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A new formula for hyper-Fibonacci numbers, and the number of occurrences

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Abstract: In this paper, we develop a new formula for hyper-Fibonacci numbers $F_n^{[k]}$, wherein the coefficients (related to Stirling numbers of the first kind) of the polynomial ingredient $p_k(n)$ are determined. As an application we investigate the number of occurrences of positive integers among $F_n^{[k]}$ and determine all the solutions in nonnegative integers x and y to the Diophantine equation $F_x^{[k]} = F_y^{[\ell]}$, where $0 \le k < \ell \le 70$. Moreover, we prove that if ℓ is fixed, then $F_x^{[k]} = F_y^{[\ell]}$ has finitely many effectively computable solutions in the nonnegative integers x, y, and $k \le \ell$.

Key words: Hyper-Fibonacci numbers, Stirling numbers of the first kind, Diophantine equation, number of occurrences

1. Introduction and results

1.1. Hyper-Fibonacci numbers

Let $\{F_n\}$ denote the sequence of Fibonacci numbers defined, as usual, by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1}+F_{n-2}$ for $n \ge 2$. The hyper-Fibonacci numbers $F_n^{[k]}$ were introduced by Dil and Mező [8] as follows. For $k \ge 0$ and $n \ge 0$ the values $F_n^{[k]}$ are arranged in an infinite matrix such that $F_n^{[k]}$ is the entry of the kth row and nth column, $F_n^{[0]} = F_n$, $F_0^{[k]} = 0$, and further

$$F_n^{[k]} = F_{n-1}^{[k]} + F_n^{[k-1]}, \qquad kn > 0.$$

Clearly, $F_n^{[k]}$ gives the sum of the first $n+1$ elements (from the 0th to the nth) of row $k-1$, i.e. $F_n^{[k]} = \sum_{i=0}^{n} F_i^{[k-1]}$ $(n \ge 0,\ k \ge 1)$. We note that [7] derived certain summatory identities valid for hyper-Fibonacci array. A consequence of Proposition 2 of [8] is

$$F_n^{[k]} = \sum_{j=1}^{n} \binom{k+n-j-1}{k-1} F_j. \tag{1}$$

Formula (1) motivated us to find a more informative and applicable expression for $F_n^{[k]}$. Particularly, we were and we are still interested in the set $\mathcal{S}$ of all solutions to the equation $F_x^{[k]} = F_y^{[\ell]}$ in non-negative integers

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x, y, k, and ℓ. In this paper, we could determine a subset of $\mathcal{S}$; we have a conjecture on $\mathcal{S}$, but we have been unable to proof the conjecture. Our method is based on giving another explicit formula for hyper-Fibonacci numbers (see Theorem 1), which eliminates the exponential ingredient F_{n+2k} and the polynomial part $p_k(n)$ with coefficients determined explicitly. Hence, this is one of the main results of this work.

Theorem 1 *For nonnegative integers n and k,*

$$F_n^{[k]} = F_{n+2k} - p_k(n) \tag{2}$$

holds, where $p_k(x)$ is a rational polynomial given by

$$p_k(x) = \sum_{t=1}^{k-1} \left(\sum_{j=1}^{t} \frac{(-1)^{t-j}}{(k-j)!} \begin{bmatrix} k-j \\ k-t \end{bmatrix} \left(\sum_{i=0}^{j-1} \binom{k}{i} F_{j-i} \right) \right) x^{k-t} + F_{2k}. \tag{3}$$

In the theorem above $\left[\begin{smallmatrix} k-j \\ k-t \end{smallmatrix}\right]$ is a Stirling number of the first kind. The first few polynomials are

$$\begin{aligned}
p_0(x) &= 0, \\
p_1(x) &= 1, \\
p_2(x) &= x+3, \\
p_3(x) &= \frac{x^2+7x+16}{2}, \\
p_4(x) &= \frac{x^3+12x^2+59x+126}{6}, \\
p_5(x) &= \frac{x^4+18x^3+143x^2+630x+1320}{24}, \\
p_6(x) &= \frac{x^5+25x^4+285x^3+1955x^2+8294x+17280}{120}.
\end{aligned}$$

Theorem 1 specifies

$$F_n^{[1]} = F_{n+2} - 1, \qquad F_n^{[2]} = F_{n+4} - (n+3), \qquad F_n^{[3]} = F_{n+6} - \frac{n^2+7n+16}{2},$$

and so on.

