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On the monodromy of Milnor open books

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Abstract: We present some techniques that can be used to factorize the monodromy of certain Milnor open books. We also describe a class of Milnor open books for which we can explicitly express the monodromy as a product of Dehn twists.

Key words: Canonical contact structures, surface singularities, Milnor open books, monodromy

1. Introduction

Let (X, x) be the germ of a normal complex analytic surface having a singularity at x. Assuming that the link M_X of (X, x) is a rational homology sphere, we can apply, without change, the construction in [1] to explicitly construct Milnor open books on M_X, which support the canonical contact structure. Although the action of the monodromy of the open book on a certain decomposition of the page was described in [1], a general method for explicitly presenting the monodromy as a product of Dehn twists was not given. Nevertheless, using the description therein, such a presentation for the monodromy of an open book decomposition supporting the canonical contact structure on the link of each type of simple singularity was given. In [10], by using a theorem of Bonatti and Paris [2], Yılmaz was able find a similar presentation of the monodromy of an open book decomposition supporting the canonical contact structure on the link of each type of quotient singularity. In that work, it should be noted that all the Milnor open books had pages of genus 1. In the present work, we present some techniques that, used in conjunction with the aforementioned result of Bonatti and Paris, allow one to explicitly present the monodromy of Milnor open books as products of Dehn twists for a wider class of examples. In particular, the pages are allowed to have genus greater than 1. These techniques are illustrated on certain Milnor open books on links of minimally elliptic singularities [5].

2. Milnor open book decompositions

In this section, for the purpose of fixing notation and giving conventions, we collect some background results on Milnor open book decompositions. For further background results and an extensive survey of the literature, the reader may refer to the notes of Popescu-Pampu [9].

Given an embedding of the normal complex analytic surface singularity germ (X, x) in $(\mathbb{C}^N, 0)$, the link of (X, x) is defined to be the 3-manifold $M_X = X \cap S_\epsilon^{2N-1}$ obtained by intersecting a sufficiently small Euclidean

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sphere centered at the origin in $\mathbb{C}^N$ with X. The complex hyperplane distribution $\xi_X = TM_X \cap JTM_X$, where J denotes the complex structure on $\mathbb{C}^N$, is called the *canonical* contact structure on the link M_X. It is known that ξ_X is independent of the embedding and ϵ for ϵ sufficiently small.

Suppose that $f\colon (X,x) \to (\mathbb{C},0)$ is a germ of a holomorphic function that defines an isolated singularity at x. Then the pair $N(f) = f^{-1}(0) \cap M_X$, $\theta(f) = \arg f\colon M_X \setminus N(f) \to S^1$ defines an open book decomposition $\mathcal{OB}(f)$ of M_X. Such open book decompositions of M_X are called *Milnor open book* decompositions. According to a theorem of Caubel et al. [3], each Milnor open book decomposition of M_X supports the canonical contact structure ξ_X.

Let $\pi\colon \tilde{X} \to X$ be a good resolution of X. This means that $\tilde{X}$ is smooth, π is biholomorphic away from the exceptional divisor $E = \pi^{-1}(x)$, and E is a normal crossing divisor. The dual resolution graph $\Gamma = \Gamma(\pi) = (\mathcal{V}, \mathcal{E})$ of π is given as follows: the vertices $\mathcal{V} = \{A_1, \ldots, A_r\}$ of Γ are in one-to-one correspondence with the irreducible components $E_1 \ldots, E_r$ of E; there is an edge (A_i, A_j) $(= (A_j, A_i)) \in \mathcal{E}$ for each transverse intersection between E_i and E_j; and each vertex A_i is decorated by a pair (g_i, e_i), where g_i is the genus of E_i and $e_i = E_i^2$. The link M_X of $(X,0)$ is diffeomorphic to the 3-manifold $M(\Gamma)$ obtained by the plumbing construction. That is, associating to each vertex A_i the circle bundle $\pi_i\colon M_i \to S_i$ of Euler number e_i over a surface of genus g_i, the 3-manifold $M(\Gamma)$ is obtained by plumbing the circle bundles π_i according to the edges of the graph Γ.

Note that the condition that M_X is a rational homology sphere implies that Γ is a tree and $g_i = 0$ for all i. In this case, open book decompositions of $M_X \cong M(\Gamma)$ are determined up to isotopy by the topological type of the binding; see [4].

Let $f\colon (X,x) \to (\mathbb{C},0)$ be a germ of a holomorphic function that defines an isolated singularity at x and consider the decomposition of the divisor $(f \circ \pi) = (f \circ \pi)_e + (f \circ \pi)_s \in \mathrm{Div}\tilde{X}$ into its exceptional and strict parts. Here $(f \circ \pi)_e$ is supported on E and $\dim |(f \circ \pi)_s| \cap E < 1$. Since, for each i, $(f \circ \pi) \cdot E_i = 0$ and $(f \circ \pi)_s \cdot E_i \geq 0$, which is implied by positivity of intersections, it follows that $(f \circ \pi)_e \cdot E_i \leq 0$. Following Lipman [6], let $\mathcal{E}^+$ denote the set of nonzero effective divisors $Y = \sum m_i E_i$ supported on E such that $Y \cdot E_i \leq 0$ for all i. Then $(f \circ \pi)_e \in \mathcal{E}^+$. On the other hand, even though each divisor $Y \in \mathcal{E}^+$ may not arise in this way, any such divisor Y can be realized in this way by choosing a suitable analytic structure on the cone over M_X; see [7, 8].

Let $I(\Gamma)$ denote the intersection matrix of the dual resolution graph of π. Given $Y = \sum m_i E_i \in \mathcal{E}^+$, let $\underline{n} = (n_1, \ldots, n_r)$ be the r-tuple of nonnegative integers satisfying

$$I(\Gamma)\underline{m}^t = -\underline{n}^t, \tag{2.1}$$

where $\underline{m} = (m_1, \ldots, m_r)$. Then any Milnor open book corresponding to the divisor $Y = \sum m_i E_i$ will have binding consisting of n_i fibers of the circle bundle $\pi_i\colon M_i \to S_i$ for each i. For rational homology sphere links, by uniqueness, it follows that there is precisely one such Milnor open book, up to isotopy, which we will denote by $\mathcal{OB}(\underline{m})$. A description of $\mathcal{OB}(\underline{m})$ in the setting we are considering is provided in [1]. To fix the notation, we provide some details of the construction below.

