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Lower and upper solutions method for a problem of an elastic beam whose one end is simply supported and the other end is sliding clamped

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Abstract: In this paper we develop the lower and upper solutions method for the fourth-order boundary value problem of the form

$$\begin{cases} y^{(4)}(x) + (k_1+k_2)y''(x) + k_1k_2y(x) = f(x, y(x)), \quad x \in (0,1), \\ y(0) = y'(1) = y''(0) = y'''(1) = 0, \end{cases}$$

which models a statically elastic beam with one of its ends simply supported and the other end clamped by sliding clamps, where $k_1 < k_2 < 0$ are the real constants and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The proof of the main result is based on the Schauder fixed point theorem.

Key words: Elastic beam, Green's function, lower and upper solutions, Schauder fixed point theorem

1. Introduction

The aim of this paper is to establish the existence of solutions for the fourth-order differential equation of the form

$$y^{(4)}(x) + (k_1+k_2)y''(x) + k_1k_2y(x) = f(x, y(x)), \quad x \in (0,1) \tag{1}$$

with the boundary conditions

$$y(0) = y'(1) = y''(0) = y'''(1) = 0, \tag{2}$$

where $k_1 < k_2 < 0$ are the real constants and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Such a boundary value problem describes the equilibrium state of the deformation of an elastic beam whose one end is simply supported and the other end sliding clamped, where y'' is the bending moment stiffness and $y^{(4)}$ is the load density stiffness; see Agarwal [1], Gupta [12], and Lazer and McKenna [14] and the references therein.

The uniqueness, existence, and multiplicity of solutions for the nonlinear fourth-order ordinary differential equation (and its special case) with one of the boundary conditions

$$y(0) = y(1) = y''(0) = y''(1) = 0, \tag{3}$$

$$y(0) = y(1) = y'(0) = y'(1) = 0, \tag{4}$$

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have been extensively studied by several authors, and many techniques for treating such problems have appeared, such as the fixed point in cones [2,18], the bifurcation theory [13,17,22], and the lower and upper solutions method [3–5,20,24].

It is well known that for a second-order differential equation, with Neumann or Dirichlet boundary conditions, the existence of a lower solution α and an upper solution β with $\alpha(x) \leq \beta(x)$ in $[0,1]$ can ensure the existence of solutions in the order interval $[\alpha(x), \beta(x)]$; see Coster and Habets [8]. However, this result is not true for fourth-order boundary value problems; see the counterexample of Cabada et al. [3, p. 1607]. The reason for this is that the use of lower and upper solutions in the fourth-order boundary value problems is heavily dependent on the positiveness properties of the corresponding linear operators, but research in this area faces many difficulties. For the results concerning the positiveness properties of fourth-order linear operators, we refer the reader to Schröder [23], Cabada et al. [3], Drábek [9,10], and Ma et al. [19] and the references therein.

To the best of our knowledge, there is no appropriate lower and upper solutions method for the problem (1)–(2) and the research has proceeded relatively slowly; see Fialho et al. [11] and Minhós et al. [21] and the references therein. The likely reasons for this are that the boundary conditions (2) are not symmetric and the positiveness properties of the corresponding linear fourth-order operator are unknown.

In particular, in [21], by using the lower and upper solutions method and degree theory, Minhós et al. considered the existence of solutions for a fully nonlinear beam equation

$$y^{(4)} = g(x, y, y', y'', y'''), \quad x \in (0,1)$$

with the boundary conditions (2), where $g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function satisfying a Nagumo-type condition. However, the second derivatives of the lower and upper solutions must be ordered. Fialho et al. [11] proved the existence and location result in the presence of not necessarily ordered lower and upper solutions for the higher order functional boundary value problem (here we only state the special case with $n = 4$)

$$\begin{cases} y^{(4)}(x) = f(x, y, y', y''(x), y'''(x)), & \text{for a.e. } x \in (0,1), \\ L_0(y, y', y'', y''', y(0)) = 0, & L_1(y, y', y'', y''', y'(0)) = 0, \\ L_2(y, y', y'', y''', y''(0)) = 0, & L_3(y, y', y'', y''', y''(1)) = 0, \end{cases} \tag{5}$$

where $f : [0,1] \times (C([0,1]))^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function satisfying a Nagumo-type growth assumption, and $L_i : (C([0,1]))^4 \times \mathbb{R} \to \mathbb{R}, i = 0,1,2,3$ are continuous functions. It is worth remarking that in order to include a lower and upper solution ordered (well or in reverse order) or not ordered at all, and to consider very general functional boundary conditions without monotone assumptions, the definitions for lower and upper solutions are restrictive: the lower and upper solutions must be interdependent of each other and the corresponding second derivatives must be ordered.

