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Nearly Kähler and nearly Kenmotsu manifolds

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Abstract: We study the class of strict nearly Kenmotsu manifolds and prove that there is no Einstein manifold or locally symmetric or locally ϕ-symmetric in this class of manifolds. We describe strict nearly Kenmotsu manifolds in low dimensions. Finally, we obtain a relation between the curvature of nearly Kenmotsu manifolds and nearly Kähler manifolds.

Key words: Almost Hermitian manifold, Kähler manifold, Kenmotsu manifold, nearly Kähler manifold, nearly Kenmotsu manifold

1. Introduction

In the 1940s, Ehresmann and Hopf introduced almost complex manifolds, which are even-dimensional manifolds furnished with a smooth linear complex structure on each tangent space. Almost complex manifolds are closely related to symplectic manifolds and have many applications in mathematics and physics [2, 16]. On the other hand, in odd dimensions, almost contact manifolds were introduced by Boothby and Wang in the 1950s. Lie used contact transformations in studying differential equations [6].

It is well known that almost complex structures are closely related to almost contacts ones. In [11], Kenmotsu showed that there exists (locally) a correspondence between the class of Kenmotsu manifolds and that of Kähler manifolds, which is one of the most important classes of almost Hermitian manifolds that appear naturally in Gray–Hervella classification [9]. These manifolds are closely related to Killing spinors, weak holonomy, and string theory [5]. In [12], the authors established a correspondence between nearly Kähler manifolds and nearly Kenmotsu manifolds. Moreover, they gave the first proper nearly Kenmotsu manifold examples and they proved that there exists proper nearly Kenmotsu manifolds for dimensions greater than 5. Using this correspondence, we prove some nonexistence theorems stating that there is no strict nearly Kenmotsu manifold among Einstein or locally symmetric or locally ϕ-symmetric manifolds (see Theorem 3.1 and Proposition 4.2). Then we define nearly Kenmotsu manifolds of constant type and prove that all 7-dimensional nearly Kenmotsu manifolds are of constant type. Using this notion, we give a description of 9- and 11-dimensional nearly Kenmotsu manifolds (see Propositions 5.3 and 5.4). Finally, we get a result on the lower bound of sectional curvature of nearly Kenmotsu manifolds (see Proposition 5.5).

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2. Preliminaries

An almost Hermitian manifold (F^{2n}, J, g) is said to be a nearly Kähler manifold if the $(2,1)$-tensor ∇J is totally skew-symmetric, i.e. for any vector fields X, Y, and Z on F the tensor $g((\nabla_X J)Y, Z)$ is skew-symmetric or equivalently the following holds:

$$(\nabla_X J)X = 0, \tag{1}$$

where ∇ is the Riemannian connection of g. Moreover, if F satisfies

$$(\nabla_X J)Y = 0, \tag{2}$$

then it is called a Kähler manifold [7]. A nearly Kähler manifold that is not a Kähler manifold is called a strict nearly Kähler manifold.

We recall some important identities holding in every $(2n+1)$-dimensional almost contact metric manifold (M, ϕ, η, ξ, g):

$$\phi\xi = 0, \;\; \eta(\xi) = 1, \;\; \phi^2 X = -X + \eta(X)\xi, \;\; \eta \circ \phi = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; g(X, \xi) = \eta(X), \tag{4}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{5}$$

where ϕ is a $(1,1)$-tensor field, η is a 1-form, ξ is a vector field called the Reeb vector field, and g is a Riemannian metric on manifold M (for more details, see [2]). An almost contact metric manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is called a nearly Kenmotsu manifold by Shukla [17] if the following relation holds:

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y, \tag{6}$$

where ∇ is the Levi-Civita connection of g. Moreover, if we have

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{7}$$

then it is called a Kenmotsu manifold. A nearly Kenmotsu manifold that is not a Kenmotsu manifold is called a strict nearly Kenmotsu manifold [14]. Kenmotsu manifolds have been studied by Jun et al. [10], De and Pathak [4], and others.

We recall that ϕ is constant along integral curves of the Reeb vector field ξ and give a formula for $\nabla\xi$.

