omega |\mathrm{curl}\,\boldsymbol{B}_j|^p dx \\ &\le \limsup_{j\to\infty} \int_\Omega |\mathrm{curl}\,\boldsymbol{B}_j|^p dx \\ &\le \frac{1}{\lambda} \limsup_{j\to\infty} \int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{B}_j|^2)|\mathrm{curl}\,\boldsymbol{B}_j|^2 dx = 0,\end{aligned} \tag{3.16}$$

thus we can see that $\operatorname{curl}\boldsymbol{B}=\mathbf{0}$, so $\boldsymbol{B}\in\mathbb{K}_T^p(\Omega)$. From (3.13) with $\boldsymbol{z}=\boldsymbol{B}$, we have

$$0=\int_\Omega|\boldsymbol{B}|^{p-2}\boldsymbol{B}\cdot\boldsymbol{B}dx=\int_\Omega|\boldsymbol{B}|^p dx.$$

Therefore, $\boldsymbol{B}=\mathbf{0}$ in Ω, so $\boldsymbol{B}_j\to\mathbf{0}$ weakly in $W^{1,p}(\Omega,\mathbb{R}^3)$ and strongly in $L^p(\Omega,\mathbb{R}^3)$. From (3.16), we can see that $\|\operatorname{curl}\boldsymbol{B}_j\|_{L^p(\Omega)}\to 0$. By (2.3),

$$\|\boldsymbol{B}_j\|_{W^{1,p}(\Omega)}\le c_2(\Omega)(\|\boldsymbol{B}_j\|_{L^p(\Omega)}+\|\operatorname{curl}\boldsymbol{B}_j\|_{L^p(\Omega)}+\|\mathcal{H}_{j,T}\|_{W^{1-1/p,p}(\partial\Omega)})\to 0$$

as $j\to\infty$. This contradicts $\|\boldsymbol{B}_j\|_{W^{1,p}(\Omega)}=1$. □

Proof of Theorem 1.1 The proof of Theorem 1.1 follows from Lemma 2.1, Proposition 3.6, and Proposition 3.10.

Remark 3.11 *Instead of minimizing*

$$\frac{1}{2}\int_\Omega S(t,|\operatorname{curl}\boldsymbol{u}|^2)dx-\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}dx,$$

it is also interesting to minimize

$$\frac{1}{2}\int_\Omega S(x,|\operatorname{div}\boldsymbol{u}|^2)dx-\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}dx.$$

This problem is related to the mathematical theory of liquid crystals. For $p=2$ and $S(x,t)=t$ and $\boldsymbol{f}=\mathbf{0}$, see Aramaki [3].

4. Continuous dependence on the data of minimizers

In this section, in addition to (1.1) we assume that there exists a constant $c>0$ such that

$$(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})\ge\begin{cases}c|\boldsymbol{a}-\boldsymbol{b}|^p\\ \qquad\text{if } p\ge 2,\\ c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^2\\ \qquad\text{if } 1<p<2\end{cases}\tag{4.1}$$

for all $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$ and Ω has no holes. We note that (4.1) implies (1.2).

Then we have the following.

Theorem 4.1 *Let $\boldsymbol{B}_0(\mathcal{H}_T,\boldsymbol{f})\in B(\Omega,\mathcal{H}_T)$ and $\boldsymbol{B}_0(\mathcal{H}'_T,\boldsymbol{f}')\in B(\Omega,\mathcal{H}'_T)$ be minimizers of $R_t^p(\mathcal{H}_T,\boldsymbol{f})$ and $R_t^p(\mathcal{H}'_T,\boldsymbol{f}')$ in (3.9), respectively. Then there exists a constant*

$$C=C(p,\Omega,\|\boldsymbol{f}\|_{L^{p'}(\Omega)},\|\boldsymbol{f}'\|_{L^{p'}(\Omega)},\|\mathcal{H}_T\|_{W^{1-1/p,p}(\partial\Omega)},\|\mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)})$$

such that

$$\begin{aligned}&\|\boldsymbol{B}_0(\mathcal{H}_T,\boldsymbol{f})-\boldsymbol{B}_0(\mathcal{H}'_T,\boldsymbol{f}')\|_{W^{1,p}(\Omega)}^{p\vee 2}\\ &\le C\left(\|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)}+\max\{\|\mathcal{H}_T-\mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)},\|\mathcal{H}_T-\mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)}^{p\vee 2}\}\right),\end{aligned}$$

where $p\vee 2=\max\{p,2\}$.