Motivated by the interesting results of [11,21] and some earlier works, in this paper, we develop a new lower and upper solutions method for (1) and (2). Our lower and upper solutions are independent of each other and can be constructed more easily.

More precisely, we develop the lower and upper solutions method for the problem (1)–(2) under the assumption of $k_1 < k_2 < 0$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous, monotone increasing function with respect to the second variable. To do that, we first construct Green's function by decomposing the fourth-order operator in equation (1) into two operators of the second order, and then we get the positiveness properties of the fourth-order differential operator

$$L(y) := y^{(4)} + (k_1 + k_2)y'' + k_1 k_2 y \tag{6}$$

with the boundary conditions (2) in an easier way and finally we deduce the sign of the solutions of the nonhomogeneous problems

$$L(y(x)) = 0, \quad x \in (0,1), \quad y(0) = 1, \quad y'(1) = y''(0) = y'''(1) = 0,$$

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Since the general solution of the above homogeneous equation is very complex, we have to face tedious computation in the process of getting the sign of the solutions.

For other results concerning the existence and multiplicity of positive solutions or sign-changing solutions of the fourth-order elastic beam problems, we refer the reader to [6,7,15,16] and the references therein.

The rest of the paper is organized as follows. In Section 2, we construct Green's function for (1)–(2) and prove it possesses the positiveness properties under the condition of $k_1 < k_2 < 0$. Section 3 is devoted to developing the lower and upper solutions method for (1)–(2) via the Schauder fixed point theorem. Finally, in Section 4, we give an example and some remarks to illustrate our main result.

2. Green's function in the case of $k_1 < k_2 < 0$

Denote

$$k_1 = -m^2, \qquad k_2 = -r^2$$

with some $m > 0$ and $r > 0$. Define a linear operator $\mathcal{L} : D(\mathcal{L}) \to C([0,1])$ by

$$\mathcal{L}(y) := y^{(4)} - (m^2 + r^2)y'' + m^2 r^2 y, \qquad y \in D(\mathcal{L}), \tag{7}$$

where

$$D(\mathcal{L}) := \{y \in C^4([0,1]) : y(0) = y'(1) = y''(0) = y'''(1) = 0\}.$$

Lemma 2.1 *The linear boundary value problem*

$$\begin{cases} L(y) = y^{(4)} - (m^2 + r^2)y'' + m^2 r^2 y = 0, \quad x \in (0,1), \\ y(0) = y'(1) = y''(0) = y'''(1) = 0 \end{cases} \tag{8}$$

with $m > r > 0$ has only trivial solution.

Proof The roots of the characteristic equation for (8) are the real numbers m, $-m$, r, $-r$. Now the claim of the lemma follows from the fact that the determinant

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ m\cosh m & m\sinh m & r\cosh r & r\sinh r \\ 0 & m^2 & 0 & r^2 \\ m^3\cosh m & m^3\sinh m & r^3\cosh r & r^3\sinh r \end{vmatrix} = mr(m^2 - r^2)^2 \cosh m \cosh r$$

is nonzero. □

Theorem 2.2 *Let $m > r > 0$. Then Green's function for the linear problem (8) is*

$$H(x,s) = \begin{cases} \dfrac{1}{m^2-r^2}\Big[\dfrac{\sinh rx \cosh r(1-s)}{r\cosh r} - \dfrac{\sinh mx \cosh m(1-s)}{m\cosh m}\Big], & 0 \le x < s \le 1, \\ \dfrac{1}{m^2-r^2}\Big[\dfrac{\sinh rs \cosh r(1-x)}{r\cosh r} - \dfrac{\sinh ms \cosh m(1-x)}{m\cosh m}\Big], & 0 \le s < x \le 1. \end{cases}$$

Moreover,

$$H(x,s) \ge 0 \quad \textit{for} \quad 0 \le x, s \le 1.$$

Proof Let us define a linear operator $\mathcal{L}_1 : C^2([0,1]) \to C([0,1])$ by

$$\mathcal{L}_1(y) := y'' - m^2 y, \quad y \in D(\mathcal{L}_1) := \{y \in C^2([0,1]) : y(0) = y'(1) = 0\}.$$

Then Green's function of $\mathcal{L}_1(y) = 0$ is

$$G_1(t,s) = \begin{cases} \dfrac{\cosh m(1-t)\sinh ms}{m\cosh m}, & 0 \le s < t \le 1, \\ \dfrac{\cosh m(1-s)\sinh mt}{m\cosh m}, & 0 \le t < s \le 1, \end{cases} \tag{9}$$

and $G_1(t,s) \ge 0$ for every $(s,t) \in [0,1] \times [0,1]$.