For each vertex A_i of Γ take a copy of S^2 with $v_i + n_i$ small disks removed, where v_i is the valency of A_i. Call this surface $\overline{S}_i$ and label the boundary components of $\overline{S}_i$, $\gamma_1^i, \ldots \gamma_{v_i+n_i}^i$. Pick a base point $q_i \in \overline{S}_i$ and, for $k = 1, \ldots, v_i + n_i$, choose a loop α_k^i, based at q_i, having winding number δ_{kl} around the boundary

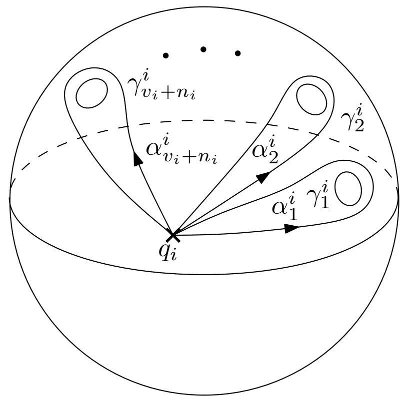


Figure 1. The surface $\overline{S}_i$.

component γ^i_l; see Figure 1. Here δ_{kl} denotes the Kronecker delta function. Now let $p_i\colon F_i \to \overline{S}_i$ denote the degree m_i regular covering associated to the monodromy representation $\rho_i\colon \pi_1(\overline{S}_i, q_i) \to \mathbb{Z}/m_i\mathbb{Z}$ given by $\rho_i([\alpha^i_k]) = -\mu_k + m_i\mathbb{Z}$, where

$$\mu_k = \begin{cases} m_{i_k} & \text{if } 1 \leq k \leq v_i \\ 1 & \text{if } k > v_i. \end{cases} \tag{2.2}$$

Here $i_1, \ldots, i_{v_i}$ denote the indices of the vertices that are connected to A_i by an edge.

As explained in [1], a page Σ of $\mathcal{OB}(\underline{m})$ is given by gluing together the possibly disconnected surfaces F_i, for $i = 1, \ldots, r$; an annulus U^i_t, for $i = 1, \ldots, r$ and $t = 1, \ldots, n_i$, for each binding component of $\mathcal{OB}(\underline{m})$; and a collection of annuli $U^{i,j}_l$, $l = 1, \ldots, (m_i, m_j)$ for each pair (i,j) with $1 \leq i < j \leq r$ such that $(A_i, A_j) \in \mathcal{E}$. We recall here the action of the monodromy ϕ of $\mathcal{OB}(\underline{m})$ restricted to the surfaces F_i. Note that there is a canonical $\mathbb{Z}/m_i\mathbb{Z}$-action on each fiber of $p_i\colon F_i \to \overline{S}_i$. For $x \in F_i$, $\phi(x)$ is defined to be the image of x under the action of $1 + m_i\mathbb{Z}$.

We close this section by quoting the following theorem of Bonatti and Paris [2], which will play an important role in the computation of the monodromy of Milnor open books.

Theorem 1 *Let Σ be a surface of genus 1 with nonempty boundary. If f and g are elements of the mapping class group $Mod(\Sigma)$ of Σ such that $f^m = g^m$ for some $m \geq 1$, then f is conjugate to g.*

3. Mondromy of Milnor open books

In this section, we present some techniques that can be used to factorize the monodromy of certain Milnor open books. In particular, in Theorem 3, we give a class of dual resolution graphs Γ and divisors $Y = \sum m_i E_i$ in $\mathcal{E}^+$ for which we can explicitly write the monodromy of $\mathcal{OB}(\underline{m})$ as a product of Dehn twists.

We begin with a definition.

Definition 2 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a dual resolution graph and $\underline{m}$ be an r-tuple of positive integers corresponding to a divisor $Y = \sum m_i E_i \in \mathcal{E}^+$. Let $\underline{n}$ be given by (2.1). A vertex A_i of Γ is called a* rupture vertex *of the pair $(\Gamma, \underline{m})$ if $n_i + v_i \geq 3$. A rupture vertex A_i is called* simple *if $n_i > 0$ or $\gcd(\{m_i\} \cup \{m_j \,|\, (A_i, A_j) \in \mathcal{E}\}) = 1$,* *and is called* good *if, in addition,* $m_i = 2$ *or* $m_i > 2$ *and the following conditions are satisfied:* $m_i \nmid m_j$ *for all* j *such that* $(A_i, A_j) \in \mathcal{E}$ *;* $v_i + n_i = g_i + 2$ *, where*

$$g_i = 1 + \frac{(v_i + n_i - 2)m_i - \sum_{(A_i,A_j)\in\mathcal{E}}(m_i, m_j) - n_i}{2}$$

is the genus of F_i *; and there is a filtration* $I_1 \subset I_2 \subset \cdots \subset I_{v_i+n_i-3}$ *of the set* $\{1, 2, \ldots, v_i + n_i\}$ *such that* $|I_j| = j + 1$ *and* $(m_i, \sum_{k\in I_j} \mu_k) = 1$ *for every* j *, where* μ_k *is given by* (2.2)*.*

Our main result is the following theorem.

