

Dynamic optimal contract under parameter uncertainty with risk-averse agent and principal

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Abstract: We consider a continuous-time principal–agent model on a finite time horizon, where we look for the existence of an optimal contract that both parties agreed on. Contrary to the mainstream, where the principal is modeled as risk-neutral, we assume that both the principal and the agent have exponential utility and are risk-averse with same risk awareness level. Moreover, the agent’s quality is unknown and is modeled as a filtering term in the problem, which is revealed as time passes. The principal cannot observe the agent’s real action, but can only recommend action levels to the agent. Hence, we have a *moral hazard* problem. In this setting, we give an explicit solution to the optimal contract problem.

Key words: Dynamic principal–agent problem, moral hazard, optimal control

1. Introduction

We consider optimal contracting between two parties, a principal (“she”) and an agent (“he”), in continuous time, when the agent’s actual effort cannot be observed by the principal. In economics, this type of problem is called a “hidden action” or “moral hazard” problem, where the agent’s control of the drift of the output process cannot be contracted upon. To give an example of a moral hazard problem, we can consider a scenario in which an investor (“principal”) hires a portfolio manager (“agent”) to manage her savings. The investor cannot observe the actual effort (or action) of the portfolio manager, only the current wealth of the portfolio. Hence, in the case that the investor is not satisfied about the performance of the portfolio manager, the manager could blame the market and argue that he gave the best performance for her savings, since the investor cannot observe the actual efforts of her portfolio manager.

The seminal paper on the continuous-time principal–agent problem is [9], where both parties have exponential utilities and agree on a linear optimal contract. Those results were generalized and extended by several authors (see, e.g., [4, 6, 8, 11, 12, 15, 16], among others). A nice survey of the literature was provided by Sung in [19]. Recently, [3] considered a general formulation of the principal–agent problem with a lump-sum payment on a finite horizon, where the agent influences both the agent and the volatility of the output, where the proofs use techniques based on a backward stochastic differential equations approach to non-Markovian stochastic control.

Another seminal paper, [14], presented a continuous-time moral hazard model with infinite horizon where the payments are paid continuously, rather than as a lump-sum payment at the terminal time. In [14], the

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principal is risk-neutral and the agent is risk-averse and the agent only controls the drift of the output. [13] extended [14] to the case of unobserved drift and made use of the stochastic maximum principle. [20] solved a principal–agent moral hazard problem, where both the principal and the agent are risk-averse, the payments are continuous, and the agent controls the unobserved drift term of the output. The main difference in the contract between a model with a risk-averse and a risk-neutral principal is whether there is ongoing consumption and dividends, as would be the case with risk aversion, or only “lumpy” consumption and dividends, which would be the case with risk neutrality. While most of the literature focuses on a risk-neutral principal, there are exceptions like the seminal work [9] and the recent work [20]. The current manuscript is another work in this direction with a risk-averse principal.

In this paper, we solve a moral hazard problem in a continuous-time Brownian model, where there is an additional endogenous learning term representing the agent’s unknown quality added to the model. To represent the unknown quality of the agent, we follow the framework introduced in [13]. However, we do not assume that the principal is risk-neutral as in [13], but instead both the principal and the agent have exponential utility with the same risk-awareness level as in [20]. Our model is both quantitatively and qualitatively different from the one in [13] and from [20]. In the risk-neutral case, as in [13], contracting is profitable because the principal can extract profits by providing insurance to the agent. In our case, combined with the unknown quality of the agent, the interaction is more complicated. We cannot conclude directly that the economic benefits will decrease as the difference in risk aversion between the principal and the agent shrinks, since we do not know the quality of the agent a priori; rather, his quality is revealed with time. The same obstacle can be observed in [13], where the results hold on the condition that enough time has passed to conclude qualitative results about the model. In our model, we see that as the unknown quality of the agent is revealed with time such that it does not affect the model significantly, our model converges to the analogous findings without the parameter uncertainty. On the other hand, even though the risk awareness of the principal is taken into consideration in [20], there is no endogenous learning term; hence, there are no aforementioned complications related to it. Furthermore, we also show that both parties agree on a contract, where the agent gives full effort from the beginning of the contract until the horizon T . Hence, neither the risk awareness of the principal nor the unknown quality of the agent affect the agent’s actual effort level given throughout the model, even though the principal cannot observe it, which was to be observed in [13], whereas [20] focused on interior optimal efforts.

The rest of the paper is as follows. In Section 2, we outline the general model of the problem. In Section 3, we describe the agent’s problem and find the dynamics of the continuation value function of the agent. In Section 4, we solve the principal’s optimal control problem and describe the optimal contract, and in Section 5 we further discuss and elaborate our main results and conclude the paper.

2. The model

In this section, we give the framework and dynamics of the model. Let $\{W_t\}_{t \geq 0}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathcal{F}_t is generated by the Brownian motion W_t . As in [13], we assume that the cumulative output y_t up to time T satisfies the stochastic integral equation

$$y_t = \int_0^t (\eta + a_s) ds + \int_0^t \sigma dW_s, \quad (1)$$

for $0 \leq t \leq T$, where η stands for the quality of the agent and is denoted by η , and $a_t \in [0, M]$ for $0 \leq t \leq T$ is the effort provided by the agent. η is unknown and we model it as in [13] with the common prior being

normal with mean m_0 and precision h_0 . Posterior over η , denoted by $\hat{\eta}$, depends on y_t and on cumulative effort $A_t \triangleq \int_0^t a_s ds$. Conditional on (y_t, A_t, t) , posterior belief about η is also normal with

$$\begin{aligned} \hat{\eta}(y_t - A_t, t) &\triangleq \mathbb{E}_t[\eta|y_t, A_t] \\ \hat{\eta}(y_t - A_t, t) &\triangleq \frac{h_0 m_0 + \sigma^{-2}(y_t - A_t)}{h_t} \\ h_t &\triangleq h_0 + \sigma^{-2}t \\ \hat{\eta}(0, 0) &= m_0. \end{aligned} \tag{2}$$

