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On the growth of meromorphic solutions of some higher order linear differential equations

Farid MESBOUT, Tahar ZERZAIHI*
Department of Mathematics,
Laboratory of Pure and Applied Mathematics (LMPA),
University of Mohamed Seddik Ben Yahia, Jijel, Algeria

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Abstract: Let k, m, n be integers such that $k \geq 1$, $n \geq 2$ and $1 \leq m \leq n$. In this article we study the order $\rho(f)$ and the hyperorder $\rho_2(f)$ of nonzero meromorphic solutions f of the differential equation

$$\sum_{j=1, j\neq m}^{n} A_j(z) f^{(j)}(z) + A_m(z) e^{p_m(z)} f^{(m)}(z) + \left(A_0(z)e^{p(z)} + B_0(z)e^{q(z)}\right) f(z) = 0,$$

where $B_0(z)$, $A_0(z), \cdots, A_n(z)$ are meromorphic functions such that $A_0 A_m A_n B_0 \not\equiv 0$, $\max\{\rho(B_0), \rho(A_0), \cdots, \rho(A_n)\} < k$, and $p(z), q(z), p_m(z)$ are polynomials of degree k. Under some conditions, we show that $\rho(f) = +\infty$ and $\rho_2(f) = k$. This is an extension of some recent results by Peng, Chen, Xu, and Zhang devoted to linear differential equations of the second order.

Key words: Meromorphic functions, Nevanlinna value distribution theory, linear differential equation, order of growth

1. Introduction and preliminary results

In this article, for a meromorphic function $f(z)$ in the whole complex plane $\mathbb{C}$, we use Nevanlinna value distribution theory notations such as $T(r,f)$, $m(r,f)$, and $N(r,f)$ (see, e.g., [5, 6, 9]). We also use $\rho(f)$, $\rho_2(f)$, $\lambda(f)$, and $\lambda(\frac{1}{f})$ to denote the order of growth of $f(z)$, the hyperorder of growth of $f(z)$, the exponent of convergence of the zero sequence of $f(z)$, and the exponents of convergence of the pole sequence of $f(z)$, respectively defined by:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \ \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}, \ \text{and } \lambda(f) = \limsup_{r\to\infty} \frac{\log N(r,\frac{1}{f})}{\log r}.$$

Recently Peng and Chen [7] considered some second-order differential equations with entire coefficients of order less than 1 and obtained the following theorem:

*Correspondence: zerzaihi@yahoo.com

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Theorem A *Let $A_1(z), A_2(z)$ be nonzero entire functions such that $\max\{\rho(A_1), \rho(A_2)\} < 1$. Let a_1, a_2 be two distinct nonzero complex numbers ($|a_1| \leq |a_2|$). We suppose that $\arg a_1 \neq \pi$ or $a_1 < -1$. Then, for every nonzero meromorphic solution $f(z)$ of the equation*

$$f'' + e^{-z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0,$$

we have $\rho(f) = +\infty$ and $\rho_2(f) = 1$.

More recently, Xu and Zhang [8] extended this result to the case when the above equation has meromorphic coefficients:

Theorem B *Let $A_0(z), A_1(z), A_2(z)$ be nonzero meromorphic functions such that $\max\{\rho(A_0), \rho(A_1), \rho(A_2)\} < 1$. Let a_1, a_2 be two distinct nonzero complex numbers ($|a_1| \leq |a_2|$). Let $a_0 < 0$. If $\arg a_1 \neq \pi$ or $a_1 < a_0$, then every nonzero meromorphic solution $f(z)$ whose poles are of uniformly bounded multiplicities of the equation*

$$f'' + A_0 e^{a_0 z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0$$

satisfies that we have $\rho(f) = +\infty$ and $\rho_2(f) = 1$.

Here, we mean to extend the results above to more general higher order linear differential equations with meromorphic coefficients of finite order. More precisely, we prove the following theorem:

Theorem 1.1 *Let k, m, n be integers such that $k \geq 1$, $n \geq 2$ and $1 \leq m \leq n$. Suppose that $B_0(z), A_0(z), \cdots, A_n(z)$ are meromorphic functions such that $A_0 A_m A_n B_0 \not\equiv 0$ and $\max\{\rho(B_0), \rho(A_0), \cdots, \rho(A_n)\} = \sigma < k$. Let $p(z) = \alpha z^k + \cdots$, $q(z) = \beta z^k + \cdots$, $p_m(z) = \alpha_m z^k + \cdots$ be polynomials of degree k. Suppose that $\alpha \neq \beta$ and that at least one of the two following conditions is satisfied:*

i) at least two among the numbers α, β, α_m are of distinct arguments,

ii) $|\alpha_m| < \max\{|\alpha|, |\beta|\}$, (if $\arg\alpha = \arg\alpha_m = \arg\beta$).

