An expansion theorem for q-Sturm–Liouville operators on the whole line

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Abstract: In this work, we establish a Parseval equality and an expansion formula in eigenfunctions for a singular $q-$Sturm-Liouville operator on the whole line.

Key words: q-Sturm–Liouville operator, singular point, Parseval equality, spectral function, eigenfunction expansion

1. Introduction

Nowadays, q-calculus is quite a popular subject in mathematics because the subjects of q-calculus include number theory, quantum theory, statistical mechanics, combinatorics, quantum groups, quantum exactly solvable systems, etc. The first results in this calculus belong to Euler (1707–1783)m who introduced q-calculus in the tracks of Newton's infinite series (see [4]). Since then, it has been widely studied by many authors. Some new results in q-calculus can be found in [1,3] and the references cited therein.

On the other hand, for solving various problems in mathematics, eigenfunction expansion theorems are important because we encounter the problem of expanding an arbitrary function as a series of eigenfunctions whenever we seek a solution of a partial differential equation by the Fourier method. The eigenfunction expansion is obtained by several methods, such as the method of contour integration, the method of integral equations, and the finite difference method (see [5,6,11,13]).

In [2], Annaby et al. investigated the eigenfunction expansions for singular q-Sturm–Liouville problems on $[0, \infty)$ by using Titchmarsh's method. In the present article, we exploit the results of [2] to obtain a Parseval equality and an expansion theorem for a singular q-Sturm–Liouville operator on the whole line.

2. Preliminaries

In this section, we recall some necessary fundamental concepts of q-calculus. Following the standard notations in [1] and [9], let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R} = (-\infty, \infty)$ and $a \in A$. A q-difference equation is an equation that contains q-derivatives of a function defined on A. Let y be a complex-valued function on A. The q-difference operator D_q, the Jackson q-derivative, is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{(q-1)x} \text{ for all } x \in A.$$

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We know that there is a connection between the q-deformed Heisenberg uncertainty relation and the Jackson derivative on q-basic numbers (see [12]). In the q-derivative, as $q \to 1$, the q-derivative is reduced to the classical derivative. The q-*derivative at zero* is defined by

$$D_q y(0) = \lim_{n \to \infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on x. A *right-inverse* to D_q, the *Jackson q-integration [8]*, is given by

$$\int_0^x f(t) \, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t) \, d_q t = \int_0^b f(t) \, d_q t - \int_0^a f(t) \, d_q t \quad (a, b \in A).$$

The q-*integration* for a function is defined in [7] by the formulas

$$\int_0^{\infty} f(t) \, d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n),$$

$$\int_{-\infty}^0 f(t) \, d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n f(-q^n),$$

$$\int_{-\infty}^{\infty} f(t) \, d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n [f(q^n) + f(-q^n)].$$

A function f that is defined on A, $0 \in A$, is said to be *q-regular at zero* if

$$\lim_{n \to \infty} f(xq^n) = f(0)$$

for every $x \in A$. Through the remainder of the paper, we deal only with functions that are q-regular at zero.

If f and g are q-regular at zero, then we have

$$\int_0^a g(t) D_q f(t) \, d_q t - \int_0^a f(qt) D_q g(t) \, d_q t = f(a) g(a) - f(0) g(0).$$

Let $L_q^2(\mathbb{R})$ be the space of all real-valued functions defined on $\mathbb{R}$ such that

$$\|f\| := \left(\int_{-\infty}^{\infty} f^2(x) \, d_q x \right)^{1/2} < \infty.$$

The space $L_q^2(\mathbb{R})$ is a separable Hilbert space with the inner product

$$(f, g) := \int_{-\infty}^{\infty} f(x) g(x) \, d_q x, \quad f, g \in L_q^2(\mathbb{R})$$

(see [2]).

The q-Wronskian of $y(x)$ and $z(x)$ is defined to be

$$W_q(y,z)(x) := y(x) D_q z(x) - z(x) D_q y(x), \quad x \in \mathbb{R}. \tag{1}$$

3. Main results

Let us consider the q-Sturm–Liouville problem

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + u(x) y(x) = \lambda y(x), \tag{2}$$

where λ is a complex eigenvalue parameter, u is a real-valued function defined on $\mathbb{R}$ and continuous at zero, and $u \in L^1_{q,loc}(\mathbb{R})$.

We will denote by $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ the solutions of the equation (2) that satisfy the initial conditions

$$\phi_1(0, \lambda) = 1, \ D_{q^{-1}} \phi_1(0, \lambda) = 0, \ \phi_2(0, \lambda) = 0, \ D_{q^{-1}} \phi_2(0, \lambda) = 1. \tag{3}$$

Let $[-q^{-m}, q^{-m}]$ be an arbitrary finite interval where $m \in \mathbb{N}$.