The principal does not observe the agent’s effort; she can only recommend actions \hat{a} . We denote the filtration generated by output and recommended actions (\bar{y}, \hat{a}) as

$$\mathcal{F}_t^y \triangleq \sigma(y_s, \hat{a}_s; 0 \leq s \leq t) \tag{3}$$

and $\mathbb{F}^y \triangleq \{\mathcal{F}_t^y\}_{t \geq 0}$, the \mathbb{P} -augmentation of this filtration. As in [13], we take the utility function of the agent with $\lambda \in (0, 1), \theta > 0$ as

$$u(w, a) = -e^{-\theta w + \theta \lambda a} \tag{4}$$

and actions of the agent are limited in a compact set $a_t \in [0, M]$ for $0 \leq t \leq T$. On the other hand, the agent knows the actual level of effort \bar{a} , which only he knows. Hence, the agent’s information is more than the principal’s. We denote the filtration generated by output, recommended actions, and actual actions up to time t as (\bar{y}, \bar{a}^*) as

$$\mathcal{F}_t^a \triangleq \sigma(y_s, \bar{a}_s, \bar{a}^*; 0 \leq s \leq t) \tag{5}$$

and $\mathbb{F}^a \triangleq \{\mathcal{F}_t^a\}_{0 \leq t \leq T}$, the \mathbb{P} -augmentation of this filtration. The agent is restricted to the class of control processes $\mathcal{A} \triangleq \{a_t : [0, T] \times \Omega \rightarrow [0, M]\}$ that are \mathbb{F}^a -predictable. We work with the induced distributions on the space of continuous functions. We take the sample space Ω to be the space of all continuous paths $C[0, T]$ equipped with the supremum norm $\|\cdot\|_\infty$. On $C[0, T]$, we let $W_t^0 = \omega(t)$ be the family of coordinate functions and $\mathcal{F}_t^0 = \sigma\{W_s^0, s \leq t\}$ the filtration generated by W_t^0 . We denote by P^0 the corresponding Wiener measure on $(\Omega, \mathcal{F}_t^0)$ and let \mathcal{F}_t be the completion with the null sets of \mathcal{F}_t^0 . On this space, we define the corresponding Brownian motion W_t^0 as in Equation (1). The set of admissible contracts \mathcal{C} is the set of $\mathbb{F}^Y \triangleq \{\mathcal{F}_t^Y\}_{0 \leq t \leq T}$ predictable functions $(a_t, w_t) : [0, T] \times Y \rightarrow A \times W$. Hence, the contract specifies a wage \bar{w}_t and a recommended action \bar{a}_t at date t that depend on the whole past history of the output \bar{y}_t . Then, given a contract, the agent makes his own choice of action a_t at each time t . Thus, the set of admissible actions \mathcal{A} for the agent are those \mathcal{F}_t^a -predictable functions $(\bar{a}, \bar{w}) : [0, T] \times Y \rightarrow A \times W$.

Definition 2.1 *A contract is called implementable if the agent agrees to the contract at time zero and chooses the recommended actions: $(a^*, w^*) = (\bar{a}, \bar{w})$.*

The dependence on the whole past implies that we cannot use a direct approach to the agent’s problem, since the entire past history \bar{y} would be a state variable. To overcome this difficulty, we make the problem tractable as in [1, 2],[4], [20] by taking the key state variable to be the density of the output process rather than the

output process itself. By considering different action choices corresponding to different output processes, we take the relative density process Γ_t , as defined in Equation (10). For $\sigma > 0$, we define

$$dy_t = \sigma dW_t^0, \tag{6}$$

where y_0 is given. This is the evolution of output under an effort policy \bar{a}^0 , which makes the drift of output equal to zero at each time $t \in [0, T]$. Hence, different effort choices alter the evolution of output by changing the distribution over outcomes in y with their corresponding Γ_t .

3. The agent’s problem

We impose a terminal date T on the contracting horizon. Until time T , both the principal and the agent are committed to the contract. To have an incentive-compatible contract, we need to specify what action the agent would choose when facing a given contract. First, we assume the following assumption. The agent’s continuation value is

$$v(a, t) \triangleq \mathbb{E} \left[\int_t^T e^{-\rho(s-t)} u(w_s, a_s) ds + e^{-\rho(T-t)} g(w_T) | \mathcal{F}_t^a \right], \tag{7}$$

where $\bar{y}_t \triangleq \{y_s; 0 \leq s \leq t\}$ is the output history, $\rho \in (0, 1)$ is the constant discount rate, and the functions u and g are defined below. The history dependence on the past makes it necessary to change the relevant state variable. We denote

$$f(t, \bar{y}, a_t) = \hat{\eta}(y_t - A_t, t) + a_t, \tag{8}$$

where

$$A_t = \int_0^t a_s ds, \tag{9}$$

and here we recall that \bar{y}_t means y depends on the whole path. We denote the density depending on action \bar{a} of \mathcal{F}_t -predictable processes:

$$\Gamma_t(\bar{a}) = \exp \left(\int_0^t \sigma^{-1} f(s, \bar{y}, a_s) dW_s^0 - \frac{1}{2} \int_0^t |\sigma^{-1} f(s, \bar{y}, a_s)|^2 ds \right), \tag{10}$$