Then, for every nonzero meromorphic solution $f(z)$ of the equation

$$\sum_{\substack{j=1 \\ j\neq m}}^{n} A_j(z) f^{(j)}(z) + A_m(z) e^{p_m(z)} f^{(m)}(z) + \left(A_0(z) e^{p(z)} + B_0(z) e^{q(z)}\right) f(z) = 0, \tag{1.1}$$

we have $\rho(f) = +\infty$ and $\rho_2(f) \geq k$.

Remark 1.1 *1) Our conditions on the numbers α, β, α_m in Theorem 1.1 are weaker than those of Theorems A and B.*

2) Theorem 1.1 provides sufficient (but not necessary) conditions to have the stated properties. Indeed, let us consider the equation $f'''(z) + f''(z) - e^{2z} f'(z) - (2e^z + 4e^{2z}) f(z) = 0$. It is easily checked that the function $f(z) = e^{e^z}$ is a solution of this equation and $\rho(f) = +\infty$ and $\rho_2(f) = k = 1$. However, conditions (i) and (ii) of the theorem are not fulfilled.

Corollary 1.1 *Under the assumptions of Theorem 1.1, we show that if $f(z)$ is a nonzero meromorphic solution of Equation (1.1), all of whose poles are of uniformly bounded multiplicity, then $\rho_2(f) = k$. This is, for instance, the case when f has only a finite number of poles and particularly when f is entire.*

Theorem 1.2 *Suppose that the conditions of Theorem 1.1 are satisfied. Then, for every nonzero meromorphic solution $f(z)$ of (1.1), we have*

1) $\lambda(f+h) = \lambda(\frac{1}{f+h}) = \infty$, for every nonzero meromorphic function h of finite order.

2) $\lambda(f'+h) = \lambda(\frac{1}{f'+h}) = \infty$, for every nonzero meromorphic function h of order $< k$.

3) $\lambda(f''+h) = \lambda(\frac{1}{f''+h}) = \infty$, for every nonzero meromorphic function h of order $< k$.

2. The proofs

We need the following lemmas:

Lemma 2.1 *[4] Let $f(z)$ be a meromorphic function of finite order ρ, $H = \{(k_1, j_1), (k_2, j_2), ..., (k_q, j_q)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ $(i = 1, ..., q)$, and ϵ be a positive constant. Then there exists a subset $E_1 \subset]1, \infty[$, of finite logarithmic measure such that, for every $z \in \mathbb{C}$ such that $|z| \notin [0, 1] \cup E_1$ and every $(k_i, j_i) \in H$, we have:*

$$\left| \frac{f^{(k_i)}(z)}{f^{(j_i)}(z)} \right| \leq |z|^{(k_i - j_i)(\rho - 1 + \epsilon)}.$$

Lemma 2.2 *[3] Let $f(z)$ be a meromorphic function of finite order ρ; then, for any given $\epsilon > 0$, there exists a set $E_2 \subset]1, \infty[$, of finite logarithmic measure such that, for every $z \in \mathbb{C}$ such that $|z| \notin [0, 1] \cup E_2$, we have: $|f(z)| \leq \exp\{|z|^{\rho+\epsilon}\}$.*

Lemma 2.3 *[4] Let f be a transcendental meromorphic function. Let $\lambda > 1$ be a constant, and let k and j be integers satisfying $k > j \geq 0$. Then there exist a subset $E_3 \subset]1, \infty[$ of finite logarithmic measure and a constant $C > 0$ such that for every $z \in \mathbb{C}$ such that $|z| = r \notin [0, 1] \cup E_3$, we have $\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq C \left[\frac{T(\lambda r, f)}{r} (\log r)^{\lambda} \log T(\lambda r, f) \right]^{(k-j)}$.*

Lemma 2.4 *Let $\varphi, \psi \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right[$ and let k be an integer ≥ 1. Let $E \subset \left[-\frac{\pi}{2k}, \frac{3\pi}{2k} \right[$ be a set of linear measure zero. If $\varphi \neq \psi$, then there exist infinitely many $\theta \in \left[-\frac{\pi}{2k}, \frac{3\pi}{2k} \right[\backslash E$, such that $\cos(\varphi + k\theta)\cos(\psi + k\theta) < 0$.*

Proof Without loss of generality, we may assume that $-\frac{\pi}{2} \leq \varphi < \psi < \frac{3\pi}{2}$ and we distinguish the following cases:

Case 1. Suppose that $-\frac{\pi}{2} \leq \varphi < \psi \leq \frac{\pi}{2}$. Then we have

$$0 \leq \frac{\pi}{2} - \psi < \frac{\pi}{2} - \varphi \leq \pi \leq \frac{3\pi}{2} - \psi. \tag{2.1}$$