Now we will consider the boundary value problem (2) with the boundary conditions

$$\begin{aligned}
D_{q^{-1}} y\left(-q^{-m}\right) \cos \alpha + y_1\left(-q^{-m}\right) \sin \alpha &= 0, \\
D_{q^{-1}} y_2\left(q^{-m}\right) \cos \beta + y_1\left(q^{-m}\right) \sin \beta &= 0, \ \alpha, \beta \in \mathbb{R}, \ m \in \mathbb{N}.
\end{aligned} \tag{4}$$

Let $\lambda_0, \lambda_{\pm 1}, \lambda_{\pm 2}, \ldots$ be the eigenvalues and $y_0, y_{\pm 1}, y_{\pm 2}, \ldots$ the corresponding eigenfunctions of the problem (2)–(4). Since the solutions of this problem are linearly independent, we get

$$y_n(x) = c_n \phi_1(x, \lambda_n) + d_n \phi_2(x, \lambda_n).$$

There is no loss of generality in assuming that $|c_n| \leq 1$ and $|d_n| \leq 1$. Set

$$z_n^2 = \int_{-q^{-m}}^{q^{-m}} y_n^2(x) d_q x.$$

Let $f(.) \in L_q^2(-q^{-m}, q^{-m})$. If we apply the Parseval equality (see [2]) to $f(x)$, then we obtain

$$\int_{-q^{-m}}^{q^{-m}} f^2(x) d_q x = \sum_{m=-\infty}^{\infty} \frac{1}{z_n^2} \left\{ \int_{-q^{-m}}^{q^{-m}} f(x) y_n(x) d_q x \right\}^2$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{z_n^2} \left\{ \int_{-q^{-m}}^{q^{-m}} f(x) c_n \phi_1(x, \lambda_n) + d_n \phi_2(x, \lambda_n) \, d_q x \right\}^2$$

$$= \sum_{m=-\infty}^{\infty} \frac{c_n^2}{z_n^2} \left\{ \int_{-q^{-m}}^{q^{-m}} f(x) \phi_1(x, \lambda_n) \, d_q x \right\}^2 \tag{5}$$

$$+2 \sum_{m=-\infty}^{\infty} \frac{c_n d_n}{z_n^2} \prod_{j=1}^{2} \left\{ \int_{-q^{-m}}^{q^{-m}} f(x) \phi_j(x, \lambda_n) \, d_q x \right\}$$

$$+ \sum_{m=-\infty}^{\infty} \frac{d_n^2}{z_n^2} \left\{ \int_{-q^{-m}}^{q^{-m}} f(x) \phi_2(x, \lambda_n) \, d_q x \right\}^2 .$$

Now we will define the step function $\mu_{ij,q^{-m}}$ $(i,j = 1, 2)$ on $(-q^{-m}, q^{-m})$ by

$$\mu_{11,q^{-m}}(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_n < 0} \frac{c_n^2}{z_n^2}, & \text{for } \lambda \leq 0 \\ \sum_{0 \leq \lambda_n < \lambda} \frac{c_n^2}{z_n^2} & \text{for } \lambda > 0 \end{cases} ,$$

$$\mu_{12,q^{-m}}(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_n < 0} \frac{c_n d_n}{z_n^2}, & \text{for } \lambda \leq 0 \\ \sum_{0 \leq \lambda_n < \lambda} \frac{c_n d_n}{z_n^2} & \text{for } \lambda > 0 \end{cases} ,$$

$$\mu_{12,q^{-m}}(\lambda) = \mu_{21,q^{-m}}(\lambda),$$

$$\mu_{22,q^{-m}}(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_n < 0} \frac{d_n^2}{z_n^2}, & \text{for } \lambda \leq 0 \\ \sum_{0 \leq \lambda_n < \lambda} \frac{d_n^2}{z_n^2} & \text{for } \lambda > 0. \end{cases}$$

From (5), we obtain

$$\int_{-q^{-m}}^{q^{-m}} f^2(x) \, d_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda) \, d\mu_{ij,q^{-m}}(\lambda), \tag{6}$$

where

$$F_i(\lambda) = \int_{-q^{-m}}^{q^{-m}} f(x) \phi_i(x, \lambda) \, d_q x \ (i = 1, 2).$$

Now we will prove a lemma.

Lemma 1 *There exists a positive constant* $\Lambda = \Lambda(\xi), \ \xi > 0$ *such that*

$$\bigvee_{-\xi}^{\xi} \left\{ \mu_{ij,q^{-m}}(\lambda) \right\} < \Lambda \ (i, j = 1, 2), \tag{7}$$

where Λ *does not depend on* q^{-m}.