Theorem 3 *Let* Γ *be a dual resolution graph corresponding a normal surface singularity with rational homology sphere link and* $Y = \sum m_i E_i$ *be an element of* $\mathcal{E}^+$ *. Suppose that each rupture vertex of* $(\Gamma, \underline{m})$ *is good. Suppose further that each path* Γ_t *in* Γ *that connects a pair of rupture vertices and does not contain any rupture vertex in its interior satifies either*

(i) $(m_i, m_j) = 1$ *for some pair* A_i, A_j *of adjacent vertices of* Γ_t *;*

(ii) $(m_i, m_j) = 2$ *for some pair* A_i, A_j *of adjacent vertices of* Γ_t *, and* Γ_t *connects rupture vertices* A_k *and* A_l *with* $m_k = m_l = 2$ *.*

Then there is an algorithm to present the monondromy of $\mathcal{OB}(\underline{m})$ *explicitly in terms of Dehn twists.*

Our proof of Theorem 3 rests on two observations. The first observation is that the monodromy ϕ of $\mathcal{OB}(\underline{m})$ can be decomposed along the collection of annuli in a page Σ of the open book $\mathcal{OB}(\underline{m})$ that correspond to the paths Γ_t in Γ, between rupture vertices, that satisfy condition (i) of the theorem. Presently, we explain this.

To begin with, note that if (A_i, A_j) is an edge of Γ with $(m_i, m_j) = 1$, then the edge (A_i, A_j) corresponds to a single boundary component, which we shall denote by $c_{i,j}$, of F_i. Observe, however, that the monodromy ϕ of $\mathcal{OB}(\underline{m})$ does not fix the boundary component $c_{i,j}$ pointwise; rather, it rotates $c_{i,j}$ by $a_{i,j}/m_i$ of a full turn, where $0 \le a_{i,j} < m_i$ satisfies $a_{i,j} m_j \equiv -1 \pmod{m_i}$. (Note that by a "rotation of a boundary component by a/b of a full turn" we simply mean that, for some identification of an invariant neighborhood of the boundary component with $[0, 1) \times \mathbb{R}/2\pi\mathbb{Z}$, the map in question corresponds to the diffeomorphism $(t, \theta) \mapsto (t, \theta + 2a\pi/b)$.) To see this, observe from the definition of the covering $p_i \colon F_i \to \overline{S}_i$ that a rotation of $c_{i,j}$ by $1/m_i$ of a full turn (which is obtained by a lifting of the circle α^i_k in the base to the covering, where $i_k = j$) corresponds to moving in the negative direction of the fibers m_j sheets (from the definition of the monodromy representation). (Note that we orient $c_{i,j}$ by the orientation coming from α^i_k, where $i_k = j$; this is the negative of the usual boundary orientation.) Since the restriction of the monodromy $\phi|_{F_i}$ is given by mapping the point $x \in F_i$ to the image of x under the action of $1 + m_i\mathbb{Z}$, which corresponds to moving in the positive direction of the fibers 1 sheet, it follows that ϕ corresponds to an a/m_i-turn about $c_{i,j}$, where $a(m_j/m_i) = [(m_i - 1)/m_i] + n$ for some $n \in \mathbb{Z}$ and $0 \le a < m_i$. The assertion follows.

We now give a convenient formula for the twist induced by the monodromy in an annulus in the page Σ if it corresponds to a sequence of edges.

Lemma 4 *Let A and A' be two vertices of Γ. Suppose that, after relabeling the vertices, if necessary, there is a path $A = A_1, A_2, \ldots, A_l = A'$ such that $v_i + n_i \leq 2$ for $i = 2, \ldots, l-1$ and $(m_1, m_2) = 1$. Then the monodromy ϕ of $\mathcal{OB}(\underline{m})$ twists the annulus $U^{1,l}$ connecting F_1 to F_l by*

$$1 - \frac{a_{1,2}}{m_1} - \frac{a_{l,l-1}}{m_l} + \#\{m_i \,|\, 1 < i < l, m_i = 1\} \tag{3.1}$$

full twists relative to the ends, where $0 \leq a_{i,j} < m_i$ satisfies $a_{i,j} m_j \equiv -1 \pmod{m_i}$.

Proof Note that $v_i + n_i \leq 2$ for $2 \leq i \leq l-1$ implies that for each i in this range $v_i = 2$ and $n_i = 0$. We thus have $e_i m_i = m_{i-1} + m_{i+1}$ for $2 \leq i \leq l-1$. Since $(m_1, m_2) = 1$, it follows that $(m_i, m_{i+1}) = 1$ for all $1 \leq i \leq l-1$. Hence, there is precisely one annulus connecting F_1 to F_l. By the discussion in [1], the monodromy twists this annulus by $\sum_{i=1}^{l-1} 1/m_i m_{i+1}$ full twists relative to the ends. It suffices thus to show that the value of (3.1) is equal to $\sum_{i=1}^{l-1} 1/m_i m_{i+1}$. We will show this by induction on the number of vertices $l \geq 2$ in the path joining A and A'.

First suppose that $l = 2$. Since, by definition, $a_{2,1} m_1 \equiv -1 \pmod{m_2}$ with $0 \leq a_{2,1} < m_2$, there exists an integer $k \geq 1$ such that

$$a_{2,1} m_1 = k m_2 - 1.$$

Similarly, there exists an integer $t \geq 1$ such that

$$a_{1,2} m_2 = t m_1 - 1.$$

Multiplying these two expressions together and rearranging gives

$$a_{2,1} m_1 + a_{1,2} m_2 = h m_1 m_2 - 1,$$

where $h = kt - a_{2,1} a_{1,2}$. The result will follow once we show that $h = 1$.

Clearly $h \geq 1$. To see that we also have $h \leq 1$, note that, since $a_{2,1} < m_2$, $a_{2,1} m_1 \leq m_1 m_2 - 1$. Similarly, $a_{1,2} m_2 \leq m_1 m_2 - 1$. It thus follows that

$$h m_1 m_2 - 1 = a_{2,1} m_1 + a_{1,2} m_2 \leq 2 m_1 m_2 - 2$$

and hence $h \leq 1$.