Define a linear operator $\mathcal{L}_2 : C^2([0,1]) \to C([0,1])$ by

$$\mathcal{L}_2(y) := y'' - r^2 y, \quad y \in D(\mathcal{L}_2) := \{y \in C^2([0,1]) : y(0) = y'(1) = 0\}.$$

Then Green's function of $\mathcal{L}_2(y) = 0$ is

$$G_2(t,s) = \begin{cases} \dfrac{\cosh r(1-t)\sinh rs}{r\cosh r}, & 0 \le s < t \le 1, \\ \dfrac{\cosh r(1-s)\sinh rt}{r\cosh r}, & 0 \le t < s \le 1, \end{cases} \tag{10}$$

and it is clear that $G_2(t,s) \ge 0$ for every $(s,t) \in [0,1] \times [0,1]$.

It follows from

$$\mathcal{L}(y) = (\mathcal{L}_1 \circ \mathcal{L}_2)(y)$$

that the Green's function of $\mathcal{L}(y) = 0$ is

$$H(x,s) := \int_0^1 G_2(x,t) G_1(t,s) dt. \tag{11}$$

For any $0 \le x < s \le 1$, it follows from (9), (10), and (11) that we have

$$
\begin{aligned}
H(x,s) = &\int_0^x \frac{\sinh rt \cosh r(1-x)}{r\cosh r}\cdot\frac{\sinh mt \cosh m(1-s)}{m\cosh m}dt + \int_x^s \frac{\sinh rx \cosh r(1-t)}{r\cosh r}\cdot\frac{\sinh mt\cosh m(1-s)}{m\cosh m}dt \\
&+ \int_s^1 \frac{\sinh rx \cosh r(1-t)}{r\cosh r}\cdot\frac{\sinh ms \cosh m(1-t)}{m\cosh m}dt \\
=&\frac{\cosh m(1-s)\cosh r(1-x)}{mr\cosh m\cosh r}\int_0^x \sinh mt \sinh rt dt + \frac{\cosh m(1-s)\sinh rx}{mr\cosh m\cosh r}\int_x^s \sinh mt\cosh r(1-t)dt \\
&+ \frac{\sinh ms \sinh rx}{mr\cosh m\cosh r}\int_s^1 \cosh m(1-t)\cosh r(1-t)dt \\
=&\frac{\cosh m(1-s)\cosh r(1-x)}{2mr\cosh m\cosh r}[\frac{\sinh(m+r)x}{m+r}-\frac{\sinh(m-r)x}{m-r}] + \frac{\cosh m(1-s)\sinh rx}{2mr\cosh m\cosh r}[\frac{\cosh((m-r)s+r)}{m-r} \\
&+\frac{\cosh((m+r)s-r)}{m+r}-\frac{\cosh((m-r)x+r)}{m-r}-\frac{\cosh((m+r)x-r)}{m+r}] \\
&+\frac{\sinh ms\sinh rx}{2mr\cosh m\cosh r}[\frac{\sinh(m+r-(m+r)s)}{m+r}+\frac{\sinh(m-r-(m-r)s)}{m-r}] \\
=&\frac{\cosh m(1-s)}{2mr\cosh m\cosh r}[\frac{\sinh(m+r)x\cosh r(1-x)}{m+r}-\frac{\sinh(m-r)x\cosh r(1-x)}{m-r}-\frac{\cosh((m-r)x+r)\sinh rx}{m-r} \\
&-\frac{\cosh((m+r)x-r)\sinh rx}{m+r}] + \frac{\sinh rx}{2mr\cosh m\cosh r}[\frac{\cosh((m-r)s+r)\cosh m(1-s)}{m-r} \\
&+\frac{\cosh((m+r)s-r)\cosh m(1-s)}{m+r}+\frac{\sinh(m+r-(m+r)s)\sinh ms}{m+r}+\frac{\sinh(m-r-(m-r)s)\sinh ms}{m-r}] \\
=&\frac{\cosh m(1-s)}{2mr\cosh m\cosh r}[\frac{\sinh mx\cosh r}{m+r}-\frac{\sinh mx\cosh r}{m-r}] \\
&+\frac{\sinh rx}{2mr\cosh m\cosh r}[\frac{\cosh m\cosh r(1-s)}{m+r}+\frac{\cosh m\cosh r(1-s)}{m-r}] \\
=&\frac{1}{m^2-r^2}[\frac{\sinh rx\cosh r(1-s)}{r\cosh r}-\frac{\sinh mx\cosh m(1-s)}{m\cosh m}].
\end{aligned}
$$

Similarly, it follows from (9), (10), and (11) that

$$H(x,s) = H(s,x).$$

Thus,

$$H(x,s) = \frac{1}{m^2-r^2}\left[\frac{\sinh rs\cosh r(1-x)}{r\cosh r}-\frac{\sinh ms\cosh m(1-x)}{m\cosh m}\right]$$

for $0 \le s < x \le 1$.