The properties of the polynomials $p_k(x)$ are challenging themselves and furthermore they have importance in the investigation of the problem of the number of occurrences.

First consider the sum of the coefficients. Replace n by 1 in (2), which together with $F_1^{[k]} = 1$ admits the following:

Corollary 2 *Let k be a nonnegative integer. Then $p_k(1) = F_{2k+1} - 1$.*

From Corollary 2 we can simply conclude

$$p_k(1) - F_{2k} = F_{2k-1} - 1 < F_{2k}. \tag{4}$$

The sign of the coefficients of $p_k(x)$ is described by:

Theorem 3 *For $k \geq 1$ the coefficients of $p_k(x)$ are positive.*

Combining Theorem 3 and the fact that the sum of all but the constant term F_{2k} of the coefficients of $p_k(x)$ is smaller then the constant term itself (see (4)), it implies the following:

Corollary 4 *Letting k be a nonnegative integer, the height of the polynomial $p_k(x)$ is F_{2k}.*

We have not been able to prove it and therefore we state the following property as:

Conjecture 1 *Let $k \geq 2$. The coefficients of $p_k(x)$ are strictly decreasing starting from the constant term.*

For nonnegative k and n, Belbachir and Belkhir [4] proved the formula

$$F_n^{[k]} = F_{n+2k} - \sum_{t=0}^{k-1} \binom{n-1+2k-t}{t}, \tag{5}$$

similar to (2) (see Theorem 10 in [4]), but in (2) the coefficients of the polynomial $p_k(x)$ are explicit, which offers a chance for further examinations, for instance in case of the Diophantine equation $F_x^{[k]} = F_y^{[\ell]}$ (see Subsection 1.3). We think that our approach will be useful in studying analogous questions related to hyper-Lucas, hyper-Horadam, etc. sequences as well.

Let k be fixed. Then combining the generating function

$$\sum_{n=0}^{\infty} F_n^{[k]} t^n = \frac{t}{(1-t-t^2)(1-t)^k}$$

of the kth row of the hyper-Fibonacci array (given in Proposition 14, [8]) and (2), we find the explicit formula

$$F_n^{[k]} = c_k \gamma^n - d_k \bar{\gamma}^n - p_k(n) \cdot 1^n, \tag{6}$$

where $\gamma = (1+\sqrt{5})/2$, $\bar{\gamma} = (1-\sqrt{5})/2$, and further $c_k = \gamma^{2k}/\sqrt{5}$, $d_k = \bar{\gamma}^{2k}/\sqrt{5}$. Indeed, the zeros of the characteristic polynomial $(x^2-x-1)(x-1)^k$ of $F_n^{[k]}$ are γ, $\bar{\gamma}$, and 1 (with multiplicity $k \geq 0$ for the zero 1), and further $c_k\gamma^n - d_k\bar{\gamma}^n = F_{n+2k}$. The significance of Theorem 1 is in the explicit quantification of coefficients of $p_k(n)$ by (3).

1.2. Generalized arithmetical arrays and triangles

In the literature there exist several constructions varying or extending the idea of hyper-Fibonacci numbers or their rectangular shape arrangement (for instance, hyper-Lucas [3], hyper-Pell [1], hyper-Horadam numbers [2]; Fibonacci and Lucas Pascal triangles ([6]). Many properties can be examined by having common generalizations of them. Therefore, we describe and compare two of them. It may facilitate the corresponding investigations in the future.