Lemma 2.1 *[12] Let M be a (strict) nearly Kenmotsu manifold. Then the following relations hold:*

$$\nabla_\xi \phi = 0, \tag{8}$$

$$\nabla\xi = I - \eta \otimes \xi. \tag{9}$$

Suppose that (B, g_B) and (F, g_F) are Riemannian manifolds, and let $f > 0$ be a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with the metric tensor

$$g = \pi^*(g_B) + \sigma^*(g_F), \tag{10}$$

where π and σ are the projection of $B \times F$ onto B and F, respectively (for more information about warped products, see [15]).

In [12], Küpeli Erken et al. proved that a warped product of a (strict) nearly Kähler manifold and real line gives rise to a (strict) nearly Kenmotsu manifold. Moreover, the converse is true in the following sense.

Proposition 2.1 *[12] Let M be a (strict) nearly Kenmotsu manifold. Then M is locally identified with a warped product space $(-\varepsilon, \varepsilon) \times_f F$ where $(-\varepsilon, \varepsilon)$ is an open interval, $f(t) = ce^t$, and F is a (strict) nearly Kählerian manifold.*

Example 1 *Let S^6 be the 6-dimensional sphere with its canonical nearly Kähler structure. Then the warped product $M = \mathbb{R} \times_f S^6$ is a strict nearly Kenmotsu manifold.*

Remark 2.1 *In [12], the authors proved that every nearly Kenmotsu manifold of dimensions 3 and 5 is actually a Kenmotsu manifold.*

3. Nonexistence theorems

Using Proposition 2.1, we prove the following nonexistence theorems stating that there is no strict nearly Kenmotsu manifold among Einstein or locally symmetric manifolds.

Theorem 3.1 *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a strict nearly Kenmotsu manifold. Then (M, g) is not Einstein manifold.*

Proof M is locally isometric to a warped product $B \times_f F$, where F is a strict nearly Kähler manifold. Taking into account (9) and using a standard fact, given in [1], about the Ricci tensor of a warped product manifold, we have

$$\begin{aligned} Ric(\xi, \xi) &= Ric_B(\xi, \xi) - (n-1), \\ Ric(X, \xi) &= 0, \\ Ric(X, Y) &= Ric_F(X, Y) - (n-1)g(X, Y), \end{aligned} \tag{11}$$

where X and Y are vector fields with $\eta(X) = \eta(Y) = 0$ and $n = 2m$ [1]. Suppose that g is an Einstein metric. From the first part of (11) and flatness of B, one can get $Ric = -(n-1)g$. Then the third part of (11) implies that F is Ricci-flat. On the other hand, it follows from Theorem 1.1 and Lemma 2.1 of [13] that on strictly nearly Kähler manifolds the Ricci operator has a positive eigenvalue, which is in contradiction to Ricci-flatness of F. Hence, g is not an Einstein metric. This completes the proof. □

Theorem 3.2 *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a strict nearly Kenmotsu manifold. Then (M, g) is not locally symmetric.*

Proof With the notations of Theorem 3.1, it is easy to see that the following relation holds:

$$(\nabla_X R)_{VW} U = g({}^F R_{VW} U, X)\xi + (\nabla'_X {}^F R)_{VW} U, \tag{12}$$

where ∇' is the induced connection from ∇ to F. Now, suppose that g is locally symmetric, i.e. $\nabla R = 0$. From (12), we get

$$g({}^F R_{VW} U, X) g(\xi, \xi) + g((\nabla'_X {}^F R)_{VW} U, \xi) = 0. \tag{13}$$

Taking into account $g((\nabla'_X {}^F R)_{VW} U, \xi) = 0$ and $g(\xi, \xi) = 1$, we get

$$g({}^F R_{VW} U, X) = 0, \tag{14}$$

and consequently, F is flat, but a nearly Kähler manifold with flat curvature is Kähler [3] and this is a contradiction with the strict assumption. Hence, g is not locally symmetric. This completes the proof. □

In [13], Nagy showed that every nearly Kähler manifold can be decomposed as a product of a Kähler manifold and a strict nearly Kähler manifold. Therefore, we have the following.