Combining (11) with the nonnegativity of $G_1(t,s)$ and $G_2(t,s)$ on $[0,1]\times[0,1]$, it is concluded that the Green's function $H(x,s)$ for the linear problem (8) is nonnegative on $[0,1]\times[0,1]$. □

Remark 2.3 *By a similar argument, we know that the Green's function associated to the problem of an elastic beam whose both ends are simply supported,*

$$\begin{cases} L(y(x)) = 0, \quad x\in(0,1), \\ y(0)=y(1)=y''(0)=y''(1)=0, \end{cases}$$

is

$$G(x,s)=\begin{cases}\dfrac{1}{m^2-r^2}\Big[\dfrac{\sinh m(s-1)\sinh mx}{m\sinh m}-\dfrac{\sinh r(s-1)\sinh rx}{r\sinh r}\Big], & 0\le x<s\le 1,\\ \dfrac{1}{m^2-r^2}\Big[\dfrac{\sinh m(x-1)\sinh ms}{m\sinh m}-\dfrac{\sinh r(x-1)\sinh rs}{r\sinh r}\Big], & 0\le s<x\le 1.\end{cases}$$

It is worth remarking that Vrabel got the nonnegativity of Green's function $G(x,s)$ via the monotone property of function $g(\xi,x,s)$; see [24, 2(a)]. However, in our argument, we use the nonnegativity of Green's function directly.

Remark 2.4 *We may show the nonnegativity of Green's function in the case of $k_1<0<k_2$ and $0<k_1<k_2<\pi^2$ by a similar method.*

Remark 2.5 *(Maximum principle) Let*

$$L(y(x))\ge 0$$

for

$$y\in\Phi:=\{y\in C^4([0,1]):y(0)=y'(1)=y''(0)=y'''(1)=0\}.$$

Then $y(x)\ge 0$ on $[0,1]$.

Proof Let $y\in\Phi$ satisfy $L(y(x))\ge 0$ on $[0,1]$. Then y is a solution of boundary value problem $L(y(x))=g(x)$ for an appropriate continuous function $g(x)\ge 0$. Then the function y is a solution of an integral equation

$$y=\int_0^1 H(x,s)g(s)ds.$$

From Theorem 2.2, the function y is nonnegative on $[0,1]$. □

3. Lower and upper solutions method

In this section, we will develop the lower and upper solutions method for (1)–(2) under the condition of $k_1<k_2<0$. The following result can be deduced by a direct computation.

Lemma 3.1 *(i) The function*

$$p(\gamma,x)=\frac{\cosh\gamma(1-x)}{\gamma^2\cosh\gamma},\quad \gamma>0 \text{ and } x\in(0,1]$$

is a monotone decreasing, positive function for the variable γ.

(ii) The function

$$q(\gamma,x)=\frac{\sinh\gamma x}{\gamma^3\cosh\gamma},\quad \gamma>0 \text{ and } x\in(0,1]$$

is a monotone decreasing, positive function for the variable γ.

(iii) The function

$$\xi(\gamma, x) = \frac{\cosh \gamma(1-x)}{\cosh \gamma}, \quad \gamma > 0 \text{ and } x \in (0,1]$$

is a monotone decreasing, positive function for the variable γ.

(iv) The function

$$\eta(\gamma, x) = \frac{\sinh \gamma x}{\gamma \cosh \gamma}, \quad \gamma > 0 \text{ and } x \in (0,1]$$

is a monotone decreasing, positive function for the variable γ.

Definition 3.2 *A function* $\alpha \in C^4([0,1])$ *is said to be a lower solution of the boundary value problem (1)–(2) if*

$$L(\alpha(x)) \leq f(x, \alpha(x)) \quad \text{for} \quad x \in (0,1) \tag{12}$$

and

$$\alpha(0) \leq 0, \quad \alpha'(1) \leq 0, \quad \alpha''(0) \geq 0, \quad \alpha'''(1) \geq 0. \tag{13}$$

Similarly, an upper solution $\beta \in C^4([0,1])$ *is defined by reversing the inequalities in (12) and (13).*

Remark 3.3 *We first define an operator* $T : C([0,1]) \to C^4([0,1])$ *by*

$$T(y)(x) = \int_0^1 H(x,s) f(s, y(s)) ds, \quad x \in [0,1], \tag{14}$$

where $H(x,s)$ *is the Green's function of the linear homogeneous problem (8).*

If we let

$$h_\alpha(x) = L(\alpha(x)) - f(x, \alpha(x)), \quad h_\beta(x) = L(\beta(x)) - f(x, \beta(x)), \quad x \in [0,1],$$

then from Definition 3.2, we have

$$h_\alpha(x) \leq 0, \quad h_\beta(x) \geq 0 \quad \text{for} \quad x \in [0,1]. \tag{15}$$