Proof For brevity of notations, we write $\boldsymbol{B} = \boldsymbol{B}_0(\mathcal{H}_T, \boldsymbol{f})$ and $\boldsymbol{B}' = \boldsymbol{B}_0(\mathcal{H}'_T, \boldsymbol{f}')$. Then $\boldsymbol{B}$ and $\boldsymbol{B}'$ are weak solutions of the following equations:

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{B}|^2)\operatorname{curl}\boldsymbol{B}] = \boldsymbol{f}, \ \operatorname{div}\boldsymbol{B} = 0 & \text{in } \Omega, \\ \boldsymbol{B}_T = \mathcal{H}_T & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{B}'|^2)\operatorname{curl}\boldsymbol{B}'] = \boldsymbol{f}', \ \operatorname{div}\boldsymbol{B}' = 0 & \text{in } \Omega, \\ \boldsymbol{B}'_T = \mathcal{H}'_T & \text{on } \partial\Omega. \end{cases}$$

Then we have

$$\begin{aligned} \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{B}-\boldsymbol{B}')dx &= \int_\Omega S_t(x, |\operatorname{curl}\boldsymbol{B}|^2)\operatorname{curl}\boldsymbol{B}\cdot\operatorname{curl}(\boldsymbol{B}-\boldsymbol{B}')dx \\ &\quad + \int_{\partial\Omega}\langle \mathcal{H}_T - \mathcal{H}'_T, \boldsymbol{\nu}\times S_t(x, |\operatorname{curl}\boldsymbol{B}|^2)\operatorname{curl}\boldsymbol{B}\rangle dS, \end{aligned}$$

and

$$\begin{aligned} \int_\Omega \boldsymbol{f}'\cdot(\boldsymbol{B}-\boldsymbol{B}')dx &= \int_\Omega S_t(x, |\operatorname{curl}\boldsymbol{B}'|^2)\operatorname{curl}\boldsymbol{B}'\cdot\operatorname{curl}(\boldsymbol{B}-\boldsymbol{B}')dx \\ &\quad + \int_{\partial\Omega}\langle \mathcal{H}_T - \mathcal{H}'_T, \boldsymbol{\nu}\times S_t(x, |\operatorname{curl}\boldsymbol{B}'|^2)\operatorname{curl}\boldsymbol{B}'\rangle dS. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\int_\Omega (S_t(x, |\operatorname{curl}\boldsymbol{B}|^2)\operatorname{curl}\boldsymbol{B} - S_t(x, |\operatorname{curl}\boldsymbol{B}'|^2)\operatorname{curl}\boldsymbol{B}')\cdot(\operatorname{curl}\boldsymbol{B} - \operatorname{curl}\boldsymbol{B}')dx \\ &= \int_\Omega (\boldsymbol{f}-\boldsymbol{f}')\cdot(\boldsymbol{B}-\boldsymbol{B}')dx \\ &\quad - \int_{\partial\Omega}\langle \mathcal{H}_T - \mathcal{H}'_T, \boldsymbol{\nu}\times\big(S_t(x, |\operatorname{curl}\boldsymbol{B}|^2)\operatorname{curl}\boldsymbol{B} \\ &\quad - S_t(x, |\operatorname{curl}\boldsymbol{B}'|^2)\operatorname{curl}\boldsymbol{B}'\big)\rangle dS \\ &\le \|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)}\|\boldsymbol{B}-\boldsymbol{B}'\|_{L^p(\Omega)} \\ &\quad + C\|\mathcal{H}_T - \mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)}(\|S_t(x, |\operatorname{curl}\boldsymbol{B}|^2)\operatorname{curl}\boldsymbol{B} \\ &\quad - S_t(x, |\operatorname{curl}\boldsymbol{B}'|^2)\operatorname{curl}\boldsymbol{B}'\|_{L^{p'}(\Omega)} + \|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)}) \\ &\le \|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)}(\|\boldsymbol{B}\|_{L^p(\Omega)} + \|\boldsymbol{B}'\|_{L^p(\Omega)}) \\ &\quad + C\|\mathcal{H}_T - \mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)}(\|\operatorname{curl}\boldsymbol{B}\|^{p/p'}_{L^p(\Omega)} + \|\operatorname{curl}\boldsymbol{B}'\|^{p/p'}_{L^{p'}(\Omega)} \\ &\quad + \|\boldsymbol{f}\|_{L^{p'}(\Omega)} + \|\boldsymbol{f}'\|_{L^{p'}(\Omega)}) \\ &\le C_1(\|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)} + \|\mathcal{H}_T - \mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)}). \end{aligned}$$