We now check the induction step. Let $k \geq 2$ and suppose that the result is true for all $l \leq k$. Consider the case $l = k+1$. We note that

$$\begin{aligned} 1 - \frac{a_{1,2}}{m_1} - \frac{a_{k+1,k}}{m_{k+1}} + \#\{m_i \,|\, 1 < i < k+1, m_i = 1\} =& 1 - \frac{a_{1,2}}{m_1} - \frac{a_{k,k-1}}{m_k} + \#\{m_i \,|\, 1 < i < k, m_i = 1\} \\ &+ \left(\frac{a_{k,k-1}}{m_k} + \frac{a_{k,k+1}}{m_k} - 1 + \delta_k \right) + \left(1 - \frac{a_{k,k+1}}{m_k} - \frac{a_{k+1,k}}{m_{k+1}} \right), \end{aligned}$$

where

$$\delta_k = \begin{cases} 1 & \text{if } m_k = 1 \\ 0 & \text{if } m_k \neq 1. \end{cases}$$

By the induction hypothesis

$$1-\frac{a_{1,2}}{m_1}-\frac{a_{k,k-1}}{m_k}+\#\{m_i\,|\,1<i<k, m_i=1\}=\sum_{i=1}^{k-1}\frac{1}{m_im_{i+1}}$$

and

$$1-\frac{a_{k,k+1}}{m_k}-\frac{a_{k+1,k}}{m_{k+1}}=\frac{1}{m_km_{k+1}}.$$

Hence, it suffices to show that

$$\frac{a_{k,k-1}}{m_k}+\frac{a_{k,k+1}}{m_k}-1+\delta_k=0. \tag{3.2}$$

Note that (3.2) is clear if $m_k=1$ and hence assume that $m_k\neq 1$. Now

$$a_{k,k-1}m_{k-1}=xm_k-1,$$
$$a_{k,k+1}m_{k+1}=ym_k-1$$

for some integers $x,y\geq 1$. Also,

$$e_km_k=m_{k-1}+m_{k+1}.$$

Hence,

$$(a_{k,k+1}+a_{k,k-1})m_{k+1}=(y-x+a_{k,k-1}e_k)m_k.$$

Since $(m_k,m_{k+1})=1$,

$$a_{k,k+1}+a_{k,k-1}=tm_k$$

for some $t\geq 0$, and thus it suffices to check that $t=1$. The latter follows immediately from the fact that $0\leq a_{k,j}<m_k$ for $j=k-1,k+1$. □

Now note that, under the hypotheses of Lemma 4, if $m_i\neq 1$ for $i=1$ and l, the monodromy ϕ twists the annulus connecting F_1 and F_l by

$$1-\frac{a_{1,2}}{m_1}-\frac{a_{l,l-1}}{m_l}+N=\left(1-\frac{a_{1,2}}{m_1}\right)+(N-1)+\left(1-\frac{a_{l,l-1}}{m_l}\right)$$

full twists relative to the ends, where $N=\#\{m_i\,|\,1<i<l, m_i=1\}$. Also notice that the monodromy restricted to F_1 followed by a twist of $1-\frac{a_{1,2}}{m_1}$ along the boundary component of F_1 connected to $U^{1,l}$ fixes that boundary component of F_1. Similarly, the monodromy restricted to F_l followed by a twist of $1-\frac{a_{l,l-1}}{m_l}$ along the boundary component of F_l connected to $U^{1,l}$ fixes the corresponding boundary component of F_l. This allows one to cut along the cylinder $U^{1,l}$, putting the $N-1$ twists to one side of the cut; see Figure 2.

If $m_1=1$ and $m_l\neq 1$, then $a_{1,2}=0$ and ϕ twists the annulus connecting F_1 and F_l by

$$\left(1-\frac{a_{l,l-1}}{m_l}\right)+N$$

full twists relative to the ends. Again this allows one to cut along the cylinder $U^{1,l}$. The situation is illustrated in Figure 3. The situation is similar if $m_l=1$.

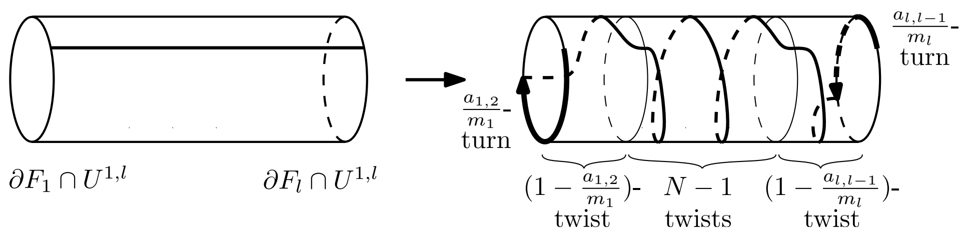


Figure 2. The action of the monodromy on the cylinder connecting F_1 to F_l if $m_i \neq 1$ for $i = 1$ and l.

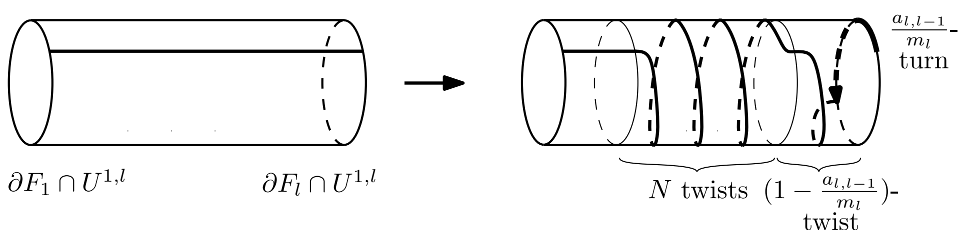


Figure 3. The action of the monodromy on the cylinder connecting F_1 to F_l if $m_1 = 1$ and $m_l \neq 1$.