Now let $u_\alpha(x)$ *be the solution of the nonhomogeneous problem*

$$\begin{cases} L(u_\alpha(x)) = 0, & x \in (0,1), \\ u_\alpha(0) = \alpha(0), \ u'_\alpha(1) = \alpha'(1), \ u''_\alpha(0) = \alpha''(0), \ u'''_\alpha(1) = \alpha'''(1). \end{cases} \tag{16}$$

Then, due to Lemma 2.1, $u_\alpha(x)$ *is uniquely determined as*

$$u_\alpha(x) = \alpha(0)\varphi(x) + \alpha'(1)\chi(x) + \alpha''(0)\psi(x) + \alpha'''(1)\omega(x), \tag{17}$$

where $\varphi(x)$, $\chi(x)$, $\psi(x)$, *and* $\omega(x)$ *are defined respectively as the unique solutions of the following nonhomogeneous problems*

$$L(\varphi(x)) = 0, \quad x \in (0,1), \qquad \varphi(0) = 1, \quad \varphi'(1) = \varphi''(0) = \varphi'''(1) = 0,$$

$$L(\chi(x)) = 0, \quad x \in (0,1), \qquad \chi'(1) = 1, \quad \chi(0) = \chi''(0) = \chi'''(1) = 0,$$

$$L(\psi(x)) = 0, \quad x \in (0,1), \qquad \psi''(0) = 1, \quad \psi(0) = \psi'(1) = \psi'''(1) = 0,$$

$$L(\omega(x)) = 0, \quad x \in (0,1), \qquad \omega'''(1) = 1, \quad \omega(0) = \omega'(1) = \omega''(0) = 0,$$

and they can be explicitly given by

$$\varphi(x) = \frac{1}{m^2 - r^2}\Big[\frac{m^2 \cosh m \cosh r(1-x) - r^2 \cosh r \cosh m(1-x)}{\cosh m \cosh r}\Big],$$

$$\chi(x) = \frac{1}{m^2 - r^2}\Big[\frac{m^3 \cosh m \sinh rx - r^3 \cosh r \sinh mx}{mr \cosh m \cosh r}\Big],$$

$$\psi(x) = \frac{1}{m^2 - r^2}\Big[\frac{\cosh r \cosh m(1-x) - \cosh m \cosh r(1-x)}{\cosh m \cosh r}\Big],$$

$$\omega(x) = \frac{1}{m^2 - r^2}\Big[\frac{r \cosh r \sinh mx - m \cosh m \sinh rx}{mr \cosh m \cosh r}\Big].$$

It follows from Lemma 3.1 that we have

$$\varphi(x) \ge 0, \qquad \chi(x) \ge 0, \qquad \psi(x) \le 0, \qquad \omega(x) \le 0, \qquad for \;\; x \in [0,1]. \tag{18}$$

Let $u_\beta(x)$ be the solution of the problem

$$\begin{cases} L(u_\beta(x)) = 0, \qquad x \in (0,1), \\ u_\beta(0) = \beta(0), \;\; u_\beta'(1) = \beta'(1), \;\; u_\beta''(0) = \beta''(0), \;\; u_\beta'''(1) = \beta'''(1). \end{cases} \tag{19}$$

Then $u_\beta(x)$ is uniquely determined as

$$u_\beta(x) = \beta(0)\varphi(x) + \beta'(1)\chi(x) + \beta''(0)\psi(x) + \beta'''(1)\omega(x). \tag{20}$$

By (18) and Definition 3.2, we have

$$u_\alpha(x) \le 0 \quad and \quad u_\beta(x) \ge 0 \quad for\ x \in [0,1]. \tag{21}$$

Hence, for a lower solution α of boundary value problem (1)–(2), it follows from Theorem 2.2, (14), (15), and (21) that

$$\begin{aligned} L(\alpha) = f(x,\alpha) + h_\alpha(x) \Rightarrow & \alpha(x) = u_\alpha(x) + \int_0^1 H(x,s) f(s,\alpha(s)) ds + \int_0^1 H(x,s) h_\alpha(s) ds \\ & \Rightarrow \alpha(x) \le T(\alpha)(x) \quad on \;\; [0,1], \end{aligned}$$

and similarly $\beta(x) \ge T(\beta)(x)$ on $[0,1]$.

The proof of our main result is based on the following important result; see [24].

Lemma 3.4 *Let X be a Banach space, $B \subset X$ be a closed and convex subset, $T : B \to B$ be a continuous map, and $T(B) = \{Tx : x \in B\}$ be precompact. Then T has at least one fixed point in B.*

Lemma 3.5 *Let there exists a constant K such that*

$$|f(x,y)| \leq K$$

for $(x,y) \in [0,1] \times \mathbb{R}$. Then the boundary value problem (1)–(2) has a solution.