A natural generalization of the hyper-Fibonacci numbers (to create a *generalized arithmetical array*) was described by [8], where the leftmost column sequence $\{F_0^{[k]}\} = \{0\}$ and the topmost row sequence $\{F_n^{[0]}\} = \{F_n\}$ were replaced by two arbitrary sequences, $\{a_n\}$ and $\{b_n\}$, respectively. The output generated by the two sequences is an infinite matrix

$$\mathbf{M} = (M_{k,n})_{k\geq 0, n\geq 0} \tag{7}$$

with the property $M_{k,0} = a_k$, $M_{0,n} = b_n$, and $M_{k,n} = M_{k-1,n} + M_{k,n-1}$ if $kn > 0$.

A similar approach in constructing a sort of *generalized arithmetical triangle* (in short GAT) was used in [5] with $\{a_n\}$ and $\{b_n\}$, and additionally with $A,\ B \in \mathbb{R}$. This GAT is structurally identical to Pascal's original triangle (he called his object an arithmetical triangle), and it also contains rows labeled by $0,\ 1,\ 2,\ \ldots$ such that the nth row possesses the elements $\left\langle {n \atop k} \right\rangle$ in the positions (say columns) $k = 0,\ 1,\ \ldots,\ n$ as follows.

Let $\left\langle {0 \atop 0} \right\rangle$ be arbitrary, denoted by Ω (since generally $a_0 \neq b_0$, and it has no influence on the triangle at all), and for any positive integer n put

$$\left\langle {n \atop 0} \right\rangle = A^n a_n \qquad \text{and} \qquad \left\langle {n \atop n} \right\rangle = B^n b_n, \tag{8}$$

and further for $n \geq 2$ and $1 \leq k \leq n-1$ let

$$\left\langle {n \atop k} \right\rangle = B\left\langle {n-1 \atop k-1} \right\rangle + A\left\langle {n-1 \atop k} \right\rangle.$$

Illustrating the GAT, for the first few rows we have

$$\begin{array}{ccccccccc} & & & & \Omega & & & & \\ & & & Aa_1 & & Bb_1 & & & \\ & & A^2a_2 & & AB(a_1+b_1) & & B^2b_2 & & \\ & A^3a_3 & & A^2B(a_1+a_2+b_1) & & AB^2(a_1+b_1+b_2) & & B^3b_3 & \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots, \end{array} \tag{9}$$

using our notation $\left\langle {2 \atop 1} \right\rangle = AB(a_1+b_1)$, $\left\langle {3 \atop 1} \right\rangle = A^2B(a_1+a_2+b_1)$, $\left\langle {3 \atop 2} \right\rangle = AB^2(a_1+b_1+b_2)$, etc. Theorem 1 of [5] admits a direct formula,

$$\left\langle {n \atop k} \right\rangle = A^{n-k}B^k \left(\sum_{i=1}^{n-k} \binom{n-1-i}{k-1} a_i + \sum_{j=1}^{k} \binom{n-1-j}{n-k-1} b_j \right), \tag{10}$$

if $1 \leq k \leq n-1$. (For $k=0$ and $k=n$ we have (8).) This GAT extends Ensley's GAT [9], since here we allow $a_0 \neq b_0$ in the generator sequences; furthermore, we also vary the rule of addition by the parameters A and B.

Approaching the rectangular structure of Dil and Mező [8], observe that the infinite matrix

$$\mathbf{M}^{(A,B)} = (M_{k,n}^{(A,B)})_{k\geq 0, n\geq 0} = \begin{bmatrix} \Omega & Bb_1 & B^2b_2 & B^3b_3 & \cdots \\ Aa_1 & AB(a_1+b_1) & AB^2(a_1+b_1+b_2) & \ddots & \\ A^2a_2 & A^2B(a_1+a_2+b_1) & \ddots & & \\ A^3a_3 & \ddots & & & \\ \vdots & & & & \end{bmatrix}$$

with $M_{k,0}^{(A,B)} = A^k a_k$, $M_{0,n}^{(A,B)} = B^n b_n$, and $M_{k,n}^{(A,B)} = AM_{k-1,n}^{(A,B)} + BM_{k,n-1}^{(A,B)}$, if $kn > 0$, and the triangular shape GAT (9) with entries $\left\langle {n \atop k} \right\rangle$ differ only in their appearance. Indeed, apart from the geometrical display, the identity

$$M_{k,n}^{(A,B)} = \left\langle {k+n \atop n} \right\rangle \tag{11}$$

transmits them to each other for $k+n \geq 1$.