where W_t^0 is as defined in Equation (6). Γ_t is an \mathcal{F}_t -martingale (as the assumptions on f ensure that Novikov’s condition is satisfied) with $\mathbb{E}[\Gamma_T(\bar{a})] = 1$ for all $\bar{a} \in \mathcal{A}$ where \mathcal{A} stands for the set of admissible actions. Thus, by the Girsanov theorem, we define a new measure $P^{\bar{a}}$ via:

$$\frac{dP^{\bar{a}}}{dP^0} = \Gamma_T(\bar{a}), \tag{11}$$

and by the filtering theorem of Fujisaki [7], the process $W_t^{\bar{a}}$ is defined by

$$W_t^{\bar{a}} = W_t^0 - \int_0^t \bar{\sigma}^{-1} f(s, \bar{y}, a_s) ds, \tag{12}$$

a Brownian motion under $P_{\bar{a}}$. Thus, we have

$$dy_t = \sigma dW_t^0 \tag{13}$$

$$= \sigma [dW_t^{\bar{a}} + \sigma^{-1} f(t, \bar{y}, a_t) dt] \tag{14}$$

$$= f(t, \bar{y}, a_t) dt + \sigma dW_t^{\bar{a}}. \tag{15}$$

Hence, each effort choice \bar{a} results in a different Brownian motion $W_t^{\bar{a}}$. Γ_t defined above satisfies $\Gamma_t = \mathbb{E}[\Gamma_T | \mathcal{F}_t^{\bar{a}}]$. Moreover, via derivation as in [13] by the Ito lemma we have that $\hat{\eta}$ is a $P^{\bar{a}}$ -martingale with decreasing variance

$$d\hat{\eta}(y_t - A_t, t) = \frac{\sigma^{-1}}{h_t} dW_t^{\bar{a}}. \tag{16}$$

Using the state variable as the density process Γ_t , we rewrite the optimization problem as

$$v(a, t) = \mathbb{E}_t^0 \left[\int_t^T \Gamma_{s,T}^a e^{-\rho(s-t)} u(w_s, a_s) ds + e^{-\rho(T-t)} g(w_T) | \mathcal{F}_t^{\bar{a}} \right], \tag{17}$$

where the terminal value function being $g(w_T) = -e^{\frac{1-\rho}{r} - \lambda \theta r w_T}$ of the agent is to be derived below. This approach makes our optimization problem tractable with optimal control techniques. The agent’s problem then is to solve

$$v^*(t) = \sup_{\bar{a} \in \mathcal{A}} v(\bar{a}, \bar{w}). \tag{18}$$

Theorem 3.1 *For each fixed action process $a(\cdot)$, there exists a unique decomposition for the agent’s continuation value, Equation (7), that satisfies*

$$dv_t = [\rho v_t - u(w_t, a_t)] dt + \sigma \gamma_t^a dW_t^{\bar{a}}, \tag{19}$$

$$v_T = g(w_T), \tag{20}$$

for some square integrable process γ_t^a , namely $\mathbb{E}^a[\int_0^T (\gamma_t^a)^2 dt] < \infty$. The process γ_t^a is denoted as the “incentive-compatibility parameter” in the moral hazard literature (see, e.g., [5]).

Proof Recall that for each action a we have for that action a

$$v(a, t) = \mathbb{E}_t^a \left[\int_t^T e^{-\rho(s-t)} u(w_s, a_s) ds + e^{-\rho(T-t)} g(w_T) | \mathcal{F}_t^a \right] \tag{21}$$

$$= e^{\rho t} \mathbb{E}_t^a \left[\int_0^T e^{-\rho s} u(w_s, a_s) ds + e^{-\rho(T-t)} g(w_T) | \mathcal{F}_t^a \right]$$

$$- e^{\rho t} \int_0^t e^{-\rho s} u(w_s, a_s) ds$$

$$dv(a, t) = \rho v_t dt + \gamma_t^a \sigma dW_t^a - u(w_t, a_t) dt,$$

where we appeal to the martingale representation theorem by Fujisaki [7] for square integrable martingales. We note here that

$$\mathbb{E}_t^a \left[\int_0^T e^{-\rho(s-t)} u(w_s, a_s) ds + e^{-\rho(T-t)} g(w_T) | \mathcal{F}_t^a \right] \tag{22}$$

is a square integrable martingale since the functions $g(\cdot), u(\cdot, \cdot)$ are bounded for $a \in [0, M]$ and $t \in [0, T]$. \square

Next, we characterize the necessary and sufficient conditions to maximize the value function of the agent in Equation (7). Our result is analogous to Proposition 4 in [13].

Lemma 3.2 *Maximizing the Hamiltonian defined as*

$$H(t, y, a, A, \gamma) \triangleq u(w_t, a_t) + \gamma(\hat{\eta}(A_t, y) + a_t) \tag{23}$$

is sufficient for the agent to maximize his value function as in Equation (7). Furthermore, it is necessary for the incentive compatibility parameter γ_t^a to satisfy

$$\gamma_t^a(a, w) = -u_a(w_t, a_t^*) + \frac{\sigma^{-2}}{h_t} p_t, \tag{24}$$

where the term p_t is as defined in Equation (31).