As E is a set of linear measure zero, we see that $\left]\frac{1}{k}\left(\frac{\pi}{2}-\psi\right), \frac{1}{k}\left(\frac{\pi}{2}-\varphi\right)\right[\setminus E$ is an infinite subset of $\left]-\frac{\pi}{2k}, \frac{3\pi}{2k}\right[\setminus E$. Then, using Inequalities (2.1), we see that for every $\theta \in \left]\frac{1}{k}\left(\frac{\pi}{2}-\psi\right), \frac{1}{k}\left(\frac{\pi}{2}-\varphi\right)\right[\setminus E$, we have:

$$0 \leq \frac{\pi}{2} - \psi < k\theta < \frac{\pi}{2} - \varphi \text{ and } \frac{\pi}{2} - \psi < k\theta < \frac{\pi}{2} - \varphi \leq \frac{3\pi}{2} - \psi,$$

Hence, we have $-\frac{\pi}{2} \leq \varphi < \varphi + k\theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \psi + k\theta < \frac{3\pi}{2}$.

Therefore, in this case, we have $\cos(\varphi + k\theta) > 0$ and $\cos(\psi + k\theta) < 0$.

Case 2. Suppose that $\frac{\pi}{2} \leq \varphi < \psi < \frac{3\pi}{2}$, and let $\varphi' = \varphi - \pi$ and $\psi' = \psi - \pi$. Thus, we have $-\frac{\pi}{2} \leq \varphi' < \psi' < \frac{\pi}{2}$. From the previous case it is seen that, for every $\theta \in \left]\frac{1}{k}\left(\frac{\pi}{2}-\psi'\right), \frac{1}{k}\left(\frac{\pi}{2}-\varphi'\right)\right[\setminus E \subset \left]-\frac{\pi}{2k}, \frac{3\pi}{2k}\right[\setminus E$, we have $\cos(\varphi' + k\theta)\cos(\psi' + k\theta) < 0$. Since $\cos(\varphi' + k\theta) = -\cos(\varphi + k\theta)$ and $\cos(\psi' + k\theta) = -\cos(\psi + k\theta)$, it follows that $\cos(\varphi + k\theta)\cos(\psi + k\theta) < 0$.

Case 3. Suppose that $-\frac{\pi}{2} \leq \varphi < \frac{\pi}{2} < \psi < \frac{3\pi}{2}$. Thus, we have $0 < \frac{\pi}{2} - \varphi \leq \pi$ and $0 < \frac{3\pi}{2} - \psi < \pi$. Let $\alpha = \min\{\frac{\pi}{2} - \varphi, \frac{3\pi}{2} - \psi\}$. Then $\left]0, \frac{\alpha}{k}\right[\setminus E$ is an infinite subset of $\left]-\frac{\pi}{2k}, \frac{3\pi}{2k}\right[\setminus E$. Moreover, for every $\theta \in \left]0, \frac{\alpha}{k}\right[\setminus E$, we have $-\frac{\pi}{2} \leq \varphi < \varphi + k\theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \psi < \psi + k\theta < \frac{3\pi}{2}$. It follows that, in this case, $\cos(\varphi + k\theta) > 0$ and
$\cos(\psi + k\theta) < 0$. □

Lemma 2.5 *[1] Let $k \geq 2$ and $A_0, A_1, ..., A_{k-1}$ be meromorphic functions.*
Let $\rho = \max\{\rho(A_j), j = 0, 1, ..., k-1\}$. Let $f(z)$ be a meromorphic solution of the differential equation

$$f^{(k)} + A_{(k-1)} f^{(k-1)} + \cdots + A_0 f = 0.$$

If all poles of $f(z)$ are of uniformly bounded multiplicity, then we have $\rho_2(f) \leq \rho$.

Lemma 2.6 *[2] Let k be an integer ≥ 1 and let $A_0, A_1, ..., A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If $g(z)$ is an infinite order meromorphic solution of the equation*

$$g^{(k)} + A_{(k-1)} g^{(k-1)} + \cdots + A_0 g = F,$$

then $g(z)$ satisfies $\lambda(g) = \lambda(\frac{1}{g}) = \infty$.

Lemma 2.7 *[9] Suppose that $f_1(z), f_2(z), ..., f_{n+1}(z)$ $(n \geq 1)$ are meromorphic functions and $g_1(z), g_2(z), ..., g_n(z)$ are entire functions satisfying the following conditions:*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$.

(2) For $1 \leq j \leq n+1$, $1 \leq k \leq n$, the order of f_j is less than the order of $e^{g_k(z)}$, and furthermore the order of $f_j(z)$ is less than the order of $e^{g_h - g_k}$ for $n \geq 2$ and $1 \leq j \leq n+1$, $1 \leq h < k \leq n$

Then $f_j(z) \equiv 0$ $(j = 1, 2, ..., n+1)$.