Assume now that $A = B = 1$. Then $\mathbf{M}^{(A,B)}$ returns with (7), and apparently formulae (10) and (1) are equivalent via (11).

1.3. The number of occurrences and the equation $F_x^{[k]} = F_y^{[\ell]}$

Obviously, to investigate the number of occurrences is equivalent to considering the Diophantine equation

$$F_x^{[k]} = F_y^{[\ell]} \tag{12}$$

in the nonnegative integers x, y, k, and ℓ. The explicit formula in Theorem 1 makes it possible to provide an algorithm for the resolution of (12) if $0 \le k \le \ell$ are given (see the last section). Note that apart from the equality $F_1^{[0]} = 1 = F_2^{[0]}$ the row sequences of the hyper-Fibonacci array are strictly monotone increasing, so we may assume $k < \ell$. Clearly, $F_0^{[k]} = 0 = F_0^{[\ell]}$ and $F_1^{[k]} = 1 = F_1^{[\ell]}$ are trivial solutions, but we even have $F_2^{[0]} = 1 = F_1^{[\ell]}$, and moreover by $F_2^{[k]} = k+1$

$$F_x^{[k]} = F_2^{\left[F_x^{[k]}-1\right]} \tag{13}$$

also holds. Varying k and ℓ, we conjecture that there exist only 12 nontrivial solutions to (12) given by the following list.

Conjecture 2 *Besides the trivial solutions given above, the equation*

$$F_x^{[k]} = F_y^{[\ell]} \tag{14}$$

possesses only the solutions

$$\begin{aligned}(k,\ell,x,y) \quad = \quad & (0,11,14,4),(0,16,16,4),(0,17,55,3),(1,2,4,3),(1,7,12,5),(1,20,11,3),\\ & (2,8,6,3),(2,11,7,3),(2,33,11,3),(4,6,5,4),(4,12,5,3),(6,12,4,3).\end{aligned} \tag{15}$$

Using the approach described in the last section we proved only:

Theorem 5 *List (15) contains all nontrivial solution to (14) if* $0 \le k < \ell \le 70$.

We also proved:

Theorem 6 *Given the positive integer* ℓ*, the equation* $F_x^{[k]} = F_y^{[\ell]}$ *has finitely many solutions in the nonnegative integers* x*,* y*, and* $k \le \ell$*, which are effectively computable.*

For fixed k and ℓ there is a short but ineffective way, by the result of Schmidt and Schlickewei [11] (Proposition 1), to show that the number of solutions of $F_x^{[k]} = F_y^{[\ell]}$ is finite. If $k = 0 < \ell$, then the number of zeros of the characteristic polynomials differ (see the explanation after (6)) and consequently the two sequences are not related. Thus, the finiteness is obvious. If $0 < k < \ell$, then we are in a doubly related situation since $\bar{\gamma} = \gamma^{-1}$, but neither system (1.11) of [11] nor system (1.12a) together with (1.12b) of [11] is solvable. It provides again only finitely many solutions for our equation.

If $\beta(t)$ denotes the number of occurrences of the nonnegative integer t in the set $\{F_n^{[k]}\}$, we see that $\beta(0) = \beta(1) = \infty$, and furthermore Conjecture 2 together with (13) is equivalent to the conjecture

$$1 \le \beta(t) \le 4 \qquad \text{for} \qquad t \ge 2.$$

Now we will prepare the proofs of the theorems.