Proof By integrating Equation (19) for the optimal action \hat{a} and for any other action \bar{a} , we have

$$\begin{aligned} e^{-\rho T} v(T) &= e^{-\rho T} g(w_T) = e^{-\rho t} v(t, \hat{a}) - \int_t^T e^{-\rho s} u(w_s, \hat{a}_s) ds + \int_t^T \hat{\zeta}_s \sigma dW_s^{\hat{a}}, \\ e^{-\rho T} v(T) &= e^{-\rho T} g(w_T) = e^{-\rho t} v(t, \bar{a}) - \int_t^T e^{-\rho s} u(w_s, \bar{a}_s) ds + \int_t^T \bar{\zeta}_s \sigma dW_s^{\bar{a}}, \end{aligned} \tag{25}$$

where $\zeta_t \triangleq e^{-\rho t} \gamma_t$. Moreover, we have

$$\begin{aligned} dy_t &= \sigma dW_t^0, \\ dW_t^{\hat{a}} &= dW_t^0 - \frac{1}{\sigma} (\hat{\eta}(y_t - \hat{A}_t, t) + \hat{a}_t) dt, \\ dW_t^{\bar{a}} &= dW_t^0 - \frac{1}{\sigma} (\hat{\eta}(y_t - \bar{A}_t, t) + \bar{a}_t) dt, \\ dW_t^{\hat{a}} &= dW_t^{\bar{a}} + \frac{1}{\sigma} [\tilde{\eta}(y_t - \bar{A}_t, t) + \bar{a}_t - \tilde{\eta}(y_t - \hat{A}_t, t) - \hat{a}_t] dt. \end{aligned} \tag{26}$$

Hence, the following holds:

$$\begin{aligned} v(t, \bar{a}) - v(t, \hat{a}) &= e^{\rho t} \mathbb{E}_{\bar{a}} \left[\int_t^T e^{-\rho(s-t)} [u(w_t, \bar{a}_t) - u(w_t, \hat{a}_t)] dt + \right. \\ &\quad \left. + \int_t^T \hat{\gamma}_t \sigma dW_t^{\bar{a}} + \int_t^T \hat{\gamma} (\bar{a}_t - \hat{a}_t + \tilde{\eta}(y_t, \bar{A}_t) - \tilde{\eta}(y_t, \hat{A}_t)) dt \right] \\ &= e^{\rho t} \mathbb{E}_{\bar{a}} \left[\int_t^T [H(\bar{a}, \hat{\gamma}) - H(\hat{a}, \hat{\gamma})] dt + \int_t^T \hat{\gamma}_t \sigma dW_t^{\bar{a}} \right] \\ &\leq e^{\rho t} \mathbb{E}_{\bar{a}} \left[\int_t^T \hat{\gamma}_t \sigma dW_t^{\bar{a}} \right] = 0. \end{aligned} \tag{27}$$

The last term is a martingale due to square integrability of $\hat{\gamma}_t$ and $a \in [0, M]$ being bounded. Hence, we have proved the sufficient condition for the agent. Next, we prove the necessary condition for the agent's value

function in Equation (7).

$$\begin{aligned} \tilde{a}_t^\epsilon &\triangleq a_t + \epsilon \Delta a_t \\ \nabla v_t(a) &\triangleq \lim_{\epsilon \rightarrow 0} \frac{v_t(\tilde{a}) - v_t(a)}{\epsilon} \end{aligned}$$

By small perturbation, we have

$$\begin{aligned} e^{-\rho t} \nabla v_t(a) &= e^{-\rho t} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E}_{\tilde{a}} \left[\int_t^T e^{-\rho s} [u(w, \tilde{a}^\epsilon) - u(w, a)] ds \right. \\ &\quad \left. + \int_t^T \hat{\zeta}_s dW_s^{\tilde{a}^\epsilon} + \int_t^T \hat{\zeta}(\tilde{a}^\epsilon - a + \hat{\eta}(y_s, A_s^\epsilon) - \hat{\eta}(y_s, A_s)) ds \right], \end{aligned} \tag{28}$$

which gives the condition

$$\mathbb{E}_{\tilde{a}} \left[\int_t^T e^{-\rho s} u_a \Delta a_s + \zeta_s \left(\Delta a_s - \frac{\sigma^{-2}}{h_s} \int_t^s \Delta a_r dr \right) \right] \leq 0. \tag{29}$$

Integrating by parts, we get that

$$\mathbb{E}_{\tilde{a}} \left[\int_t^T \left(e^{-\rho s} u_a + \zeta_s - \int_s^T \zeta_r \frac{\sigma^{-2}}{h_r} dr \right) \Delta a_s ds \right] \leq 0.$$

By noting that Δa_s is arbitrary, we get

$$\left(\mathbb{E}_t^a \left[\int_t^T -\zeta_s \frac{\sigma^{-2}}{h_s} ds \right] + \zeta_t + e^{-\rho s} u_a(w_s, a_s) \right) (a_t - a_t^*) \leq 0. \tag{30}$$

By focusing only at time t and using that Δa_t is arbitrary, we conclude that for $a_t \in [0, M]$, we have

$$\zeta_t + e^{-\rho t} u_a(w_t, a_t^*) - \frac{\sigma^{-2}}{h_t} p_t \geq 0, \tag{31}$$

where

$$p_t = h_t \mathbb{E} \left[- \int_t^T \zeta_s \frac{1}{h_s} ds \middle| \mathcal{F}_t^a \right]. \tag{32}$$

Since increasing γ_t^a causes the volatility of the output to increase in Equation (19), the principal wants to minimize the incentive compatibility parameter γ_t^a . Hence, by multiplying the equation by $e^{\rho t}$ in Equation (31), we assume that the principal confines with

$$\gamma_t^a(a, w) = -u_a(w_t, a_t^*) + \frac{\sigma^{-2}}{h_t} p_t, \tag{33}$$

and we conclude the result. □

We rewrite the term p_t in Equation (31) in a more tractable way as in [13] as follows. First, we denote

$$\tilde{p}_t \triangleq \frac{\sigma^{-2}}{h_t} p_t.$$

Then we have

$$\tilde{p}_t = \mathbb{E} \left[- \int_t^T e^{-\rho(s-t)} \zeta_s \frac{\sigma^{-2}}{h_s} ds \right].$$