Lemma 2.8 *[6] Let $F(r)$ and $G(r)$ be nondecreasing real-valued functions on $[0, \infty[$ such that $F(r) \leq G(r)$ for all r outside of a set $E \subset]1, \infty[$ of finite linear measure or outside of a set $H \cup [0,1]$, where $H \subset]1, \infty[$ is of finite logarithmic measure. Then, for every constant $\alpha > 1$, there exists an $r_0 > 0$ such that $F(r) \leq G(\alpha r)$ for all $r > r_0$.*

2.1. Proof of Theorem 1.1

We shall see that this theorem is an immediate consequence of the following proposition in which we are reduced to the case where the polynomials $p(z), q(z)$ and $p_m(z)$ are just monomials.:

Proposition 2.1 *Let k, m, n be integers such that $k \geq 1$, $n \geq 2$ and $1 \leq m \leq n$. Suppose that $B_0(z), A_0(z), \cdots, A_n(z)$ are meromorphic functions such that $A_0 A_m A_n B_0 \not\equiv 0$ and $\max\{\rho(B_0), \rho(A_0), \cdots, \rho(A_n)\} = \sigma < k$. Let α, β, α_m be complex numbers such that $\alpha\beta \neq 0$ and $\alpha \neq \beta$. Suppose that at least one of the two following conditions is satisfied:*

i) at least two of the numbers α, β, α_m have distinct arguments,

ii) $|\alpha_m| < \max\{|\alpha|, |\beta|\}$, (if $\arg \alpha = \arg \alpha_m = \arg \beta$).

Then, for every nonzero meromorphic solution $f(z)$ of the following equation:

$$\sum_{\substack{j=1 \\ j \neq m}}^{n} A_j(z) f^{(j)}(z) + A_m(z) e^{\alpha_m z^k} f^{(m)}(z) + \left(A_0(z) e^{\alpha z^k} + B_0(z) e^{\beta z^k}\right) f(z) = 0, \tag{2.2}$$

we have $\rho(f) = +\infty$ and $\rho_2(f) \geq k$.

Proof Let us first show that every nonzero meromorphic solution of Equation (2.2), is of infinite order. Indeed, suppose that (2.2) admits a nonzero meromorphic solution $f(z)$ of finite order. Let us set $z = re^{i\theta}$, $\alpha = |\alpha| e^{i\varphi}$, $\alpha_m = |\alpha_m| e^{i\theta_m}$, and $\beta = |\beta| e^{i\psi}$ and let $0 < \epsilon < k - \sigma$. From the hypothesis, we easily check that:

$$\max\left\{\rho\left(\frac{A_j}{A_0}\right), \rho\left(\frac{A_j}{B_0}\right),\ 0 \leq j \leq n\right\} \leq \sigma.$$

Thus, by Lemma 2.2 there exists a set $E\subset]1,\infty[$ of finite logarithmic measure such that, for every z such that $|z|=r\notin[0,1]\cup E$, we have:

$$\max\left\{\left|\frac{A_j(z)}{A_0(z)}\right|,\left|\frac{A_j(z)}{B_0(z)}\right|,\ 0\leq j\leq n\right\}\leq \exp\{r^{\sigma+\epsilon}\}, \tag{2.3}$$

and we distinguish the following cases:

1) $\varphi\neq\psi$. By Lemma 2.4, there exists $\theta\in[-\frac{\pi}{2k},\frac{\pi}{2k}[$ such that, $\cos(\varphi+k\theta)\cos(\psi+k\theta)<0$. Without loss of generality, we may suppose that $\cos(\varphi+k\theta)>0$ and $\cos(\psi+k\theta)<0$. Then, from Equation (2.2), we have

$$-e^{\alpha z^k}=\sum_{\substack{j=1\\ j\neq m}}^{n}\frac{A_j(z)f^{(j)}(z)}{A_0(z)f(z)}+e^{\alpha_m z^k}\frac{A_m(z)f^{(m)}(z)}{A_0(z)f(z)}+e^{\beta z^k}\frac{B_0(z)}{A_0(z)}, \tag{2.4}$$

$$-e^{\beta z^k}=\sum_{\substack{j=1\\ j\neq m}}^{n}\frac{A_j(z)f^{(j)}(z)}{B_0(z)f(z)}+e^{\alpha_m z^k}\frac{A_m(z)f^{(m)}(z)}{B_0(z)f(z)}+e^{\alpha z^k}\frac{A_0(z)}{B_0(z)}. \tag{2.5}$$

1.1) Suppose that $\cos(\theta_m+k\theta)\leq 0$. Then by Equations (2.3), (2.4) and Lemma 2.1 we have

$$\exp\{|\alpha|\cos(\varphi+k\theta)r^k\}\leq (n+1)\exp\left\{r^{\sigma+\epsilon}\right\}r^{n(\rho(f)-1+\epsilon)},$$

which is a contradiction with $\sigma+\epsilon<k$.