Proof Denote $X = C([0,1])$. Obviously it is a Banach space equipped with the maximum norm $\|y\|_\infty = \max\limits_{x\in[0,1]} |y(x)|$. We define an operator $T : C([0,1]) \to C([0,1])$ by

$$T(\varphi)(x) = \int_0^1 H(x,s) f(s,\varphi(s))ds, \quad x \in [0,1], \tag{22}$$

where $H(x,s)$ is a Green's function of problem (8).

If we denote

$$M_1 = \max_{(x,s)\in[0,1]\times[0,1]} H(x,s) \quad \text{and} \quad M_2 = \max_{(x,s)\in[0,1]\times[0,1]} \left|\frac{\partial H(x,s)}{\partial x}\right|,$$

then it follows from (22) that $\|T\varphi\|_\infty \leq M_1 K$. Denoting

$$B =: \{\varphi \in C([0,1]) : \|\varphi\|_\infty \leq M_1 K\},$$

obviously B is a bounded, closed, and convex set in $C([0,1])$. By Theorem 2.2, problem (1)–(2) has a solution that is equivalent to T having a fixed point. Thus, now we will prove T indeed has a fixed point. First, T maps B into B, and moreover $T(B)$ is compact on the basis of the fact that $|(T\varphi)'| \leq M_2 K$ for any $\varphi \in B$. Therefore, by the Arzela–Ascoli theorem, T is a compact operator. It follows from the continuity of H and f that the operator T is continuous. Hence, by the Schauder fixed point theorem, T has a fixed point in B. □

Theorem 3.6 *Let $k_1 < k_2 < 0$. Suppose that for the problem (1)–(2) there exist a lower solution α and an upper solution β such that $\alpha(x) \leq \beta(x)$ for $x \in [0,1]$. If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies*

$$f(x,y_1) \leq f(x,y_2) \quad \textit{for} \quad \alpha(x) \leq y_1 \leq y_2 \leq \beta(x) \quad \textit{and} \quad x \in [0,1], \tag{23}$$

then there exists a solution $y(x)$ for boundary value problem (1)–(2) and it satisfies

$$\alpha(x) \leq y(x) \leq \beta(x) \quad \textit{for} \quad 0 \leq x \leq 1. \tag{24}$$

Proof Let us define a function F on $[0,1] \times \mathbb{R}$ by

$$F(x,y) = \begin{cases} f(x,\alpha(x)), & \text{for} \quad y < \alpha(x), \\ f(x,y), & \text{for} \quad \alpha(x) \leq y \leq \beta(x), \\ f(x,\beta(x)), & \text{for} \quad y > \beta(x). \end{cases}$$

Clearly, F is continuous and bounded on $[0,1] \times \mathbb{R}$. By Lemma 3.5, there exists a solution y for the boundary value problem

$$\begin{cases} L(y(x)) = F(x,y(x)), \quad x \in (0,1), \\ y(0) = y'(1) = y''(0) = y'''(1) = 0. \end{cases}$$

The following will prove that the inequality (24) is true. It follows from Definition 3.2 and (23) that

$$L(y(x)-\beta(x)) = L(y(x)) - L(\beta(x)) \leq F(x,y(x)) - f(x,\beta(x)) \leq 0 \tag{25}$$

and

$$L(y(x)-\alpha(x)) = L(y(x)) - L(\alpha(x)) \geq F(x,y(x)) - f(x,\alpha(x)) \geq 0 \tag{26}$$

for $x \in [0,1]$.

Let

$$L(y(x)-\beta(x)) := h_1(x) \leq 0, \quad L(y(x)-\alpha(x)) := h_2(x) \geq 0, \quad \text{for} \quad x \in [0,1].$$

Then, from Theorem 2.2 and (21), we have that for $x \in [0,1]$,

$$y(x) - \beta(x) = -u_\beta(x) + \int_0^1 H(x,s)h_1(s)ds \leq 0.$$

This implies that $y(x) \leq \beta(x)$ on $[0,1]$.