By differentiating with respect to time t , we get

$$\begin{aligned} \frac{d\tilde{p}_t}{dt} &= \rho\tilde{p}_t + \frac{\sigma^{-2}}{h_t} \gamma_t \\ &= \rho\tilde{p}_t - \frac{\sigma^{-2}}{h_t} (u_a(w_t, a_t) + \tilde{p}_t), \end{aligned}$$

and integrating this expression, we obtain

$$\tilde{p}_t = \frac{\sigma^{-2}}{h_t} \mathbb{E}_a \left[\int_t^T e^{-\rho(s-t)} u_a(w_s, a_s) ds \right]. \tag{34}$$

Remark 3.3 *By the above derivation we see that $\gamma_t(w, a)$ is bounded by wage process w_t being nonnegative and $a \in [0, M]$. Furthermore, we also note that when there is no term p_t in Equation (33), we have $\gamma_t = -u_a$, which corresponds to the first-order condition with respect to a of the term*

$$\tilde{H}(t, y, a, \gamma) = u(w, a) + \gamma_t a_t, \tag{35}$$

the Hamiltonian term without the filtering term in the model of Equation (1).

For the terminal date, we assume that from date T on the unknown filtering term $\hat{\eta}$ is revealed, no more production takes place, and both the principal and the agent live off their assets for the infinite future, earning the same constant rate of return r . We assume that both the principal and the agent solve the problem of the following form:

$$\begin{aligned} V_T(a_0) &= \max_{b_t} - \int_0^\infty \exp(-\rho t - \lambda\theta b_t) dt \\ &= \max_{b_t} - \int_T^\infty \exp(-\rho(t-T) - \lambda\theta b_t) dt, \end{aligned} \tag{36}$$

with c_0 given and $dc_t = (rc_t - b_t)dt$. For the agent $b_t = w_t$ and $a_0 = w_T$. The Hamilton–Jacobi–Bellman (HJB) equation for (36) reads as

$$\rho V_T(a) = \max_b \{- \exp(-\lambda\theta b) + V'_T(c)[rc - b]\}, \tag{37}$$

whose solution is

$$V_T(c) = - \exp \left(\frac{1-\rho}{r} - \lambda\theta rc \right),$$

with optimal

$$b(c) = \frac{\rho-1}{\lambda\theta r} + rc.$$

Hence, we have for the terminal time T the agent's and the principal's terminal value function v_T and V_T^p as

$$g(w_T) = -\exp\left(\frac{1-\rho}{r} - \lambda\theta r w_T\right), \tag{38}$$

$$v_T = g(w_T), \tag{39}$$

$$V_T^p = g(y_T - w_T). \tag{40}$$

4. The principal's problem

From the principal's point of view, the dynamics of the output follow

$$dy_t = (ry_t + \hat{\eta} + a_t - d_t)dt + \sigma dW_t^a. \tag{41}$$

We assume there is a common risk aversion λ between the principal and the agent. The principal discounts at the same rate ρ with the agent and has a flow utility

$$U(d_t) = -\exp(-\lambda\theta d_t) \tag{42}$$

over his consumption d_t with the value function

$$J(t, y, v, \hat{\eta}) = \max_{d, w, a} \mathbb{E}_t^a \left[\int_t^T e^{-\rho(s-t)} U(d_s) ds + e^{-\rho(T-t)} V_T^p(y_T - w_T) | \mathcal{F}_t^a \right], \tag{43}$$

where $V_T^p(y_T - w_T)$ is defined as follows:

$$\begin{aligned} V_T^p(y_T - w_T) &= -\exp\left(\frac{1-\rho}{r} - \lambda\theta r(y_T - w_T)\right) \\ &= -\exp\left(\frac{1-\rho}{r} - \lambda\theta r y_T\right) \exp(\lambda\theta r w_T). \end{aligned}$$

Using the terminal value of the agent at time T , the principal value function at time T reads as follows using Equation (38):

$$\begin{aligned} v_T &= -\exp\left(\frac{1-\rho}{r} - \lambda\theta r w_T\right), \\ J(T, y_T, v_T) &= -\frac{\exp^2\left(\frac{1-\rho}{r}\right)}{v_T} \exp(-\lambda\theta r y_T). \end{aligned}$$

For convenience, we summarize the value function dynamics of the principal as follows:

$$d\hat{\eta} = \frac{\sigma^{-1}}{h_t} dW_t^a \tag{44}$$

$$\hat{\eta}(y_t - A_t, t) = \frac{h_0 m_0 + \sigma^{-2}(y_t - A_t)}{h_t}, \tag{45}$$

$$\eta(0, 0) = m_0, \tag{46}$$

$$dv_t = [\rho v_t - u(w_t, a_t)]dt + \sigma \gamma_t^a(a, w) dW_t^a, \tag{47}$$

$$v_T = -\exp\left(\frac{1-\rho}{r}\right) \exp(-\lambda\theta r w_T), \tag{48}$$

$$dy_t = (ry_t + \hat{\eta} + a_t - d_t)dt + \sigma dW_t^a, \tag{49}$$

$$y_0 = 0. \tag{50}$$

We define the controlled value function for fixed admissible action process a_t as

$$J^u(t, v, y, \hat{\eta}) = \mathbb{E}_{t,T}^a \left[\int_t^T e^{-\rho(s-t)} e^{-\lambda \theta d_s} ds + V_T^p(y_T - w_T) | \mathcal{F}_t^a \right] \tag{51}$$

and the value function of the control problem given $(t, \hat{\eta}, v, y) \in [0, T] \times \mathbb{R}^3$ as

$$J(t, \hat{\eta}, v, y) := \sup_{a(\cdot) \in \mathcal{A}} J^a(t, \hat{\eta}, v, y). \tag{52}$$

We next state our main theorem in this section and subsequently prove it in the subsection below.