1.2) Suppose that $\cos(\theta_m+k\theta)>0$. Letting $\theta'=\theta+\dfrac{\pi}{k}$, then we have $\cos(\theta_m+k\theta')<0$ and $\cos(\psi+k\theta')>0$. By Equations (2.3) and (2.5) and Lemma 2.1, we have

$$\exp\{|\beta|\cos(\psi+k\theta')r^k\}\leq (n+1)\exp\left\{r^{\sigma+\epsilon}\right\}r^{n(\rho(f)-1+\epsilon)},$$

which is a contradiction with $\sigma+\epsilon<k$.

2) $\varphi=\psi$. Since $\alpha\neq\beta$, then $|\alpha|\neq|\beta|$. Without loss of generality, we may suppose that $|\alpha|<|\beta|$. Then we have the following subcases:

2.1) $\theta_m\neq\varphi=\psi$. Then, by Lemma 2.4, there exists $\theta\in[-\frac{\pi}{2k},\frac{\pi}{2k}[$ such that $\cos(\varphi+k\theta)\cos(\theta_m+k\theta)<0$, and we may suppose that $\cos(\varphi+k\theta)>0$. Then, by Equations (2.3) and (2.5) and Lemma 2.1, we have

$$\exp\{|\beta|\cos(\varphi+k\theta)r^k\}\leq (n+1)\exp\left\{r^{\sigma+\epsilon}\right\}r^{n(\rho(f)-1+\epsilon)}\exp\{|\alpha|\cos(\varphi+k\theta)r^k\},$$

and therefore, we have

$$\exp\{(|\beta|-|\alpha|)\cos(\varphi+k\theta)r^k\}\leq (n+1)\exp\left\{r^{\sigma+\epsilon}\right\}r^{n(\rho(f)-1+\epsilon)}.$$

This is a contradiction with $\sigma+\epsilon<k$.

2.2) $\theta_m=\varphi=\psi$. We choose $\theta\in[-\frac{\pi}{2},\frac{3\pi}{2}[$ such that $\cos(\varphi+k\theta)>0$. If $|\alpha_m|\leq|\alpha|$, then by Equations (2.3) and (2.5) and Lemma 2.1, we have

$$\exp\{(|\beta|-|\alpha|)\cos(\varphi+k\theta)r^k\}\leq (n+1)\exp\left\{r^{\sigma+\epsilon}\right\}r^{n(\rho(f)-1+\epsilon)},$$

a contradiction.

If $|\alpha| < |\alpha_m| < |\beta|$, by Equations (2.3) and (2.5) and Lemma 2.1, we have

$$\exp\{(|\beta| - |\alpha_m|)\cos(\varphi + k\theta)r^k\} \leq (n+1)\exp\left\{r^{\sigma+\epsilon}\right\} r^{n(\rho(f)-1+\epsilon)},$$

and this is a contradiction.

Let us now show that every nonzero meromorphic solution $f(z)$ of Equation (2.2) satisfies $\rho_2(f) \geq k$.

By Lemma 2.3, we know that there exists a set $E_3 \subset]1, \infty[$, with finite logarithmic measure and a constant $C > 0$ such that, for all $|z| = r \notin [0,1] \cup E_3$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C\left[\frac{T(2r,f)}{r}(\log r)^2 \log T(2r,f)\right]^j \leq C\,[T(2r,f)]^{j+1}, \quad j = 1, ..., n. \tag{2.6}$$

Using (2.6) and for the proof of the first step, we have

$$\exp\{h_1 r^k\} \leq C(n+1)\exp\{r^{\sigma+\epsilon}\}\,[T(2r,f)]^{n+1}, \tag{2.7}$$

where $h_1 > 0$ is a constant. By $h_1 > 0$, $\sigma + \epsilon < k$, (2.7), and Lemma 2.8, we know that there exists $r_0 > 0$ such that when $r > r_0$, we have $\rho_2(f) \geq k$. $\square$

We are now able to explain how to deduce Theorem 1.1 from the proposition above.

Let us set $\mathcal{A}_0(z) = A_0(z)e^{p(z)-\alpha z^k}$, $\mathcal{B}_0(z) = B_0(z)e^{q(z)-\beta z^k}$,

$\mathcal{A}_m(z) = A_m(z)e^{p_m(z)-\alpha_m z^k}$, and $\mathcal{A}_j(z) = A_j(z)$ for $j = 1, ..., n; j \neq m$.

With these notations, Equation (1.1) becomes

$$\sum_{\substack{j=1\\ j\neq m}}^{n} \mathcal{A}_j(z)f^{(j)}(z) + \mathcal{A}_m(z)e^{\alpha_m z^k} f^{(m)}(z) + \left(\mathcal{A}_0(z)e^{\alpha z^k} + \mathcal{B}_0(z)e^{\beta z^k}\right) f(z) = 0, \tag{2.8}$$

which is of the same form as Equation (2.2). Moreover, it is easy to check that the functions $\mathcal{B}_0, \mathcal{A}_0, ..., \mathcal{A}_n$ have the same proprieties as those of the functions $B_0, A_0, ..., A_n$ in Proposition (2.1). We just apply this proposition to conclude.