Proposition 3.1 *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a nearly Kenmotsu manifold. Then the universal covering of M is locally decomposed as a warped product of a strict nearly Kenmotsu manifold and a Kähler manifold or decomposed to a warped product of a strict nearly Kähler manifold and a normal contact manifold.*

4. Locally ϕ-symmetric nearly Kenmotsu manifolds

For a nearly Kenmotsu manifold $(M^{2m+1}, \phi, \xi, \eta, g)$, we define the tensor field T as follows:

$$T(X,Y) := (\nabla_X \phi)(Y) + \eta(X)\phi(Y). \tag{15}$$

Using (6) and the relation $g(\phi(X), Y) + g(X, \phi(Y)) = 0$, we conclude that T is totally antisymmetric, i.e. $g(T(X,Y), Z)$ is antisymmetric with respect to its three arguments. Moreover, using (9) yields $T(\xi, X) = \phi(X)$ and

$$T(X, \phi Y) = -\phi(T(X,Y)) + g(X,Y)\xi + \eta(Y)X - 2\eta(X)Y, \tag{16}$$

$$\eta(T(X,Y)) = g(\phi X, Y). \tag{17}$$

Let us define a new connection $\bar{\nabla}$ as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{2} T(X, \phi(Y)). \tag{18}$$

Then (16) and (17) imply that

$$-2(\bar{\nabla}_X\, g)(Y,Z) = g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2g(Z,Y)\eta(Z) \tag{19}$$

and

$$(\bar{\nabla}_X \phi)Y = -\eta(Y)\phi(X) + \frac{1}{2} g(\phi X, Y)\xi. \tag{20}$$

Now, suppose X and Y are tangent to the fiber F. Then we have

$$\bar{\nabla}_X Y = \frac{1}{2}(\nabla_X Y - \phi \nabla_X \phi Y). \tag{21}$$

Let ∇' be the Levi-Civita connection of F induced from ∇. Then (21) can be written as follows:

$$\bar{\nabla}_X Y = \frac{1}{2}\big\{\nabla'_X Y - \phi \nabla'_X \phi Y + I\!I(X,Y) - \phi(I\!I(X, \phi(Y))\big\}, \tag{22}$$

where $I\!I$ is the second fundamental form of F in M. It is known that F is a totally umbilic submanifold. Thus, we have

$$I\!I(X,Y) = g(X,Y)\xi. \tag{23}$$

Plugging (23) into (22) and using $\phi(\xi)=0$ implies that

$$\bar{\nabla}_X Y = \frac{1}{2}\big\{\nabla'_X Y - \phi\nabla'_X \phi Y + g(X,Y)\xi\big\}. \tag{24}$$

The restriction of ϕ to F is the almost complex structure of F. In [13], the canonical Hermitian connection of (F,J) is defined as follows:

$$\bar{\nabla}_X Y = \frac{1}{2}\big\{\nabla'_X Y - J\nabla'_X JY\big\}. \tag{25}$$

Plugging (25) into (24), we get

$$\bar{\nabla}_X Y = \bar{\nabla}'_X Y + \frac{1}{2}g(X,Y)\xi. \tag{26}$$

The Nijenhuis torsion of ϕ is defined as follows:

$$N^{\phi}(X,Y) := \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]. \tag{27}$$

A direct computation and using (16) implies that $N^{\phi} = -4(\phi T + \eta\wedge I)$, i.e.

$$N^{\phi}(X,Y) = 4\big\{-\phi(T(X,Y)) - \eta(X)Y + \eta(Y)X\big\}. \tag{28}$$

Suppose that $N^{\phi}=0$. Then it follows from (28) that

$$-\phi(T(X,Y)) - \eta(X)Y + \eta(Y)X = 0. \tag{29}$$

Applying ϕ on both sides of (29) and using (17), we get (7). This means that (M,ϕ,η,ξ,g) is a Kenmotsu manifold. Thus, a nearly Kenmotsu manifold (M,ϕ,η,ξ,g) is a Kenmotsu manifold if and only if $N^{\phi}=0$, which was already proved in [12].

Gray studied a special kind of homogeneous spaces and denominated them 3-symmetric spaces. He proved that a (semi)-Riemannian 3-symmetric space with its canonical complex structure is nearly Kähler if and only if it is naturally reductive [7]. Here we give a well-known characterization result about naturally reductive 3-symmetric spaces, which was proved in [7].

Proposition 4.1 *Suppose that (M,g,J) is a complete and simply connected nearly Kähler manifold. Then the following conditions are equivalent:*

(1) M is a 3-symmetric space and J is its canonical almost complex structure.

(2) For every vector field X on M, we have $g((\nabla_X R)_{XJX}X, JX)=0$.

(3) The torsion of the canonical Hermitian connection $T_X Y = \nabla_X JY$ is a homogeneous structure.