The second observation concerns the possibility of decomposing the monodromy ϕ along some closed curve in a piece F_i of the page Σ corresponding to a simple rupture vertex A_i. For this, suppose that $v_i + n_i \geq 4$ and let I be any subset of $\{1, 2, \ldots, v_i + n_i\}$ of size $2 \leq |I| \leq v_i + n_i - 2$. Let α_I be a loop in S_i based at q_i that has winding number -1 around γ^i_j if $j \in I$ and winding number 0 around γ^i_j otherwise (see Figure 4). Then, since α_I is homotopic to the product of, possibly, conjugates of the loops $\overline{\gamma^i_k}$, $k \in I$, in some order,

$$\rho([\alpha_I]) = \sum_{k \in I} \mu_k + m_i\mathbb{Z},$$

where μ_k is given by (2.2). Here $\overline{\gamma^i_k}$ denotes the curve γ^i_k with the orientation reversed. Thus, in F_i, the preimage $p_i^{-1}(\alpha_I)$ will consist of $(m_i, \sum_{k\in I} \mu_k)$ components, which are cyclically permuted by the restriction of the monodromy $\phi|_{F_i}$. In particular, if $(m_i, \sum_{k\in I} \mu_k) = 1$, then F_i will necessarily be connected and $p_i^{-1}(\alpha_I)$ will consist of 1 component.

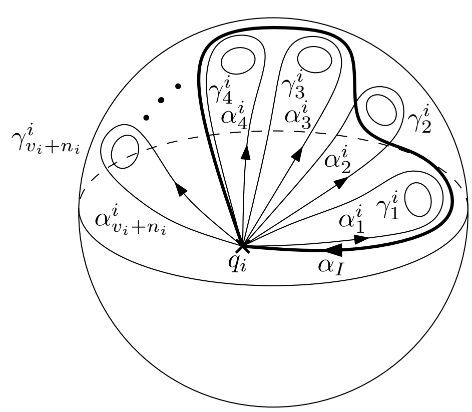


Figure 4. The surface $\overline{S}_i$ and a curve α_I. In the drawing, $I = \{1, 3, 4\}$.

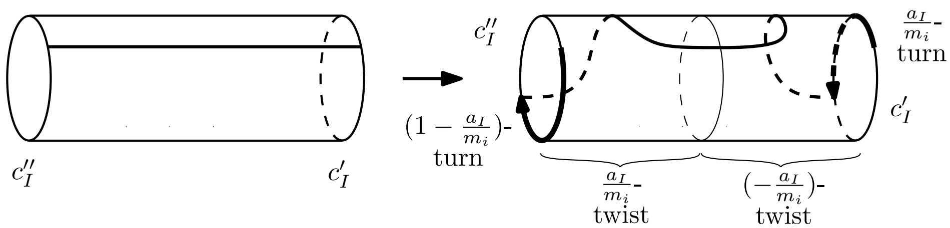


Figure 5. The action of the monodromy on the cylinder U_I.

Assume now that $(m_i, \sum_{k\in I}\mu_k)$ and let U_I denote a closed regular neighborhood of $p_i^{-1}(\alpha_I)$. Then the closure $\overline{F_i \setminus U_I}$ will consist of two connected components F_i', F_i'' whose genera may be computed using the Riemann–Hurwitz formula. For definiteness, let F_i' denote the component of $\overline{F_i \setminus U_I}$ that contains the preimages of the boundary curves α_k^i, $k \in I$. Let c_I' (resp. c_I'') denote $\partial F_i' \cap U_I$ (resp. $\partial F_i'' \cap U_I$). Note that, as we observed before, the restriction of the monodromy ϕ to F_i' rotates c_I' by a_I/m_i of a full turn, where $0 \le a_I < m_i$ satisfies $a_I \sum_{k\in I}\mu_k \equiv 1 \pmod{m_i}$. Thus, the restriction of the monodromy ϕ to the cylinder U_I can be decomposed as indicated in Figure 5. Note, in particular, that the monodromy restricted to F_i' followed by a $(-a_I/m_i)$-twist along c_I' fixes the boundary component c_I' of F_i'. Similarly, the monodromy restricted to F_i'' followed by an a_I/m_i-twist along c_I'' fixes the boundary component c_I'' of F_i''.

With these observations in hand, we now prove Theorem 3.

Proof [Proof of Theorem 3] First decompose the monodromy ϕ along meridian circles in the annuli in Σ corresponding to the paths Γ_t in Γ, between rupture vertices, that satisfy (i) of the theorem, as in the first observation above.

Next consider one factor ϕ_ν of the decomposition of the monodromy considered above and suppose that the part F_i of the page Σ corresponding to the rupture vertex A_i is in the domain of ϕ_ν. We now decompose ϕ_ν along certain curves in F_i. For this there are two cases to consider. Suppose first that $m_i = 2$. Letting μ_k, $1 \le k \le v_i + n_i$ be defined by (2.2), note that $|\{k \mid \mu_k \equiv 1 \pmod 2\}|$ is nonzero and even since A_i is simple. Assume, after relabeling the μ_k, if necessary, that $\mu_1 \equiv \mu_2 \equiv \cdots \equiv \mu_{2l} \equiv 1 \pmod 2$ and $\mu_{2l+1} \equiv \mu_{2l+2} \equiv \cdots \equiv \mu_{v_i+n_i} \equiv 0 \pmod 2$. Consider the filtration of the set $\{1,2,\ldots,v_i+n_i\}$ given by $I_k = \{2,3,\ldots,k+2\}$ for $1 \le k \le v_i+n_i-3$ if $l=1$, and

$$I_k = \begin{cases} \{2,3,\ldots,2k+2\} & \text{if } 1 \le k \le l-2 \\ \{2,3,\ldots,l+k+1\} & \text{if } l-1 \le k \le v_i+n_i-l-2 \end{cases}$$

if $l > 1$. Note that $(m_i, \sum_{s\in I_k}\mu_s) = 1$ for each k. Thus, $p_i^{-1}(\alpha_{I_k})$ has a single component for each k. The second observation above shows how to decompose the monodromy along the curves $p_i^{-1}(\alpha_{I_k})$. Note that, after cutting along the curves $p_i^{-1}(\alpha_{I_k})$, each piece of F_i has genus at most 1 and the pieces corresponding to edges (A_i, A_{i_k}) with $(m_i, m_{i_k}) = 2$ have genus 0.