Similarly, from Theorem 2.2 and (21), we have that for $x \in [0,1]$,

$$y(x) - \alpha(x) = -u_\alpha(x) + \int_0^1 H(x,s)h_2(s)ds \geq 0.$$

Hence, the inequality (24) is true. □

4. Further remarks

Remark 4.1 *The problem (1)–(2) presents a particular form of the problem (5) for $n=4$ and $I=[0,1]$; however, the current result is different from that of [11]. In fact, for problem (5), if we let*

$$f(x,y,y',y''(x),y'''(x)) = f(x,y(x)) - (k_1+k_2)y''(x) - k_1k_2y(x), \quad x \in (0,1) \tag{27}$$

and

$$\begin{aligned} L_0(y,y',y'',y''',y(0)) &= y(0) = 0, \\ L_1(y,y',y'',y''',y'(0)) &= y'(1) = 0, \\ L_2(y,y',y'',y''',y''(0)) &= -y''(0) = 0, \\ L_3(y,y',y'',y''',y''(1)) &= -y'''(1) = 0, \end{aligned} \tag{28}$$

we can get that

$$\begin{cases} y^{(4)}(x) + (k_1+k_2)y''(x) + k_1k_2y(x) = f(x,y(x)), & x \in (0,1), \\ y(0) = y'(1) = y''(0) = y'''(1) = 0. \end{cases} \tag{29}$$

However, in [11, Definition 3], we let

$$\begin{aligned} v_0 &= \alpha(x), & v_1 &= \alpha'(x), \\ w_1 &= \alpha''(x), & w_2 &= \alpha'''(x), \end{aligned} \tag{30}$$

and

$$v_0 = \beta(x), \qquad v_1 = \beta'(x),$$
$$w_1 = \beta''(x), \qquad w_2 = \beta'''(x), \tag{31}$$

respectively. It follows from the definition of $\alpha_i, \beta_i : [0,1] \to \mathbb{R}, i = 0,1$ *, that if we have*

$$\alpha''(x) \leq \beta''(x), \quad on \quad [0,1], \tag{32}$$

then

$$\alpha(x), \quad \beta(x) \in [\alpha_0(x), \beta_0(x)], \quad x \in [0,1] \tag{33}$$

$$\alpha'(x), \quad \beta'(x) \in [\alpha_1(x), \beta_1(x)], \quad x \in [0,1] \tag{34}$$

and subsequently we have that (30) and (31) are well defined.

Therefore, for the functions $\alpha, \beta \in W^{4,1}(0,1)$ *satisfying (32), it follows from (30), (31), and [11, Definition 3] that the following inequalities hold for a.e.* $x \in [0,1]$ *,*

$$L(\alpha(x)) \geq f(x, \alpha(x)),$$
$$L(\beta(x)) \leq f(x, \beta(x)), \tag{35}$$

and

$$\alpha(0) \geq 0, \qquad \alpha'(1) \geq 0, \qquad \alpha''(0) \leq 0, \qquad \alpha'''(1) \leq 0. \tag{36}$$

$$\beta(0) \leq 0, \qquad \beta'(1) \leq 0, \qquad \beta''(0) \geq 0, \qquad \beta'''(1) \geq 0. \tag{37}$$

Obviously, this is different from our Definition 3.2. In our Definition 3.2, α *is a lower solution of (29) means that*

$$L(\alpha(x)) \leq f(x, \alpha(x)) \quad x \in (0,1)$$

and β *is an upper solution of (29) means that*

$$L(\beta(x)) \geq f(x, \beta(x)) \quad x \in (0,1).$$

Theorem 3.1 requires that lower solution α *and upper solution* β *be well ordered, i.e.*

$$\alpha(x) \leq \beta(x) \qquad x \in (0,1),$$

while in [11], the lower solution γ *of (29) is defined by*

$$L(\gamma(x)) \geq f(x, \gamma(x)) \quad x \in (0,1)$$

and the upper solution σ *of (29) is defined by*

$$L(\sigma(x)) \leq f(x, \sigma(x)) \quad x \in (0,1).$$

[11, Theorem 5] requires that the second derivatives of lower solution γ *and upper solution* σ *are well ordered, i.e.*

$$\gamma''(x) \leq \sigma''(x) \qquad x \in (0,1).$$

Since there are very large differences between the restrictions $\alpha(x) \leq \beta(x)$ *and* $\gamma''(x) \leq \sigma''(x)$*, the same is true for the definitions introduced in [11] and in Definition 3.2.*

Remark 4.2 *The current result is not covered by [21].*

Let us see the following example:

$$\begin{cases} y^{(4)}(x) + \big(-\pi^2 - \dfrac{\pi^2}{4}\big)y''(x) + \big((-\pi^2)\cdot(-\dfrac{\pi^2}{4})\big)y(x) = y(x) + \sin\dfrac{\pi x}{2}, \quad x \in (0,1), \\ y(0) = y'(1) = y''(0) = y'''(1) = 0. \end{cases} \tag{38}$$

Here we take $k_1 = -\pi^2$, $k_2 = -\frac{\pi^2}{4}$, *and* $f(x,y) = y(x) + \sin\frac{\pi x}{2}$.