(4) $\bar{\nabla}R=0$, where $\bar{\nabla}$ is the canonical Hermetical connection on M.

(5) $(\nabla_X R)_{YZ} = R_{T_X YZ} + R_{YT_X Z} - [T_X, R_{YZ}]$.

We need to compute the covariant derivative of the curvature of a warped product manifold. By a routine computation, we get the following.

Lemma 4.1 *Let F be a (strict) nearly Kähler manifold and c a nonzero constant. Consider the function $f(t) = ce^t$ on a line B. Then the warped product space $M = B \times_f F$ satisfies the following:*

$$(\bar{\nabla}_X R)_{VW} U = \frac{1}{2} g({}^F R_{VW} U, X))\xi + (\bar{\nabla}'_X {}^F R)_{VW} U. \tag{30}$$

A nearly Kenmotsu manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is said to be locally ϕ-symmetric with respect to $\bar{\nabla}$ if $\bar{\nabla} R = 0$. Using relation (30) in Lemma 4.1 one can suggest a relation between 3-symmetric nearly Kähler manifold F and locally ϕ-symmetric nearly Kenmotsu manifold $B \times_f F$ as follows.

Proposition 4.2 *Let F be a (locally) 3-symmetric nearly Kähler manifold. Suppose that c is a nonzero constant and consider the function $f(t) = ce^t$ on a line B. Then the warped product space $M = B \times_f F$ is not locally ϕ-symmetric and also each nearly Kähler manifold appearing in local decomposition (as a warped product) of a locally ϕ-symmetric strict nearly Kenmotsu is not (locally) homogeneous.*

Proof Using a similar argument to the proof of Theorem 3.1 and according to (30), the proof is concluded. □

Now we can use Nagy's decomposition on nearly Kähler manifolds, given in [13], to describe and decompose (locally) nearly Kenmotsu manifolds as warped products.

5. Relation between nearly Kenmotsu and nearly Kähler structure

For every vector field X on $M = B \times_f F$, put $X' := \nabla_X \xi$. Then, by (9), we have $\eta(X') = 0$ and $\phi(X') = \phi(X)$. The former means that X' is tangent to the fiber F. Thus, the vector field X is decomposed as $X = X' + \eta(X)\xi$.

Let us denote the induced connection on F by ∇'. Then its canonical torsion T' is given by $T'(X', Y') := (\nabla'_{X'} J)Y'$. Taking into account that F is a totally umblic submanifold and (8), one can obtain the following:

$$(\nabla_X \phi)Y = T'(X', Y') + g(X, \phi(Y))\xi - \eta(Y)\phi(X). \tag{31}$$

Suppose F is of constant type in the sense of Gray [8], i.e. for some constant real number α, the following holds:

$$||T'(X', Y')||_F^2 = \alpha\{||X'||_F^2 ||Y'||_F^2 - g_F(X', Y')^2 - g_F(J(X'), Y')^2\}. \tag{32}$$

It is easy to see that (32) can be written in the following form and in terms of (M, g):

$$\begin{aligned} ||T'(X', Y')||^2 = \frac{\alpha}{f}\{&||X||^2||Y||^2 - \eta(X)^2||Y||^2 - \eta(Y)^2||X||^2 - g(X, Y)^2 \\ &+2g(X, Y)\eta(X)\eta(Y) - g(\phi(X), Y)^2\}, \end{aligned} \tag{33}$$

where we have used $||X'||^2 = ||X||^2 - \eta(X)^2$, $g(X', Y') = g(X, Y) - \eta(X)\eta(Y)$, and $g(\phi(X'), Y') = g(\phi(X), Y)$. Using (31), we rewrite (15) as follows:

$$T(X, Y) = T'(X', Y') + g(X, \phi(Y))\xi - \eta(Y)\phi(X) + \eta(X)\phi(Y). \tag{34}$$

In [8], Gray proved that $T'(X',X')=0$ and $J(T'(X',Y'))=-T'(X',J(Y'))$. Thus, we get $g(T'(X',Y'),\phi(X))=g(T'(X',Y'),\phi(Y))=0$. Therefore, we have

$$\begin{aligned}||T(X,Y)||^2 = \frac{\alpha}{f}\{&||X||^2||Y||^2-\eta(X)^2||Y||^2-\eta(Y)^2||X||^2-g(X,Y)^2\\ &+2g(X,Y)\eta(X)\eta(Y)-g(\phi(X),Y)^2\}+g(X,\phi(Y))^2\\ &+\eta(Y)^2||X||^2+\eta(X)^2||Y||^2-2\eta(X)\eta(Y)g(X,Y).\end{aligned} \tag{35}$$

A nearly Kenmotsu manifold is said to be of constant type if for some constant real number α relation (35) holds. For example, the 7-dimensional nearly Kenmotsu manifold introduced in Example 1 is of constant type with $\alpha=1$. In general, we have the following.