Now suppose that $m_i > 2$. Then, by assumption, there is a filtration $I_1 \subset I_2 \subset \cdots \subset I_{v_i+n_i-3}$ of the set $\{1,2,\ldots,v_i+n_i\}$ such that $|I_k| = k+1$ and $(m_i, \sum_{s\in I_k}\mu_s) = 1$ for all k.

Claim. Each piece of F_i cut out by the curves $p_i^{-1}(\alpha_{I_k})$ has genus 1.

Proof of Claim. After capping off the boundary components, the restriction of p_i to each piece of F_i corresponds to an m_i-sheeted branched covering of a sphere branched over three values given by a group action $G_\tau \colon S_\tau \to S_\tau$. Fix one such covering $p_\tau \colon S_\tau \to S_\tau/G_\tau$ and let x_1, x_2, x_3 denote three fixed points of G_τ that are inequivalent under the group action. Let s_i denote the order of the stabilizer of x_i in G_τ for $1 \le i \le 3$. Then the Riemann–Hurwitz formula can be written as

$$\frac{2\sigma_\tau - 2}{|G_\tau|} = -2 + \sum_{j=1}^{3} \left(1 - \frac{1}{s_j}\right), \tag{3.3}$$

where σ_τ denotes the genus of S_τ. We will show that $\sigma_\tau = 1$.

We first check that $\sigma_\tau \ge 1$. Suppose for a contradiction that $\sigma_\tau < 1$. It follows that (s_1, s_2, s_3) is either equal to $(2, 2, s)$ for $s \ge 2$ or $(2, 3, s)$ for $s = 3, 4$ or 5. Since $p_i^{-1}(\alpha_{I_k})$ is connected for each k, it follows that $s_j = |G_\tau|$ for at least one j. Since s_1, s_2, and s_3 must divide $|G_\tau|$, this immediately rules out the case $(s_1, s_2, s_3) = (2, 3, s)$ for $s = 3, 4$, or 5. In case $(s_1, s_2, s_3) = (2, 2, s)$ for $s \ge 2$, from (3.3) we have

$$|G_\nu| = \frac{-2}{1 - (\frac{1}{2} + \frac{1}{2} + \frac{1}{s})} = 2s.$$

Hence s_1, s_2, s_3 are all different from $|G_\tau|$, a contradiction. Thus, $\sigma_\tau \ge 1$.

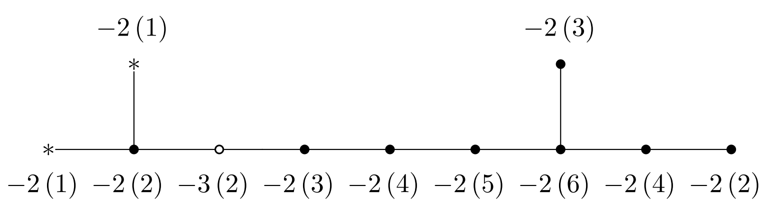
Since, by assumption, the genus of F_i is equal to $v_i + n_i - 2$ and the filtration $\{I_k\}$ cuts F_i into exactly $v_i + n_i - 2$ pieces, the genus of each piece must be 1.

Proceeding in this way for each factor ϕ_ν of the decomposition of the monodromy, we see that the page Σ will break up into a finite number of pieces, each with genus at most 1, and the monodromy will restrict to a periodic mapping class for each piece. These can be described explicitly in terms of Dehn twists with the help of Bonati and Paris' theorem and the results can be pieced together giving the conclusion. $\square$

4. Examples

We illustrate the use of Theorem 3 to compute the monodromy of Milnor open books with three examples from the list of weighted dual graphs for minimally elliptic singularities in [5]. For simplicity, if c is a simple closed curve in a surface, we denote a right-handed (resp. left-handed) Dehn twist about c by the c (resp. c^{-1}).

Example 5 *Consider the entry $A_{n,**,o} + E_{7,o}$ in Table 2 in [5] with $n = 1$ and weights $A_* \cdot A_*$ given by $-2, -2$. This corresponds to the weighted dual graph*



The weights are the numbers without parentheses and the numbers in parentheses are the coefficients of the fundamental cycle. Note that a page Σ of the corresponding Milnor open book is built up as shown in Figure 6a.

Consider the action of the monodromy on the cylinder U. By Lemma 4, the monodromy twists U by $1 - 1/2 - 1/6 = (1 - 1/2) + (-1) + (1 - 1/6)$ of a full twist relative to the ends. Cut Σ along a meridian c

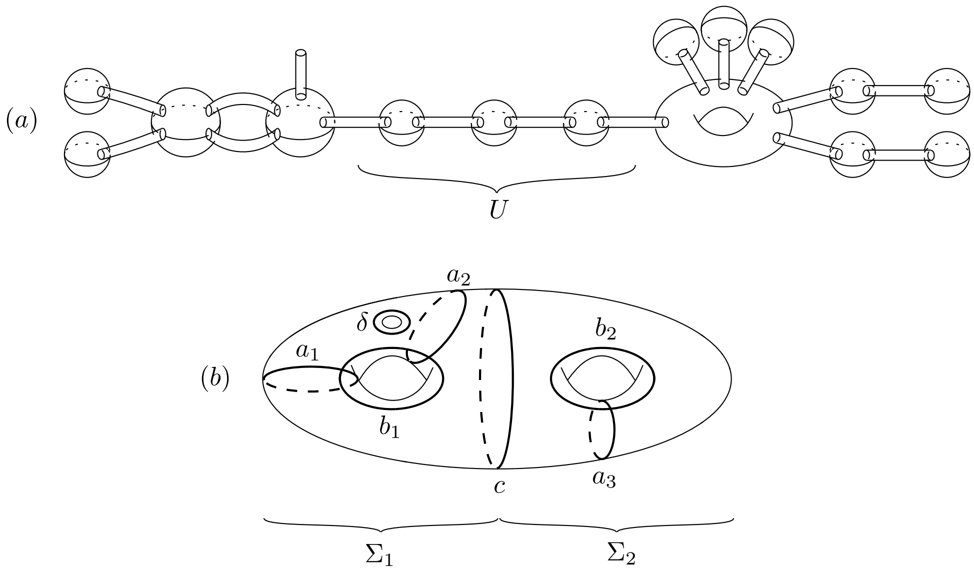
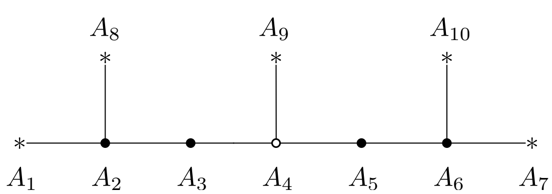