2. Auxiliary results

One way to introduce the unsigned Stirling numbers of the first kind is the polynomial

$$\binom{x}{k} = \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\begin{bmatrix}k\\ \ell\end{bmatrix}x^{\ell}. \tag{16}$$

Recall that $\left[\begin{smallmatrix}k\\0\end{smallmatrix}\right]$ is 1 if $k=0$, and 0 if $k\geq 1$. An immediate consequence of (16) is:

Lemma 1

$$\sum_{\ell=1}^{n}(-1)^{n-\ell}\begin{bmatrix}n\\ \ell\end{bmatrix} = \begin{cases} 0, & \text{if } n\geq 2;\\ 1, & \text{if } n=1.\end{cases}$$

It is known that

$$\begin{bmatrix}n\\ k\end{bmatrix} = (n-1)\begin{bmatrix}n-1\\ k\end{bmatrix} + \begin{bmatrix}n-1\\ k-1\end{bmatrix}$$

holds for $1\leq k\leq n-1$, and its successive application leads to:

Lemma 2

$$\begin{bmatrix}n\\ k\end{bmatrix} = \sum_{\ell=0}^{n-k}\binom{n-1}{\ell}\ell!\begin{bmatrix}n-1-\ell\\ k-1\end{bmatrix}.$$

The next result can be found in [12].

Lemma 3 *If* $0\leq k\leq n$, *then*

$$\sum_{\ell=0}^{n}\begin{bmatrix}n\\ \ell\end{bmatrix}\binom{\ell}{k} = \begin{bmatrix}n+1\\ k+1\end{bmatrix}, \qquad \text{especially (with } k=0) \quad \sum_{\ell=0}^{n}\begin{bmatrix}n\\ \ell\end{bmatrix} = \begin{bmatrix}n+1\\ 1\end{bmatrix} = n!.$$

Since the binomial coefficients also play an important role in this paper (see, for example, (3)), we need the following lemmas. All of them are known, or easy to prove.

Lemma 4

$$\sum_{\ell=0}^{k}(-1)^{\ell}\binom{n}{\ell} = (-1)^{k}\binom{n-1}{k}, \qquad (1\leq n,\ 0\leq k\leq n).$$

Lemma 5

$$\sum_{\ell=0}^{n}\binom{n}{\ell}F_{k-\ell} = F_{n+k}, \qquad (0\leq k,n).$$

Lemma 6 *Let* $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$. *If* $0\leq\alpha\leq n$ *is an integer, then the coefficient of* x^{α} *in* $f(x-1)$ *is*

$$\sum_{\ell=0}^{n-\alpha}(-1)^{\ell}\binom{\alpha+\ell}{\alpha}a_{\alpha+\ell}.$$

The last auxiliary result is Lemma 5 in [10]:

Lemma 7 *Let u_0 be a positive integer and further recall that $\gamma = (1+\sqrt{5})/2$ and $\bar{\gamma} = (1-\sqrt{5})/2$. Put*

$$\delta_i = \log_\gamma \left(\frac{1 + (-1)^i \left(|\bar{\gamma}|/\gamma \right)^{u_0}}{\sqrt{5}} \right)$$

for $i = 1, 2$, respectively, where $\log_\gamma$ is the logarithm in base γ. Then for all integers $u \geq u_0$ the inequality

$$\gamma^{u+\delta_1} \leq F_u \leq \gamma^{u+\delta_2}$$

holds.

In order to make the application of Lemma 7 more convenient, we shall suppose that $u_0 \geq 6$. Thus, we have $-1.68 < \delta_1 < \delta_2 < -1.66$.

3. Proof of Theorems 1–3

3.1. Proof of Theorem 1

First we verify the statement for column 0 and row 0. Obviously, we obtain

$$\begin{aligned} F_0^{[k]} &= F_{2k} - p_k(0) = F_{2k} - F_{2k} = 0, \\ F_n^{[0]} &= F_n - p_0(n) = F_n - 0 = F_n. \end{aligned}$$

For $k \geq 1$ and $n \geq 1$ we check that $F_n^{[k]} = F_{n+2k} - p_k(n)$, $F_{n-1}^{[k]} = F_{n-1+2k} - p_k(n-1)$, and $F_n^{[k-1]} = F_{n+2k-2} - p_{k-1}(n)$ satisfy the defining rule $F_n^{[k]} = F_{n-1}^{[k]} + F_n^{[k-1]}$ of hyper-Fibonacci numbers. This is a rather long computation; therefore, after the preparatory part, the verification is split into two parts (namely Subsections 3.1.1 and 3.1.2).