Proposition 5.1 *Let $(M^{2m+1},\phi,\xi,\eta,g)$ be a nearly Kenmotsu manifold. If $m=3$, then M is of constant type.*

Let $(M^{2m+1},\phi,\xi,\eta,g)$ be a nearly Kenmotsu manifold. Let us consider a ϕ-basis of M such as $\{e_i, e_{m+i}=\phi(e_i), e_{2m+1}=\xi\}$. Then, using the formula of Riemannian curvature of warped product manifolds given in [1], we have

$$R(X,e_i,Y,e_i)=g({}^FR(X',e_i)Y',e_i)-g(X',Y')+g(X',e_i)g(Y',e_i)-\eta(X)\eta(Y), \tag{36}$$

where we have used $||grad(f)||=f$ and $H^f(\xi,\xi)=f$. By definition, we have

$$Ric(X,Y)=\sum_{i=1}^{2m}R(X,e_i,Y,e_i)+R(X,\xi,Y,\xi). \tag{37}$$

Using (36), we can simplify (37) and get the following:

$$Ric(X,Y)=Ric_F(X',Y')-2mg(X,Y), \tag{38}$$

and consequently

$$Q(X)=\frac{1}{f}Q^F(X')-2mX, \tag{39}$$

where Q and Q^F are the Ricci operators of M and F, respectively. Now we define Ric^* as follows:

$$Ric^*(X,Y)=\sum_{i=1}^{2m}R(X,\phi(Y),e_i,\phi(e_i)). \tag{40}$$

A direct computation implies that

$$Ric^*(X,Y)=Ric^*_F(X',Y')-2g(X',Y'). \tag{41}$$

Let Q^* be the $(1,1)$-tensor field associated to Ric^*, i.e. $g(Q^*(X),Y)=Ric^*(X,Y)$. Then it follows from (41) that:

$$Q^*(X)=\frac{1}{f}Q^*_F(X')-2X'. \tag{42}$$

Denote by r the difference of Q and Q^*. Then we have

$$r(X) := Q(X) - Q^*(X) = \frac{1}{f} r^*_F(X') - 2(m-1)X - 2\eta(X)\xi. \tag{43}$$

By Proposition 2.5 of [8] and Proposition 5.1, we get the following.

Proposition 5.2 *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a nearly Kenmotsu manifold. If $m = 3$, then we have*

$$r(X) = (\frac{4\alpha}{f} - 4)X + (\frac{4\alpha}{f} + 2)\eta(X)\xi, \tag{44}$$

where $\frac{\alpha}{f} = \frac{s+42}{30}$ and s is the scalar curvature of M.

Remark 5.1 *It is worth mentioning that due to the J-invariant property of Ricci (Ricci*) curvature of nearly Kähler manifolds [13], as a consequence of the above propositions, one can conclude two main results of [12] (Proposition 3 and Theorem 1).*

Now we deal with describing 9-dimensional and 11-dimensional nearly Kenmotsu manifolds.

Proposition 5.3 *Every 9-dimensional nearly Kenmotsu manifold can be decomposed locally as product $M_1 \times_f M_2$ (up to universal covering space), where M_1 is 3-dimensional Kenmotsu manifold and M_2 is a 6-dimensional nearly Kähler manifold or M_1 is Kähler surface and M_2 is a 7-dimensional nearly Kenmotsu manifold.*

Proof We have the following.

$$\begin{aligned} M^9 &\overset{[12]}{\simeq} \mathbb{R} \times_{ce^t} M^8 \\ &\overset{[8]}{\simeq} \mathbb{R} \times_{ce^t} (M^2 \times M^6) \\ &\simeq \mathbb{R} \times_{ce^t} M^2 \times_{ce^t} M^6 \\ &\overset{[12]}{\simeq} M^3 \times_{ce^t} M^6 \\ &\simeq \mathbb{R} \times_{ce^t} M^6 \times_{ce^t} M^2 \\ &\overset{[12]}{\simeq} M^7 \times_{ce^t} M^2 \end{aligned} \tag{45}$$