Figure 6. (a) A page Σ of the open book corresponding to the fundamental cycle of a singularity with weighted dual graph $A_{n,**,o}+E_{7,o}$, $n=1$, $A_*\cdot A_*=-2,-2$. (b) A decomposition of the page Σ.

of U to obtain two subsurfaces Σ_1 and Σ_2; see Figure 6b. By the discussion following Lemma 4, if the total twist along U is divided into a $1/2$-twist and a $(-1/6)$-twist, then monodromy may be written as a product $\phi=\phi_1\phi_2$, where ϕ_i is in the image of the homomorphism $\eta_i\colon \mathrm{Mod}(\Sigma_i)\to\mathrm{Mod}(\Sigma)$ induced by the inclusion map $\iota_i\colon\Sigma_i\to\Sigma$. Let $a_1,a_2,a_3,b_1,b_2,\delta$ be the curves indicated in Figure 6b. Arguing as in [1], we can now check that $\phi_1^2=\delta c a_1^2$ and $\phi_2^6=c^{-1}$. Using Theorem 1, we may check that, up to conjugacy, $\phi_1=a_1^2a_2b_1a_1^2b_1$ and $\phi_2=c^{-1}(a_3b_2)^5$. Thus, we find that the monodromy is given by

$$\phi=a_1^2a_2b_1a_1^2b_1c^{-1}(a_3b_2)^5.$$

Example 6 *Consider the entry $A_{*,o}+A_{n,**,o}+A_{m,**,o}$ in Table 2 in [5] with $n=m=2$ and weights $A_*\cdot A_*$ given by $-2,-2,-2,-2,-2$. This corresponds to the weighted dual graph*



where the weights of all vertices except A_4 are -2 and the weight of A_4 is -3. The fundamental cycle corresponds to the 10-tuple of positive integers $\underline{m}=(1,2,2,2,2,2,1,1,1,1)$. A page Σ of the corresponding Milnor open book is built up as shown in Figure 7a.

Consider the rupture vertex A_4 of the weighted dual graph and let I be a subset of $\{1,2,3,4\}$ such that the preimage of α_I is the curve c indicated in Figure 7a. Explicitly, if we order the vertices adjacent to A_4 by increasing index, then we can assume that $I=\{2,3\}$. As discussed above, introducing a $+(1/2)$ and a $-(1/2)$-twist on the two sides of c, we may assume that the monodromy fixes the curve c pointwise. Now cut the page Σ along c to obtain two subsurfaces Σ_1 and Σ_2 as indicated in Figure 7b. It follows that the monodromy may

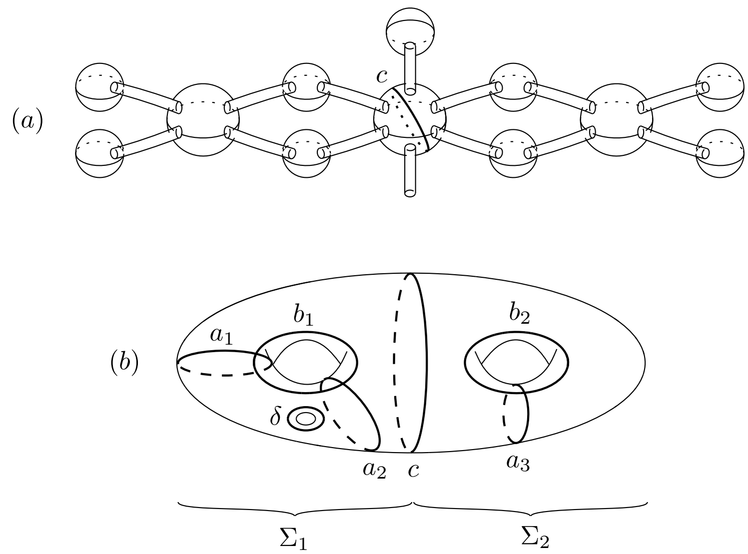
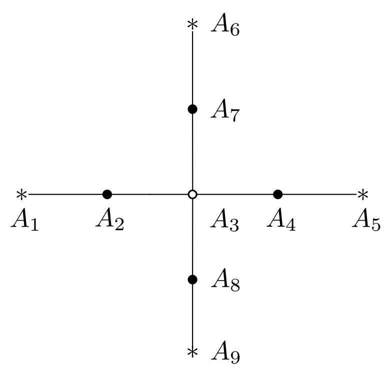


Figure 7. (a) A page Σ of the open book $\mathcal{OB}(1,2,2,2,2,2,1,1,1,1)$ for a singularity with weighted dual graph $A_{*,o}+A_{n,**,o}+A_{m,**,o}$, $n=m=2$, $A_*\cdot A_*=-2,-2,-2,-2,-2$. (b) A decomposition of the page Σ.