Clearly,

$$\underbrace{F_{n+2k} - p_k(n)}_{F_n^{[k]}} = \underbrace{F_{n-1+2k} - p_k(n-1)}_{F_{n-1}^{[k]}} + \underbrace{F_{n+2k-2} - p_{k-1}(n)}_{F_n^{[k-1]}}$$

is equivalent to

$$p_k(n-1) = p_k(n) - p_{k-1}(n), \tag{17}$$

and hence it is sufficient to prove (17). Note that for general n the values of the polynomials at n appearing in (17) can be considered as polynomials of n. In the next step we check (17) for the constant terms.

3.1.1. The constant terms

Applying Lemma 6 with $\alpha = 0$, the constant term of $p_k(n-1)$, denoted by c_0, is

$$\begin{aligned} c_0 &= F_{2k} + \sum_{t=1}^{k-1} \left(\sum_{j=1}^{t} \frac{(-1)^{t-j}}{(k-j)!} \begin{bmatrix} k-j \\ k-t \end{bmatrix} \left(\sum_{i=0}^{j-1} \binom{k}{i} F_{j-i} \right) \right) (-1)^{k-t} \\ &= F_{2k} + \sum_{j=1}^{k-1} \frac{(-1)^{k-j}}{(k-j)!} \left(\sum_{i=0}^{j-1} \binom{k}{i} F_{j-i} \right) \left(\sum_{t=0}^{k-j-1} \begin{bmatrix} k-j \\ k-j-t \end{bmatrix} \right). \end{aligned}$$

The equality above is based on a suitable rearrangement. By virtue of Lemma 3, the sum in the last brackets is $(k-j)!$. Thus,

$$\begin{aligned} c_0 &= F_{2k} + \sum_{j=1}^{k-1} (-1)^{k-j} \left(\sum_{i=0}^{j-1} \binom{k}{i} F_{j-i} \right) \\ &= F_{2k} + \sum_{j=1}^{k-1} F_j \left(\sum_{i=0}^{k-1-j} (-1)^{k-j-i} \binom{k}{i} \right) \\ &= F_{2k} - \sum_{j=1}^{k-1} F_j \binom{k-1}{k-1-j} \\ &= F_{2k} - F_{2k-2} \end{aligned}$$

follows. At the beginning simply the coefficients of distinct Fibonacci numbers are collected. In the next two steps we apply Lemma 4 and Lemma 5 consecutively. Since the constant term in $p_k(n)$ and $p_{k-1}(n)$ is F_{2k} and F_{2k-2}, respectively, the proof for the constant terms is ready.

3.1.2. The coefficients of n^α

First suppose that $\alpha = k-1$. Obviously, the leading coefficients of $p_k(n)$ and $p_k(n-1)$ coincide. One can easily compute exactly this value by inserting $t = 1$ into (3), which provides the reciprocal of $(k-1)!$.

In the sequel, assume that α is a positive integer at most $k-2$. By Lemma 6, the coefficient of n^α in $p_k(n-1)$, denoted by c_α, is

$$c_\alpha = \sum_{t=1}^{k-\alpha} (-1)^{k-\alpha-t} \binom{k-t}{\alpha} \left(\sum_{j=1}^{t} \frac{(-1)^{t-j}}{(k-j)!} \begin{bmatrix} k-j \\ k-t \end{bmatrix} \left(\sum_{i=0}^{j-1} \binom{k}{i} F_{j-i} \right) \right).$$

Now we claim to eliminate the coefficient of F_s $(1 \le s \le k-\alpha)$ in c_α. If we denote it by $c_{\alpha,s}$, we have

$$\begin{aligned} c_{\alpha,s} &= \sum_{i=0}^{k-\alpha-s} (-1)^{k-\alpha-s-i} \binom{k-s-i}{\alpha} \left(\sum_{j=0}^{i} \frac{(-1)^{i-j}}{(k-s-j)!} \begin{bmatrix} k-s-j \\ k-s-i \end{bmatrix} \binom{k}{j} \right) \\ &= \sum_{j=0}^{k-\alpha-s} \frac{(-1)^{k-\alpha-s-j}}{(k-s-j)!} \binom{k}{j} \left(\sum_{i=\alpha}^{k-s-j} \begin{bmatrix} k-s-j \\ i \end{bmatrix} \binom{i}{\alpha} \right). \end{aligned}$$