Theorem 4.1 *Suppose that the principal and the agent with an unknown quality term have the value functions of Equation (44) and Equation (7), respectively; then a contract is implementable in the sense of Definition 2.1, where both parties agree to recommend and give full effort for all times $0 \leq t \leq T$.*

4.1. Proof of Theorem 4.1

To prove Theorem 4.1, we guess an explicit solution for the value function in Equation (52) and verify our guess subsequently. Next, we guess that for $0 \leq t \leq T$ the value function of the principal is a $\mathcal{C}^{1,2}$ function of the form

$$J(t, y, v, \hat{\eta}) = \frac{e^{g(t, \hat{\eta})}}{v} \exp(-\lambda \theta r y) \tag{53}$$

and we verify its validity below. We further guess that, for the optimal action process a_t^* ,

$$e^{-\theta w + \theta \lambda a_t^*} = k(t, \hat{\eta}) v. \tag{54}$$

Then, by Theorem 3.1, we have that

$$dp_t = \theta \lambda dv_t, \tag{55}$$

$$p_t = \theta \lambda \mathbb{E}^a \left[\int_t^T e^{-\rho(s-t)} u ds \right], \tag{56}$$

$$p_t = \theta \lambda [1 - e^{\int_t^T (\rho - k(s, \hat{\eta})) ds}] v_t, \tag{57}$$

$$\varphi_t(k) = 1 - e^{\int_t^T (\rho - k(s, \hat{\eta})) ds}. \tag{58}$$

Furthermore, using our guess for the value function, we obviously have

$$J_y = -\lambda \theta r J,$$

$$J_{yy} = \lambda^2 \theta^2 r^2 J,$$

$$J_v = -\frac{1}{v} J,$$

$$J_{yv} = \lambda \theta r \frac{1}{v} J,$$

$$J_{vv} = \frac{2}{v^2} J.$$

Following our guess for the value function being in $C^{1,2}$, the HJB equation is of the form

$$\begin{aligned} \rho J - J_t = \max_{w,d,a} & \left\{ -\exp(-\lambda\theta d) + J_y[ry + \eta + a - w - d] \right. \\ & + J_v[\rho v + e^{-\theta w + \lambda\theta a}] \\ & + \frac{1}{2}J_{yy}\sigma^2 + \frac{1}{2}J_{vv}\sigma^2\gamma_t^2 \\ & + \frac{1}{2}J_{\eta\eta}\frac{\sigma^{-2}}{h_t} + J_{y\eta}\frac{1}{h_t} \\ & \left. + J_{yv}\sigma^2\gamma_t + J_{v\eta}\frac{\gamma_t}{h_t} \right\} \end{aligned} \tag{59}$$

Next, we show that our guess of the value function necessitates that the principal advises full action, namely $a_t^* \equiv M$ for $0 \leq t \leq T$.

Lemma 4.2 *The recommended action is $a^* = M$, namely the right corner is optimal for the principal.*

Proof By writing the first-order condition for wage w and action a from the HJB, we have the following pair of equations:

$$-J_y + J_v[-\theta e^{-\theta w + \lambda\theta a}] + \frac{1}{2}J_{vv}\sigma^2 2\gamma\gamma_w + J_{yv}\sigma^2\gamma_w + J_{v\eta}\frac{\gamma_w}{h_t} = 0, \tag{60}$$

$$J_y + J_v[\lambda\theta e^{-\theta w + \lambda\theta a}] + \frac{1}{2}J_{aa}\sigma^2 2\gamma\gamma_a + J_{yv}\sigma^2\gamma_a + J_{v\eta}\frac{\gamma_a}{h_t} = 0. \tag{61}$$

Then, with our guess for the value function,

$$J = e^{g(t,\eta)}\frac{e^{-\lambda\theta ry}}{v}, \tag{62}$$

$$J_y = -\lambda\theta r e^{g(t,\eta)}\frac{e^{-\lambda\theta ry}}{v}, \tag{63}$$

we see that J_y is positive, since v is negative. By the relation above,

$\frac{\partial\gamma}{\partial w} = -\lambda\frac{\partial\gamma}{\partial a} > -\frac{\partial\gamma}{\partial a}$. Hence, by noting $0 < \lambda < 1$ and by derivatives of the exponential function with respect to a and w , the first-order condition for w binds, whereas the first-order condition for a does not bind. Thus, we have either $a = 0$ or $a = M$ as optimal actions, but for $a = 0$ to be optimal, the right derivative should be less than or equal to 0 at $a = 0$, and this cannot be the case due to the first-order condition for a and w above. Similarly, the right corner's left derivative is positive whenever the first-order condition for w binds, and hence optimal action $a_t^* = M$ for all $t \in [0, T]$. \square

Using our guesses for the utility function and the value function and suppressing the arguments of the

functions for simplicity below, we rewrite the HJB equation as follows:

$$\begin{aligned} \rho J - J_t = \max_{w,d} & \left\{ -e^{-\lambda\theta d} - \lambda\theta r J[r y + \eta + \frac{\log(kv)}{\lambda\theta} + \frac{\log(\lambda\theta r e^{g(t,\eta)})}{\lambda\theta} - \frac{\log(-v)}{\lambda\theta} - r y] \right. \\ & - \frac{1}{v} J[\rho v + kv] + \frac{1}{2} \lambda^2 \theta^2 r^2 J \sigma^2 + \frac{1}{2} \frac{2}{v^2} \sigma^2 \theta^2 \lambda^2 \left(k + \frac{\sigma^{-2}}{h_t} \varphi \right)^2 v^2 \\ & \frac{1}{2} J_{\eta\eta} \frac{\sigma^{-2}}{h_t} - \lambda\theta r J_\eta \frac{1}{h_t} \\ & \left. + \lambda\theta r \frac{1}{v} J \sigma^2 \lambda \theta \left[k + \frac{\sigma^{-2}}{h_t} \varphi \right] v - \frac{1}{v} J_\eta \frac{1}{h_t} \lambda \theta v \left[k + \frac{\sigma^{-2}}{h_t} \varphi \right] \right\}. \end{aligned}$$