When $p \ge 2$, by the monotonicity (4.1), we have

$$\|\operatorname{curl}(\boldsymbol{B}-\boldsymbol{B}')\|^p_{L^p(\Omega)} \le C(\|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)} + \|\mathcal{H}_T - \mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)}).$$

Since Ω has no holes, it follows from Lemma 2.2 (ii) and the remark that

$$\begin{aligned}\|\boldsymbol{B}-\boldsymbol{B}'\|^p_{W^{1,p}(\Omega)} \leq& C(\Omega)(\|\mathrm{curl}\,(\boldsymbol{B}-\boldsymbol{B}')\|^p_{L^p(\Omega)}+\|\boldsymbol{B}_T-\boldsymbol{B}'_T\|_{W^{1-1/p,p}(\partial\Omega)})\\ \leq& C\big(\|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)}+\max\{\|\mathcal{H}_T-\mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)},\|\mathcal{H}_T-\mathcal{H}'_T\|^p_{W^{1-1/p,p}(\partial\Omega)}\}\big).\end{aligned}$$

When $1<p<2$, by the monotonicity (4.1), we have

$$\int_\Omega(|\mathrm{curl}\,\boldsymbol{B}|+|\mathrm{curl}\,\boldsymbol{B}'|)^{p-2}|\mathrm{curl}\,(\boldsymbol{B}-\boldsymbol{B}')|^2dx\leq C(\|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)}+\|\mathcal{H}_T-\mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)}).$$

If we use the reverse Hölder inequality (cf. Sobolev [14, p. 8]) with $s=p/2, s'=p/(p-2)$, we have

$$\int_\Omega(|\mathrm{curl}\,\boldsymbol{B}|+|\mathrm{curl}\,\boldsymbol{B}'|)^{p-2}|\mathrm{curl}\,(\boldsymbol{B}-\boldsymbol{B}')|^2dx\geq\left(\int_\Omega(|\mathrm{curl}\,\boldsymbol{B}|+|\mathrm{curl}\,\boldsymbol{B}'|)^p dx\right)^{(p-2)/p}\|\mathrm{curl}\,\boldsymbol{B}-\mathrm{curl}\,\boldsymbol{B}'\|^2_{L^p(\Omega)}.$$

Here we have

$$\left(\int_\Omega(|\mathrm{curl}\,\boldsymbol{B}|+|\mathrm{curl}\,\boldsymbol{B}'|)^p dx\right)^{(2-p)/p}\leq D_1(\|\mathrm{curl}\,\boldsymbol{B}\|^p_{L^p(\Omega)}+\|\mathrm{curl}\,\boldsymbol{B}'\|^p_{L^p(\Omega)})^{(2-p)/p}\leq D_2.$$

Hence,

$$\begin{aligned}\|\mathrm{curl}\,(\boldsymbol{B}-\boldsymbol{B}')\|^2_{L^2(\Omega)}&\leq D_3\int_\Omega(|\mathrm{curl}\,\boldsymbol{B}|+|\mathrm{curl}\,\boldsymbol{B}'|)^{p-2}|\mathrm{curl}\,(\boldsymbol{B}-\boldsymbol{B}')|^2dx\\ &\leq C(\|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)}+\|\mathcal{H}_T-\mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)}).\end{aligned}$$