□

Proposition 5.4 *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a nearly Kenmotsu manifold. If $m = 5$, then there exist two locally constant functions α and β on M such that $\alpha^2 \leq \beta^2$ and the tangent space of M at each point can be decomposed as follows:*

$$T_x M = V_1 \oplus V_2 \oplus V_3 \oplus <\xi>, \tag{46}$$

such that $dimV_1 = 2$ and $dimV_2 = dimV_3 = 4$. Moreover, for the tensor r on these subspaces, we have

on V_1 $\quad r(X) = (\frac{4(\alpha^2+\beta^2)}{f} - 8)X + (\frac{4(\alpha^2+\beta^2)}{f} + 2)\eta(X)\xi,$

on V_2 $\quad r(X) = (\frac{4\alpha^2}{f} - 8)X + (\frac{4\alpha^2}{f} + 2)\eta(X)\xi$,

on V_3 $\quad r(X) = (\frac{4\beta^2}{f} - 8)X + (\frac{4\beta^2}{f} + 2)\eta(X)\xi$, *and*

on V_4 $\quad r(\xi) = -10\xi$.

If $\beta = 0$ *and* $\alpha \neq 0$, *then* M *is locally isometric to a warped product* $M_1 \times_f M_2$, *where* M_1 *is a 5-dimensional Kenmotsu manifold and* M_2 *is a 6-dimensional nearly Kähler manifold or* M_1 *is a 4-dimensional Kähler manifold and* M_2 *is a 7-dimensional nearly Kenmotsu manifold. In this case, on* $V_1 \oplus V_2$ *we have*

$$r(X) = (\frac{4\alpha^2}{f} - 8)X + (\frac{4\alpha^2}{f} + 2)\eta(X)\xi \tag{47}$$

and on V_3 $\quad r(X) = -8X - 2\eta(x)\xi$.

Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a nearly Kenmotsu manifold. With the notations of Theorem 2.1, let π be the projection onto F. Put $X^{'} = \nabla_\xi X$ and $x^{'} = \pi(x)$. Let us denote the sectional curvatures of M and F by K and K^F, respectively. Then we have

$$K_x(X,Y) = f(1 - \frac{\|\eta(X)Y - \eta(Y)X\|^2}{\|X\|^2\|Y\|^2 - g(X,Y)^2})K^F_{x^{'}}(X^{'}, Y^{'}) - 1. \tag{48}$$

Moreover, ϕ-holomorphic sectional curvatures of M are related to J-holomorphic sectional curvatures of F by:

$$K_x(X, \phi(X)) = f(1 - \frac{\eta(X)^2}{\|X\|^2})K^F_{x^{'}}(X^{'}, JX^{'}) - 1. \tag{49}$$

As a result of relation (49), we conclude that ϕ-holomorphic sectional curvatures of a nearly Kenmotsu manifold M are greater than or equal to -1 if and only if J-holomorphic sectional curvatures of all nearly Kähler manifolds F that appear in locally warped product of M are nonnegative. Using Corollary 4.5 in [7], we can also get the next corollary.

Corollary 5.1 *If the* ϕ*-holomorphic sectional curvatures of a nearly Kenmotsu* M *are greater than* -1, *then at each point of* M, *the maximum of sectional curvature is obtained at* ϕ*-holomorphic sectional curvatures.*

Lemma 5.1 *In relation (48),* $1 - \frac{\|\eta(X)Y - \eta(Y)X\|^2}{\|X\|^2\|Y\|^2 - g(X,Y)^2}$ *is nonnegative and it vanishes if and only if* $X = \xi$ *or* $Y = \xi$.

Proposition 5.5 *The lower bound of the sectional curvatures of a nearly Kenmotsu manifold is* -1 *if and only if the sectional curvatures of all nearly Kähler manifolds that appear in locally warped product of* M *are nonnegative.*

Proof It is a direct consequence of relation (48) and Lemma 5.1. □

Remark 5.2 *In Proposition 5.5, the lower and nonnegative can be replaced by upper and nonpositive, respectively.*

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