It is easy to check that $\alpha = 0$, $\beta = \sin\frac{\pi x}{2}$ *are lower and upper solutions of (38), respectively, and* f *satisfies all of the assumptions in Theorem 3.6. Therefore, Theorem 3.6 guarantees that (38) has at least one solution* y *that satisfies*

$$0 \le y \le \sin\frac{\pi x}{2} \quad \text{for} \quad x \in [0,1]. \tag{39}$$

However, if we denote

$$g(x,y,p) = y(x) + \sin\frac{\pi x}{2} + \big(\pi^2 + \frac{\pi^2}{4}\big)p - \big((-\pi^2)\cdot(-\frac{\pi^2}{4})\big)y(x) \tag{40}$$

and rewrite (38) into the form

$$y^{(4)} = g(x,y,y''), \quad y(0) = y''(0) = y'(1) = y'''(1) = 0, \tag{41}$$

the obviously $\alpha = 0, \beta = \sin\frac{\pi x}{2}$ *are lower and upper solutions of (41); see [21, Definition 4]. However, the local monotony required by [21] does not hold, so we can not apply [21, Theorem 5] to deduce the existence of solutions of (41).*

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References

[1] Agarwal R. Boundary Value Problems for Higher Order Differential Equations. Singapore: World Scientific, 1986.

[2] Bai Z, Wang H. On positive solutions of some nonlinear fourth-order beam equations. J Math Anal Appl 2002; 270: 357-368.

[3] Cabada A, Cid J, Sanchez L. Positivity and lower and upper solutions for fourth order boundary value problems. Nonlinear Anal 2007; 67: 1599-1612.

[4] Cabada A, Enguiça R. Positive solutions of fourth order problems with clamped beam boundary conditions. Nonlinear Anal 2011; 74: 3112-3122.

[5] Cabada A, Fialho J, Minhós F. Extremal solutions to fourth order discontinuous functional boundary value problems. Math Nachr 2013; 286: 1744-1751.

[6] Chu J, O'Regan D. Positive solutions for regular and singular fourth-order boundary value problems. Commun Appl Anal 2006; 10: 185-199.

[7] Cid J, Franco D, Minhós F. Positive fixed points and fourth-order equations. Bull Lond Math Soc 2009; 41: 72-78.

[8] Coster C, Habets P. The lower and upper solutions method for boundary value problems. In: Cabada A, Drabek P, Fonda A, editors. Handbook of Differential Equations: Ordinary Differential Equations. Amsterdam, the Netherlands: Elsevier, 2004.

[9] Drábek P, Holubová G. Positive and negative solutions of one-dimensional beam equation. Appl Math Lett 2016; 51: 1-7.

[10] Drábek P, Holubová G. On the maximum and antimaximum principles for the beam equation. Appl Math Lett 2016; 56: 29-33.

[11] Fialho J, Minhós F. Higher order functional boundary value problems without monotone assumptions. Boundary Value Problems 2013; 2013: 81.

[12] Gupta C. Existence and uniqueness theorems for the bending of an elastic beam equation. Appl Anal 1988; 26: 289-304.

[13] Korman P. Uniqueness and exact multiplicity of solutions for a class of fourth-order semilinear problems. Proc Roy Soc Edinburgh Sect A 2004; 134: 179-190.

[14] Lazer A, McKenna P. Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. SIAM Rev 1990; 32: 537-578.

[15] Li Y. Positive solutions of fourth-order boundary value problems with two parameters. J Math Anal Appl 2003; 281: 477-484.

[16] Lin X, Jiang D, Lia X. Existence and uniqueness of solutions for singular fourth-order boundary value problems. J Comput Appl Math 2006; 196: 155-161.

[17] Ma R, Gao C, Han X. On linear and nonlinear fourth-order eigenvalue problems with indefinite weight. Nonlinear Anal 2011; 74: 6965-6969.

[18] Ma R, Wang H. On the existence of positive solutions of fourth-order ordinary differential equations. Appl Anal 1995; 59: 225-231.

[19] Ma R, Wang H, Elsanosi M. Spectrum of a linear fourth-order differential operator and its applications. Math Nachr 2013; 286: 1805-1819.

[20] Ma R, Zhang J, Fu S. The method of lower and upper solutions for fourth-order two-point boundary value problems. J Math Anal Appl 1997; 215: 415-422.

[21] Minhós F, Gyulov T, Santos AI. Lower and upper solutions for a fully nonlinear beam equation. Nonlinear Anal 2009; 71: 281-292.

[22] Rynne B. Infinitely many solutions of superlinear fourth order boundary value problems. Topol Methods Nonlinear Anal 2002; 19: 303-312.

[23] Schröder J. Operator Inequalities. Mathematics in Science and Engineering, Vol. 147. New York, NY, USA: Academic Press, 1980.

[24] Vrabel R. On the lower and upper solutions method for the problem of elastic beam with hinged ends. J Math Anal Appl 2015; 421: 1455-1468.