be written as a product $\phi=\phi_1\phi_2$, where ϕ_i is in the image of the homomorphism $\eta_i\colon \mathrm{Mod}(\Sigma_i)\to\mathrm{Mod}(\Sigma)$ induced by the inclusion map $\iota_i\colon\Sigma_i\to\Sigma$. Let $a_1,a_2,a_3,b_1,b_2,\delta$ be the curves indicated in Figure 7b. Arguing as in [1], we can check that $\phi_1^2=a_1^4c\delta$ and $\phi_2^2=c^{-1}a_3^4$. Now using Theorem 1, we find that, up to conjugation, $\phi_1=a_1^3a_2b_1a_1^2b_1$ and $\phi_2=c^{-1}(a_3b_2)^3a_3^2$. Hence,

$$\phi=a_1^3a_2b_1a_1^2b_1c^{-1}(a_3b_2)^3a_3^2.$$

Example 7 *Consider the entry $A_{1,*,o}+A_{1,*,o}+A_{1,*,o}+A_{1,*,o}$ in Table 3 in [5] with weights $A_*\cdot A_*$ given by $-2,-2,-2,-2$. This corresponds to the weighted dual graph*



where the weights of all vertices except A_3 are -2 and the weight of A_3 is -3. The fundamental cycle corresponds to the 9-tuple of positive integers $\underline{m}=(1,2,3,2,1,1,2,2,1)$. A page Σ of the corresponding Milnor open book is built up as shown in Figure 8a.

Consider the rupture vertex A_3 of the dual graph and order the vertices adjacent to A_3 arbitrarily. Let I_1 and I_2 be the subsets $\{1,2\}$ and $\{1,2,5\}$, respectively, of $\{1,2,\ldots,5\}$. Then the preimages of the α_{I_1} and α_{I_2} can be assumed to be the curves c_1 and c_2, respectively, indicated in Figure 8a. Now the curves

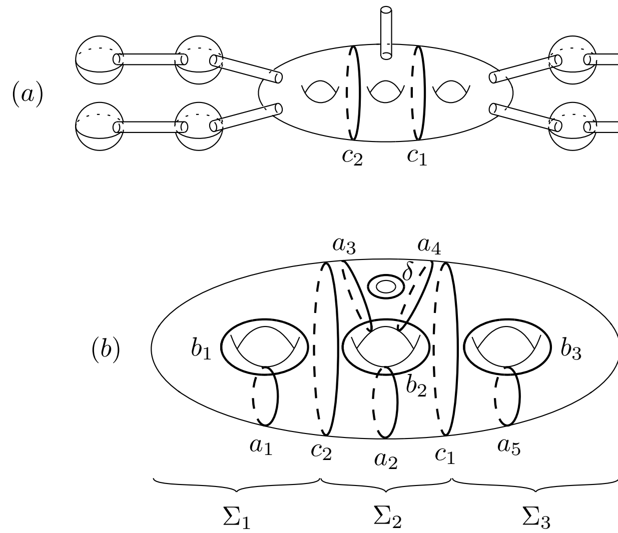


Figure 8. (a) A page Σ of the open book $\mathcal{OB}(1,2,3,2,1,1,2,2,1)$ for a singularity with weighted dual graph $A_{1,*,o}+A_{1,*,o}+A_{1,*,o}+A_{1,*,o}$, $A_*\cdot A_* = -2,-2,-2,-2$. (b) A decomposition of the page Σ.

c_1 and c_2 separate the page Σ into the three subsurfaces Σ_1, Σ_2, and Σ_3 as indicated in Figure 8b. Let $a_1,\ldots,a_5,b_1,b_2,b_3,\delta$ be the curves indicated in Figure 8b. Arguing as before, we can write the monodromy ϕ as a product $\phi_1\phi_2\phi_3$, where ϕ_i is in the image of the homomorphism $\eta_i\colon \mathrm{Mod}(\Sigma_i)\to\mathrm{Mod}(\Sigma)$ induced by the inclusion map $\iota_i\colon\Sigma_i\to\Sigma$. It is now easy to check using Theorem 1 that

$$\phi = (a_1b_1)^4c_2^{-1}a_2a_3a_4b_2c_1^{-1}(a_5b_3)^4.$$

5. Final remark

According to a theorem of Giroux, a contact structure on a 3-manifold (Y,ξ) is Stein fillable if and only if it admits a compatible open book decomposition with monodromy a product of right-handed Dehn twists. As the canonical contact structures on links of normal surface singularity are Stein fillable, and the monodromy factorizations we give above of a supporting Milnor open book decompositions often involve left-handed Dehn twists, it would be interesting to know if the monodromy of Milnor open book decompositions can always be expressed as products of right-handed Dehn twists.

References

[1] Bhupal M. Open book decompositions of links of simple surface singularities. Int J Math 2009; 20: 1527-1545.

[2] Bonatti C, Paris L. Roots in the mapping class groups. P Lond Math Soc 2009; 98: 471-503.

[3] Caubel C, Némethi A, Popescu-Pampu P. Milnor open books and Milnor fillable contact 3-manifolds. Topology 2006; 45: 673-689.

[4] Caubel C, Popescu-Pampu P. On the contact boundaries of normal surface singularities. CR Acad Sci Paris Ser I 2004; 339: 43-48.

[5] Laufer HB. On minimally elliptic singularities. Am J Math 1977; 99: 1257-1295.

[6] Lipman J. Rational singularities with applications to algebraic surfaces and unique factorization. Publ Math-Paris 1969; 36: 195-279.

[7] Neumann WD, Pichon A. Complex analytic realization of links. In: Scott Carter J, Kamada S, Kauffman LH, Kawauchi A, Kohno T, editors. Intelligence of Low Dimensional Topology; 22–26 July 2006; Hiroshima, Japan. Singapore: World Scientific Publishing Co., 2006, pp. 231-238.

[8] Pichon A. Fibrations sur le cercle et surfaces complexes. Ann Inst Fourier 2001; 51: 337374 (in French).

[9] Popescu-Pampu P. Complex Singularities and Contact Topology. Winter Braids Lecture Notes, 3. Lille, France: Winter Braids, 2016.

[10] Yılmaz E. Open book decompositions of links of quotient surface singularities. PhD, Middle East Technical University, Ankara, Turkey, 2009.