2.2. Proof of Corollary 1.1

Let f be a nonzero meromorphic solution of Equation (1.1) satisfying the conditions in the corollary. Then we have:

$$f^{(n)}(z) + \sum_{j=0}^{n-1} C_j(z)f^{(j)}(z) = 0,$$

where $C_j(z) = \dfrac{A_j(z)}{A_n(z)}$ for all $j \in \{1, ..., n\}\backslash\{0, m\}$, $C_0(z) = \dfrac{A_0(z)e^{p(z)} + B_0(z)e^{q(z)}}{A_n(z)}$, and $C_m(z) = \dfrac{A_m(z)e^{p_m(z)}}{A_n(z)}$.

It is clear that $\max\{\rho(C_j), 0 \leq j \leq n\} \leq k$. Hence, by Lemma 2.5, we have $\rho_2(f) \leq k$. On the other hand, applying Theorem 1.1, we obtain $\rho_2(f) \geq k$. Then $\rho_2(f) = k$.

2.3. Proof of Theorem 1.2

1) According to Theorem 1.1, we have $\rho(f)=\infty$. Putting $g_0=f+h$, we see that $\rho(g_0)=\rho(f)=\infty$ and we deduce from Equation (1.1) that:

$$g_0^{(n)}+C_{n-1}g_0^{(n-1)}+\cdots+C_0g_0=H_0, \tag{2.9}$$

where $C_0(z)=\dfrac{A_0(z)e^{p(z)}+B_0(z)e^{q(z)}}{A_n(z)}$, $C_m(z)=\dfrac{A_m(z)e^{p_m(z)}}{A_n(z)}$, $C_j(z)=\dfrac{A_j(z)}{A_n(z)}$, for $j\in\{1,...,n\}\backslash\{m\}$ and $H_0(z)=\sum_{j=1}^{n}C_jh^j$.

Now it is clear that $H_0\not\equiv 0$, because if $H_0\equiv 0$, we deduce by Theorem 1.1 that $\rho(h)=\infty$, which is a contradiction.

We also easily see that the functions $C_0(z),...,C_{n-1}(z)$ and $H_0(z)$ are of finite order. Thus, applying Lemma 2.6 to Equation (2.9), we have $\lambda(f+h)=\lambda(\dfrac{1}{f+h})=\infty$.

2) Suppose now that $\rho(h)<k$ and let us show that $\lambda(f'+h)=\lambda(\dfrac{1}{f'+h})=\infty$.

Letting $g_1=f'+h$, by derivation of both sides of (1.1), we obtain

$$\mathcal{A}_0'f+\sum_{j=1}^{n}\left(\mathcal{A}_{j-1}+\mathcal{A}_j'\right)f^{(j)}+\mathcal{A}_nf^{(n+1)}=0, \tag{2.10}$$

where $\mathcal{A}_0=A_0e^{p(z)}+B_0e^{q(z)}$, $\mathcal{A}_m=A_me^{p_m(z)}$, and $\mathcal{A}_j=A_j$, for $j\in\{1,...,n\}\backslash\{m\}$.

Multiplying (2.10) by $\mathcal{A}_0$ and (1.1) by $\mathcal{A}_0'$ and making the difference, we obtain

$$\sum_{j=0}^{n-1}\Big(\mathcal{A}_0(\mathcal{A}_j+\mathcal{A}_{j+1}')-\mathcal{A}_0'\mathcal{A}_{j+1}\Big)(f')^{(j)}+\mathcal{A}_0\mathcal{A}_n(f')^{(n)}=0, \text{ i.e.}$$

$$\Delta(f')=0, \tag{2.11}$$

where $\Delta(y)=\sum_{j=0}^{n-1}\Big(\mathcal{A}_0(\mathcal{A}_j+\mathcal{A}_{j+1}')-\mathcal{A}_0'\mathcal{A}_{j+1}\Big)y^{(j)}+\mathcal{A}_0\mathcal{A}_ny^{(n)}$.

Since $f'=g_1-h$, we obtain from (2.11):

$$\Delta(g_1)=\Delta(h). \tag{2.12}$$

We have $\Delta(h)\not\equiv 0$. Indeed, if $\Delta(h)\equiv 0$, using the fact $\mathcal{A}_0=A_0e^{p(z)}+B_0e^{q(z)}$ and $\mathcal{A}_m=A_me^{p_m(z)}$, we get

$$G_\alpha e^{\alpha z^k}+G_\beta e^{\beta z^k}+G_{\alpha+\alpha_m}e^{(\alpha+\alpha_m)z^k}+G_{\alpha_m+\beta}e^{(\alpha_m+\beta)z^k}+G_{\alpha+\beta}e^{(\alpha+\beta)z^k}$$
$$+G_{2\alpha}e^{2\alpha z^k}+G_{2\beta}e^{2\beta z^k}=0, \tag{2.13}$$

where the coefficients of Equation (2.13) are meromorphic functions of order $<k$, with $G_{2\alpha}=(A_0e^{p(z)-\alpha z^k})^2$ and $G_{2\beta}=(B_0e^{q(z)-\alpha z^k})^2$.