Observe that Lemma 3 implies

$$\sum_{i=\alpha}^{k-s-j} \begin{bmatrix} k-s-j \\ i \end{bmatrix} \binom{i}{\alpha} = \begin{bmatrix} k-s-j+1 \\ \alpha+1 \end{bmatrix}, \tag{18}$$

and the application of Lemma 2 for (18) and suitable rearrangements admit

$$\begin{aligned} c_{\alpha,s} &= \sum_{j=0}^{k-\alpha-s} \frac{(-1)^{k-\alpha-s-j}}{(k-s-j)!}\binom{k}{j}\left(\sum_{i=0}^{k-\alpha-s-j}\binom{k-s-j}{i} i!\begin{bmatrix} k-s-j-i \\ \alpha \end{bmatrix}\right) \\ &= \sum_{i=0}^{k-\alpha-s}\sum_{j=0}^{k-\alpha-s-i} \frac{(-1)^{k-\alpha-s-j}}{(k-s-i-j)!}\binom{k}{j}\begin{bmatrix} k-s-i-j \\ \alpha \end{bmatrix} \\ &= \sum_{i=0}^{k-\alpha-s} \frac{1}{(\alpha+i)!}\begin{bmatrix} \alpha+i \\ \alpha \end{bmatrix}\sum_{j=0}^{k-\alpha-s-i}(-1)^{k-\alpha-s-j}\binom{k}{j} \\ &= \sum_{i=0}^{k-\alpha-s} \frac{(-1)^i}{(\alpha+i)!}\begin{bmatrix} \alpha+i \\ \alpha \end{bmatrix}\binom{k-1}{k-\alpha-s-i}. \end{aligned}$$

Note that the last equality is implied by Lemma 4.

Now we show that the same amount linked to F_s in the coefficient $\hat{c}_\alpha$ of n^α in $p_k(n)-p_{k-1}(n)$ appears. Clearly, this coefficient is

$$\begin{aligned} \hat{c}_\alpha &= \sum_{j=1}^{k-\alpha} \frac{(-1)^{k-\alpha-j}}{(k-j)!}\begin{bmatrix} k-j \\ \alpha \end{bmatrix}\left(\sum_{i=0}^{j-1}\binom{k}{i}F_{j-i}\right) \\ &\qquad - \sum_{j=1}^{k-\alpha-1} \frac{(-1)^{k-\alpha-j-1}}{(k-j-1)!}\begin{bmatrix} k-j-1 \\ \alpha \end{bmatrix}\left(\sum_{i=0}^{j-1}\binom{k-1}{i}F_{j-i}\right) \\ &= \sum_{j=1}^{k-\alpha} \frac{(-1)^{k-\alpha-j}}{(k-j)!}\begin{bmatrix} k-j \\ \alpha \end{bmatrix}\left(\sum_{i=0}^{j-1}\binom{k-1}{i}F_{j-i}\right). \end{aligned}$$

Rearranging the last sum by the Fibonacci numbers, a short calculation shows exactly $c_{\alpha,s}$ belonging to F_s. Hence, the proof of Theorem 1 is complete.

3.2. Proof of Theorem 3

It comes immediately from (5) and the fact that the coefficient of any possible monomial x^τ in

$$\binom{x-1+2k-t}{t}$$

is positive for arbitrary $0 \le t \le k-1$.

4. The equation $F_x^{[k]} = F_y^{[\ell]}$ and proof of Theorems 5 and 6

4.1. Proof of Theorem 5

Apparently, with fixed $0 \le k < \ell$, we need to solve

$$F_{2k+x} - p_k(x) = F_{2\ell+y} - p_\ell(y) \tag{19}$$

in the nonnegative integers $x \ge 3$ and $y \ge 3$. Let us distinguish three cases, which are the basement of the resolution of the equation. Recall that $\gamma = (1+\sqrt{5})/2$.