By canceling the terms and by the first-order condition on d , i.e. $e^{-\lambda\theta d} = -rJ$, we get

$$\begin{aligned} \rho J - J_t = & \left\{ rJ - \lambda\theta J \left[\eta + \frac{\log(-k)}{\lambda\theta} + \frac{\log(\lambda\theta r e^{g(t,\eta)})}{\lambda\theta} \right] \right. \\ & - J[\rho + k] + \sigma^2 \theta^2 \lambda^2 \left[k + \frac{\sigma^{-2}}{h_t} \varphi \right]^2 J \\ & + \frac{1}{2} J_{\eta\eta} \frac{\sigma^{-2}}{h_t} - \lambda\theta r J_\eta \frac{1}{h_t} \\ & \left. + \lambda^2 \theta^2 \sigma^2 r J \left[k + \frac{\sigma^{-2}}{h_t} \varphi \right] - \lambda\theta \frac{1}{h_t} \left[k + \frac{\sigma^{-2}}{h_t} \varphi \right] J_\eta \right\}. \end{aligned} \tag{64}$$

By our guess for the value function, we have

$$\begin{aligned} J &= \frac{e^{g(t,\eta)}}{v} e^{-\lambda\theta r y}, \\ J_t &= g_t J, \\ J_\eta &= g_\eta J. \end{aligned}$$

Hence, the HJB equation (64) reads as

$$\begin{aligned} \rho - g_t = & r - \lambda\theta \left[\eta + \frac{\log(-k)}{\lambda\theta} + \frac{\log(\lambda\theta r)}{\lambda\theta} + \frac{g(t,\eta)}{\lambda\theta} \right] \\ & - [\rho + k] + \sigma^2 \theta^2 \lambda^2 \left[k + \frac{\sigma^{-2}}{h_t} \varphi(k) \right]^2 \\ & + \frac{1}{2} g_{\eta\eta} \frac{\sigma^{-2}}{h_t} - \lambda\theta r g_\eta \frac{1}{h_t} \\ & + \lambda^2 \theta^2 \sigma^2 r \left[k + \frac{\sigma^{-2}}{h_t} \varphi(k) \right] - \lambda\theta \frac{1}{h_t} \left[k + \frac{\sigma^{-2}}{h_t} \varphi(k) \right] g_\eta \end{aligned} \tag{65}$$

with the terminal condition $g(T, \hat{\eta}) = e^{2(\frac{1-p}{r})}$ for all $\hat{\eta} \in \mathbb{R}$.

For ease of notation, we introduce the following expressions.

$$K_1(t, \hat{\eta}) = -\rho + r - \lambda\theta\hat{\eta} - \log(-k(t, \hat{\eta})) - \log(\lambda\theta r) - \rho - k(t, \eta) \tag{66}$$

$$+ \sigma^2\theta^2\lambda^2[k(t, \hat{\eta}) + \frac{\sigma^{-2}}{h_t}\varphi(k(t, \hat{\eta}))]^2 + \lambda^2\theta^2\sigma^2r[k(t, \hat{\eta})] \tag{67}$$

$$K_2(t, \hat{\eta}) = -\lambda\theta r \frac{1}{h_t} - \lambda\theta \frac{1}{h_t} [k(t, \hat{\eta}) + \frac{\sigma^{-2}}{h_t}\varphi(k(t, \hat{\eta}))] \tag{68}$$

$$K_3(t) = \frac{\sigma^{-1}}{h_t} \tag{69}$$

By the Feynman–Kac formula, the existence and uniqueness of the PDE above is guaranteed as

$$g(t, \hat{\eta}) = \mathbb{E}_t^a \left[\int_t^T e^{-(T-s)} K_1(s, \hat{\eta}_s) ds + e^{-(T-t)} e^{2\frac{1-\rho}{T}} | \mathcal{F}_t^a \right], \tag{70}$$

under the action a such that $\hat{\eta}$ is an Ito process driven by the equation

$$d\hat{\eta} = K_2(t, \hat{\eta})dt + K_3(t, \hat{\eta}_t)dW_t^a. \tag{71}$$

Moreover, using our guesses for the value function and utility function, we rewrite the first order condition for w as

$$\lambda\theta r J - \frac{1}{v}[-\theta k v]J + \frac{1}{2} \frac{2}{v^2} J \sigma^2 2\theta \lambda v [k + \frac{\sigma^{-2}}{h_t} \varphi] (-\theta)^2 \lambda^2 k v \tag{72}$$

$$+ \lambda\theta r \frac{1}{v} J \sigma^2 \theta \lambda v [k + \frac{\sigma^{-2}}{h_t} \varphi] + g_\eta \frac{1}{v} J \theta^2 \lambda^2 \frac{k v}{h_t} = 0. \tag{73}$$

Hence, by canceling v from the equation and dividing by J , Equation (72) reads as

$$\lambda\theta r + \theta k - \sigma^2 2\theta \lambda [k + \frac{\sigma^{-2}}{h_t} \varphi] \theta^2 \lambda^2 k \tag{74}$$

$$+ \lambda\theta r \sigma^2 \theta \lambda [k + \frac{\sigma^{-2}}{h_t} \varphi] + g_\eta \theta^2 \lambda^2 \frac{k}{h_t} = 0. \tag{75}$$

With Equation (65) and Equation (74), $k(t, \hat{\eta})$ and $g(t, \hat{\eta})$ are implicitly defined and can be found numerically.