Since Ω has no holes, it follows from Lemma 2.2 (ii) and the remark that

$$\begin{aligned}\|\boldsymbol{B}-\boldsymbol{B}'\|^2_{W^{1,p}(\Omega)} \leq& C(\Omega)(\|\mathrm{curl}\,(\boldsymbol{B}-\boldsymbol{B}')\|^2_{L^p(\Omega)}+\|\boldsymbol{B}_T-\boldsymbol{B}'_T\|^2_{W^{1-1/p,p}(\partial\Omega)})\\ \leq& C\big(\|\boldsymbol{f}-\boldsymbol{f}'\|_{L^{p'}(\Omega)}+\max\{\|\mathcal{H}_T-\mathcal{H}'_T\|_{W^{1-1/p,p}(\partial\Omega)},\|\mathcal{H}_T-\mathcal{H}'_T\|^2_{W^{1-1/p,p}(\partial\Omega)}\}\big).\end{aligned}$$

This completes the proof. □

Corollary 4.2 *The minimizer* $\boldsymbol{B}_0$ *of* (3.9) *is continuous in* $W^{1,p}(\Omega,\mathbb{R}^3)$ *with respect to the date* $\boldsymbol{f}\in L^{p'}(\Omega,\mathbb{R}^3)$ *and* $\mathcal{H}_T\in W^{1-1/p,p}(\partial\Omega)$*; that is to say, if* $\boldsymbol{f}_j\to\boldsymbol{f}$ *in* $L^{p'}(\Omega,\mathbb{R}^3)$ *and* $\mathcal{H}_{j,T}\to\mathcal{H}_T$ *in* $W^{1-1/p,p}(\partial\Omega)$*, then*

$$\boldsymbol{B}_0(\mathcal{H}_{j,T},\boldsymbol{f}_j)\to\boldsymbol{B}_0(\mathcal{H}_T,\boldsymbol{f})\ \mathit{in}\ W^{1,p}(\Omega,\mathbb{R}^3).$$

Acknowledgment

The author would like to thank the anonymous referee(s) for indicating some errors and giving some advice on a previous version of this article.

References

[1] Amrouche C, Seloula NH. L^p-theory for vector potentials and Sobolev's inequalities for vector fields. C R Acad Sci Paris Ser 1 2011; 349: 529-534.

[2] Amrouche C, Seloula NH. L^p-theory for vector potentials and Sobolev's inequalities for vector fields: application to the Stokes equations with pressure boundary conditions. Math Models Methods Appl Sci 2013; 23: 37092.

[3] Aramaki J. Minimizing divergence of vector fields in a multi-connected domain. Far East J Math Sci 2010; 38: 65-84.

[4] Aramaki J. Regularity of weak solutions for degenerate quasilinear elliptic equations involving curl. J Math Anal Appl 2015; 425: 872-892.

[5] Bates P, Pan XB. Nucleation of instability of the Meissner state of 3-dimensional superconductors. Commun Math Phys 2007; 276: 571-609.

[6] Dautray R, Lions JL. Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 3. New York, NY, USA: Springer-Verlag, 1990.

[7] DiBenedetto E. Degenerate Parabolic Equations. Berlin, Germany: Springer-Science+Business Media, 1993.

[8] Evans LC. Weak Convergence Methods for Nonlinear Partial Differential Equations. Regional Conference Series in Mathematics Number 74. Providence, RI, USA: AMS, 1988.

[9] Fujita H, Kuroda N, Ito S. Functional Analysis. Tokyo, Japan: Iwanami Shoten, 1990 (in Japanese).

[10] Giraut V, Raviart PA. Finite Element Methods for Navier-Stokes Equations. Berlin, Germany: Springer-Verlag, 1986.

[11] Miranda F, Rodrigues JF, Santos L. On a p-curl system arising in electromagnetism. Discrete Cont Dynamical System Ser S 2012; 5: 605-629.

[12] Pan XB. Minimizing curl in a multiconnected domain. J Math Phys 2009; 50: 1-10.

[13] Pan XB, Qi YW. Asymptotics of minimizers of variational problem involving curl functional. J Math Phys 2000; 41: 5033.

[14] Sobolev S. Applications of Functional Analysis in Mathematical Physics. Translations of Mathematical Monographs 7. Providence, RI, USA: American Mathematical Society, 1963.