Case 1. $2k + x = 2\ell + y$.

This condition implies $y = 2k - 2\ell + x$. Thus, (19) leads to

$$p_k(x) = p_\ell(2k - 2\ell + x),$$

which is an equation only in the variable x.

Case 2. $2k + x < 2\ell + y$.

First note that

$$0 < F_{2\ell+y-2} \le F_{2\ell+y} - F_{2k+x} = p_\ell(y) - p_k(x) \le p_\ell(y) < c_\ell y^{\ell-1},$$

where c_ℓ is a suitable positive constant depending on the polynomial $p_\ell(y)$. Thus, Lemma 7 implies

$$\gamma^{2\ell+y-2-1.68-\log_\gamma c_\ell} < \frac{F_{2\ell+y-2}}{c_\ell} < y^{\ell-1}, \tag{20}$$

which leads to an upper bound $y \le y_{0,\ell}$.

Case 3. $2k + x > 2\ell + y$.

Similarly to the previous case, we have

$$0 < F_{2k+x-2} \le F_{2k+x} - F_{2\ell+y} = p_k(x) - p_\ell(y) \le p_k(x) < c_k y^{k-1},$$

with a suitable positive constant c_k (depending on the polynomial $p_k(x)$). Subsequently,

$$\gamma^{2k+x-2-1.68-\log_\gamma c_k} < \frac{F_{2k+x-2}}{c_k} < x^{k-1} \tag{21}$$

implies $x \le x_{0,k}$.

Case 1 may provide solutions in a direct manner. For Cases 2 and 3, if k and ℓ are both given, then the determination of c_k and c_ℓ works. Instead, we will use Corollary 4, since a general bound facilitates the work in the range $0 \le k \le 70$.

Assume $x \ge 3$. Then

$$p_k(x) < F_{2k}(x^{k-1} + \cdots + x + 1) = F_{2k}\frac{x^k - 1}{x - 1} < F_{2k}x^k$$

holds. Hence, according to Lemma 7, we can slightly specify the estimations (20) and (21). Indeed,

$$\gamma^{x-2.02} < \frac{F_{2k+x-2}}{F_{2k}} < x^k,$$

and then

$$\frac{\log\gamma}{k} < \frac{\log x}{x - 2.02}. \tag{22}$$

Hence, x is bounded, and one has to verify only the x values in question. The worst case occurs for $k = 70$, when $x < 1008.1$.

4.2. Example: $F_x^{[4]} = F_y^{[6]}$

To illustrate the details of the procedure, we work them out for $(k,\ell) = (4,6)$. Observe that the equation $F_x^{[4]} = F_y^{[6]}$ has no solution when $x+8 = y+12$. Indeed, looking at the list of the polynomials $p_k(x)$ after Theorem 1, with $x = y+4$ $(y \geq 0)$ we must verify

$$\frac{(y+4)^3 + 12(y+4)^2 + 59(y+4) + 126}{6} = \frac{y^5 + 25y^4 + 285y^3 + 1955y^2 + 8294y + 17280}{120}.$$

It simplifies the equation

$$0 = \frac{(y^2+10y+41)(y+6)(y+5)(y+4)}{120},$$

which has no nonnegative integer solution y.

Assume now, that $x+8 < y+12$. Then, by (22), we need to check (19) for $y < 51.1$ and $x < 55.1$. It provides only the nontrivial solution $(x,y) = (5,4)$.

The last case, when $x+8 > y+12$, is similar. Now $x < 30.5$ and consequently $y < 26.5$. This branch has no contribution to the set of nontrivial solutions.

4.3. Proof of Theorem 6

A fixed ℓ entails finitely many k. Hence, we may assume that $k < \ell$ is also fixed. With a pair (k,ℓ), only finitely many solutions is possible. The right-hand side of (22) is strictly decreasing; therefore,

$$\frac{\log\gamma}{\ell} \leq \frac{\log\gamma}{k} < \frac{\log x}{x - 2.02}$$

provides an effective bound on x depending only on ℓ. Consequently, y is also bounded effectively. Clearly, the proof is complete.

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