4.2. Verification theorem

By the discussion above, we have the following converse relation. Our guess

$$J(t, y, v, \hat{\eta}) = \frac{e^{g(t, \hat{\eta})}}{v} \exp(-\lambda\theta r y) \tag{76}$$

is a $C^{1,2}$ function. It also satisfies the HJB equation (59)

$$\begin{aligned} \rho J - J_t = \max_{w,d,a} \{ & -\exp(-\lambda\theta d) + J_y[ry + \eta + a - w - d] \\ & + J_v[\rho v + e^{-\theta w + \lambda\theta a}] \\ & + \frac{1}{2} J_{yy} \sigma^2 + \frac{1}{2} J_{vv} \sigma^2 \gamma_t^2 \\ & + \frac{1}{2} J_{\eta\eta} \frac{\sigma^{-2}}{h_t} + J_{y\eta} \frac{1}{h_t} \\ & + J_{yv} \sigma^2 \gamma_t + J_{v\eta} \frac{\gamma_t}{h_t} \} \end{aligned}$$

with boundary condition

$$J(T, y_T, v_T) = -\frac{\exp^2\left(\frac{1-\rho}{r}\right)}{v_T} \exp(-\lambda\theta r y_T). \tag{77}$$

Using our guesses for the utility function, we also have by the discussion above for optimal action a^* the following:

$$\begin{aligned} e^{-\theta w + \lambda\theta a^*} &= k(t, \eta)v, \\ -\lambda\theta w + \lambda\theta a^* &= \log(kv), \\ -\lambda\theta w &= \log(kv) - \lambda\theta a^*, \\ -w &= \frac{\log kv}{\lambda\theta} - M, \\ a^* - w &= M - w = \frac{\log(kv)}{\lambda\theta}, \\ -e^{-\lambda\theta d} &= \lambda\theta r J, \\ e^{-\lambda\theta d} &= -r \frac{e^{g(t, \hat{\eta})}}{v} e^{-\lambda\theta r y}, \\ -\lambda\theta d &= \log(r g(t, \hat{\eta})) - \log(-v) - \lambda\theta r y, \\ -d &= \frac{\log(r j^1)}{\lambda\theta} - \frac{\log(-v)}{\lambda\theta} - r y. \end{aligned}$$

Hence, for each fixed $(t, y, v, \hat{\eta})$, the expression

$$\begin{aligned} \max_{w,d,a} \{ & -\exp(-\lambda\theta d) + J_y[ry + \eta + a - w - d] \\ & + J_v[\rho v + e^{-\theta w + \lambda\theta a}] \\ & + \frac{1}{2} J_{yy} \sigma^2 + \frac{1}{2} J_{vv} \sigma^2 \gamma_t^2 \\ & + \frac{1}{2} J_{\eta\eta} \frac{\sigma^{-2}}{h_t} + J_{y\eta} \frac{1}{h_t} \\ & + J_{yv} \sigma^2 \gamma_t + J_{v\eta} \frac{\gamma_t}{h_t} \} \end{aligned} \tag{78}$$

attains its maximum (a^*, w^*, d^*) at

$$a^* = M, \tag{79}$$

$$w^* = M - \frac{\log k(t, \hat{\eta}^{a^*})v}{\lambda\theta}, \tag{80}$$

$$d^* = yr - \frac{\log(-v)}{\lambda\theta} - \frac{\log(r)}{\lambda\theta} - \frac{g(t, \hat{\eta}^{a^*})}{\lambda\theta}, \tag{81}$$

where the functions $g(t, \hat{\eta})$ and $k(t, \hat{\eta})$ are determined by Equation (74) and Equation (65). Hence, Equation (76) is indeed the solution of the HJB equation (78) and we conclude the verification theorem.

5. Discussion and conclusion

In this paper, we have studied a principal–agent problem with moral hazard. Contrary to the mainstream, where the principal is assumed to be risk-neutral, we have assumed that both the principal and the agent have exponential utility as in Equation (4), and they are risk-averse with the same risk awareness level λ as in [20]. We also take an unknown endogenous learning term representing the unknown quality of the agent into account, which is revealed as time passes. We see that both parties agree on a contract, where the agent gives full effort from the beginning until the finite horizon T . Full effort of the agent in the optimal contract is also observed in [13], where the authors assumed the principal to be risk-neutral. Hence, we see that the risk awareness level of the principal does not play a role in the agent’s actual effort in the contract; only the agent’s utility is determinant in that respect. The optimality of the right corner of the admissible action interval of the agent as seen in Lemma 4.2 is due to the specific nature of the utility function chosen in Equation (4). Changing the utility function of the agent would cause the arguments in Lemma 4.2 to not hold anymore. In that case, one usually *assumes* that the optimal effort is in the interior of the effort interval. On the other hand, we see that the payments to the agents in terms of wages and dividends are affected by posterior belief on the unknown quality of the agent $\hat{\eta}$, as well as on the risk awareness λ of the principal, as seen in the Equations (79), (80), and (81). However, we cannot conclude that there is a direct negative effect on the payment to the agent due to the nonlinear nature of the parameters. We further note that since the uncertainty on the quality of the agent decreases as seen in Equation (2), its effect on the dynamics of the problem decreases as time t passes.

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