Using the conditions of the theorem we easily show that $2\alpha \notin \{\alpha, \beta, \alpha + \alpha_m, \alpha_m + \beta, \alpha + \beta, 2\beta\}$ or $2\beta \notin \{\alpha, \beta, \alpha + \alpha_m, \alpha_m + \beta, \alpha + \beta, 2\alpha\}$. Indeed:

If $2\alpha \in \{\alpha, \beta, \alpha + \alpha_m, \alpha_m + \beta, \alpha + \beta, 2\beta\}$, we will show that $2\beta \notin \{\alpha, \beta, \alpha + \alpha_m, \alpha_m + \beta, \alpha + \beta, 2\alpha\}$. Since $\alpha\beta(\alpha - \beta) \neq 0$, we have $2\alpha \neq \alpha, \alpha + \beta, 2\beta$ and $2\beta \neq \beta, \alpha + \beta, 2\alpha$, so we have $2\alpha \in \{\beta, \alpha + \alpha_m, \alpha_m + \beta\}$ and it is sufficient to show that $2\beta \notin \{\alpha, \alpha + \alpha_m, \alpha_m + \beta\}$.

By Lemma 2.7, we get $A_0^2 \equiv 0$ or $B_0^2 \equiv 0$, a contradiction because $A_0 B_0 \not\equiv 0$. Therefore, $\Delta(h) \not\equiv 0$. Now applying Lemma (2.6) to Equation (2.12), we obtain

$$\lambda(f' + h) = \lambda(\frac{1}{f' + h}) = \infty.$$

3) Let us now prove that $\lambda(f'' + h) = \infty$. We pose $g_2 = f'' + h$, and then $\rho(g_2) = \rho(f'') = \infty$.

By derivation of (2.10), we have

$$\mathcal{A}_n f^{(n+2)} + (2\mathcal{A}_n' + \mathcal{A}_{n-1}) f^{(n+1)} + \sum_{j=2}^{n} (\mathcal{A}_{j-2} + 2\mathcal{A}_{j-1}' + \mathcal{A}_j'') f^{(j)}$$
$$+ (2\mathcal{A}_0' + \mathcal{A}_1'') f' + \mathcal{A}_0'' f = 0. \tag{2.14}$$

Equation (1.1) enables us to express f as function of $f', f'', ..., f^{(n)}$. Then a substitution of this in Equation (2.14) gives

$$\mathcal{A}_0 \mathcal{A}_n f^{(n+2)} + \mathcal{A}_0 (2\mathcal{A}_n' + \mathcal{A}_{n-1}) f^{(n+1)} + \sum_{j=2}^{n} \Big(\mathcal{A}_0 (\mathcal{A}_{j-2} + 2\mathcal{A}_{j-1}' + \mathcal{A}_j'') - \mathcal{A}_0'' \mathcal{A}_j \Big) f^{(j)}$$
$$+ \Big(\mathcal{A}_0 (2\mathcal{A}_0' + \mathcal{A}_1'') - \mathcal{A}_0'' \mathcal{A}_1 \Big) f' = 0. \tag{2.15}$$

We put $D_0 = \mathcal{A}_0(\mathcal{A}_0 + \mathcal{A}_1') - \mathcal{A}_0' \mathcal{A}_1$ and $D_1 = \mathcal{A}_0 (2\mathcal{A}_0' + \mathcal{A}_1'') - \mathcal{A}_0'' \mathcal{A}_1$.

Multiplying (2.15) by D_0 and (2.11) by D_1 and making the difference, we have

$$\Gamma(f'') = 0, \tag{2.16}$$

where $\Gamma(y) = \mathcal{A}_0 D_0 \mathcal{A}_n y^{(n)} + \Big(\mathcal{A}_0 D_0 (2\mathcal{A}_n' + \mathcal{A}_{n-1}) - \mathcal{A}_0 D_1 \mathcal{A}_n \Big) y^{(n-1)}$
$+ \sum_{j=0}^{n-2} \Big(D_0 \left(\mathcal{A}_0 (\mathcal{A}_j + 2\mathcal{A}_{j+1}' + \mathcal{A}_{j+2}'') - \mathcal{A}_0'' \mathcal{A}_{j+2} \right) - \left(\mathcal{A}_0 (\mathcal{A}_{j+1} + \mathcal{A}_{j+2}') - \mathcal{A}_0' \mathcal{A}_{j+2} \right) D_1 \Big) y^{(j)}$. Since $f'' = g_2 - h$, we obtain from (2.16):

$$\Gamma(g_1) = \Gamma(h). \tag{2.17}$$

We have $\Gamma(h) \not\equiv 0$. Indeed, if $\Gamma(h) \equiv 0$, then

$$\frac{\Gamma(h)}{A_0} \equiv 0. \tag{2.18}$$

Let us distinguish two cases:

Case 1 If $m = 1$, replacing $\mathcal{A}_0$ and $\mathcal{A}_1$ by $A_0 e^{p(z)} + B_0 e^{q(z)}$ and $A_1 e^{p_1(z)}$ in Equation (2.18), we obtain

$$f_1(z)e^{(\alpha+\alpha_1)z^k} + f_2(z)e^{(\alpha_1+\beta)} + f_3(z)e^{2\alpha z^k} + f_4(z)e^{2\beta z^k} + f_5(z)e^{(\alpha+\beta)z^k}$$
$$+ f_6(z)e^{(2\alpha+\beta)z^k} + f_7(z)e^{(\alpha+2\beta)z^k} + f_8(z)e^{(2\alpha_1+\alpha)z^k} + f_9(z)e^{(2\alpha_1+\beta)z^k}$$
$$+ f_{10}(z)e^{(2\alpha+\alpha_1)z^k} + f_{11}(z)e^{(\alpha_1+2\beta)z^k} + f_{12}(z)e^{(\alpha+\alpha_1+\beta)z^k} + f_{13}(z)e^{3\alpha z^k} + f_{14}(z)e^{3\beta z^k} = 0,$$

where the functions $f_1, ..., f_{14}$ are all of order $< k$ and particularly $f_{13}(z) = (A_0 e^{p(z)-\alpha z^k})^3$ and $f_{14}(z) = (B_0 e^{q(z)-\beta z^k})^3$.

Let $\Omega = \{3\alpha, \alpha+\alpha_1+\beta, \alpha_1+2\beta, 2\alpha+\alpha_1, 2\alpha_1+\beta, 2\alpha_1+\alpha, \alpha+2\beta, 2\alpha+\beta, \alpha+\beta, 2\beta, 2\alpha, \alpha_1+\beta, \alpha+\alpha_1\}$. Since $\alpha \neq \beta$, we have $3\alpha \neq 3\beta, \alpha+2\beta, 2\alpha+\beta, 2\alpha$ and $3\beta \neq 3\alpha, \alpha+2\beta, 2\alpha+\beta, 2\beta$.

Setting $\Omega_1 = \{\alpha+\alpha_1+\beta, \alpha_1+2\beta, 2\alpha+\alpha_1, 2\alpha_1+\beta, 2\alpha_1+\alpha, \alpha+\beta, 2\beta, \alpha_1+\beta, \alpha+\alpha_1\}$, then we have:

If $3\alpha \notin \Omega_1$, we deduce by Lemma 2.7 that $f_{13} \equiv 0$, i.e. $A_0 \equiv 0$, which is a contradiction.

If $3\alpha \in \Omega_1$, we have $3\beta \notin \Omega$. Then by Lemma 2.7, $f_{14} \equiv 0$, i.e. $B_0 \equiv 0$, which is a contradiction also.

Therefore, $\Gamma(h) \not\equiv 0$. By Equation (2.18), since $\Gamma(h) \not\equiv 0$ and $\rho(g_2) = \infty$, according to Lemma (2.6) we have $\lambda(g_2) = \lambda(f'' + h) = \lambda(\frac{1}{f''+h}) = \infty$.

Case 2 $m > 1$.
Using the fact $\mathcal{A}_0 = A_0 e^{p(z)} + B_0 e^{q(z)}$ and $\mathcal{A}_m = A_m e^{p_m(z)}$, we obtain from Equation (2.18):

$$f_1(z)e^{(\alpha+\alpha_m)z^k} + f_2(z)e^{(\alpha_m+\beta)} + f_3(z)e^{2\alpha z^k} + f_4(z)e^{2\beta z^k} + f_5(z)e^{(\alpha+\beta)z^k}$$
$$+ f_6(z)e^{(2\alpha+\beta)z^k} + f_7(z)e^{(\alpha+2\beta)z^k} + f_8(z)e^{\alpha z^k} + f_9(z)e^{\beta z^k}$$
$$+ f_{10}(z)e^{(2\alpha+\alpha_m)z^k} + f_{11}(z)e^{(\alpha_m+2\beta)z^k} + f_{12}(z)e^{(\alpha+\alpha_m+\beta)z^k} + f_{13}(z)e^{3\alpha z^k} + f_{14}(z)e^{3\beta z^k} = 0,$$

where $f_{13}(z) = (A_0 e^{p(z)-\alpha z^k})^3$, $f_{14}(z) = (B_0 e^{q(z)-\beta z^k})^3$, and $\rho(f_j) < k$ $(j = 1, ..., 14)$. Then we conclude in the same way as in Case 1.

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