Proof From (3), we have

$$D_{q^{-1}}^{(j-1)} \phi_i (0, \lambda) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Thus, there exists a $k > 0$ such that

$$\left| D_{q^{-1}}^{(j-1)} \phi_{ij} (0, \lambda) - \delta_{ij} \right| < \varepsilon, \ \varepsilon > 0, \ |\lambda| < \xi, \ x \in [0, k]. \tag{8}$$

Let $f_k (x)$ be a nonnegative function such that $f_k (x)$ vanishes outside the interval $[0, k]$ with

$$\int_0^k f_k (x) \, d_q x = 1. \tag{9}$$

Now, if we apply the Parseval equality (6) to $D_{q^{-1}}^{(h-1)} f_k (x) \ (h = 1, 2)$, then we get

$$\int_0^k \left| D_{q^{-1}}^{(h-1)} f_k (x) \right|^2 d_q x = \int_{-\xi}^{\xi} \sum_{i,j=1}^2 F_{ih} (\lambda) F_{jh} (\lambda) \, d\mu_{ij,q^{-m}} (\lambda),$$

where

$$F_{ih} (\lambda) = \int_0^k D_{q^{-1}}^{(h-1)} f_k (x) \phi_i (x, \lambda) \, d_q x = \pm \int_0^k f_k (x) D_{q^{-1}}^{(h-1)} \phi_i (x, \lambda) \, d_q x.$$

Using (8) and (9), we obtain

$$|F_{ih} (\lambda) - \delta_{ih}| < \varepsilon, \ i, h = 1, 2, \ |\lambda| < \xi. \tag{10}$$

Now, if we apply the Parseval equality (6) to $f_k (x) \ (h = 1, 2)$, then we get

$$\int_0^k \left| D_{q^{-1}}^{(h-1)} f_k (x) \right|^2 d_q x \geq \int_{-\xi}^{\xi} \sum_{i,j=1}^2 (\delta_{ih} - \varepsilon)(\delta_{jh} - \varepsilon) \left| d\mu_{ij,q^{-m}} (\lambda) \right|. \tag{11}$$

If we take $h = 1$ in (11), we have

$$\int_0^k f_k^2 (x) \, d_q x$$

$$> \ (1 - \varepsilon)^2 \int_{-\xi}^{\xi} \left| d\mu_{11,q^{-m}} (\lambda) \right| + \varepsilon (1 + \varepsilon) \int_{-\xi}^{\xi} \left| d\mu_{12,q^{-m}} (\lambda) \right|$$

$$+ \varepsilon (1 + \varepsilon) \int_{-\xi}^{\xi} \left| d\mu_{21,q^{-m}} (\lambda) \right| + \varepsilon^2 \int_{-\xi}^{\xi} \left| d\mu_{22,q^{-m}} (\lambda) \right|$$

$$= \ (1 - \varepsilon)^2 \left(\mu_{11,q^{-m}} (\xi) - \mu_{11,q^{-m}} (-\xi) \right)$$

$$+ 2\varepsilon (1 + \varepsilon) \bigvee_{-\xi}^{\xi} \left\{ \mu_{12,q^{-m}} (\lambda) \right\} + \varepsilon^2 \left(\mu_{22,q^{-m}} (\xi) - \mu_{22,q^{-m}} (-\xi) \right).$$

Since

$$\bigvee_{-\xi}^{\xi} \left\{ \mu_{12,q^{-m}} (\lambda) \right\} \leq \frac{1}{2} \left[\mu_{11,q^{-m}} (\xi) - \mu_{11,q^{-m}} (-\xi) + \mu_{22,q^{-m}} (\xi) - \mu_{22,q^{-m}} (-\xi) \right], \tag{12}$$

we get

$$\int_0^k f_k^2\left(x\right) d_q x \quad > \quad \left(2\varepsilon^2 - 3\varepsilon + 1\right)\left\{\mu_{11,q^{-m}}\left(\xi\right) - \mu_{11,q^{-m}}\left(-\xi\right)\right\}$$
$$+ \varepsilon\left(2\varepsilon - 1\right)\left\{\mu_{22,q^{-m}}\left(\xi\right) - \mu_{22,q^{-m}}\left(-\xi\right)\right\}. \tag{13}$$

Putting $h = 2$ in (11), we get

$$\int_0^k \left|D_{q^{-1}} f_k\left(x\right)\right|^2 d_q x \quad \geq \quad \left(2\varepsilon^2 - 3\varepsilon + 1\right)\left\{\mu_{22,q^{-m}}\left(\xi\right) - \mu_{22,q^{-m}}\left(-\xi\right)\right\}$$
$$+ \varepsilon\left(2\varepsilon - 1\right)\left\{\mu_{11,q^{-m}}\left(\xi\right) - \mu_{11,q^{-m}}\left(-\xi\right)\right\}. \tag{14}$$

If we add the inequalities (13) and (14), then we get

$$\int_0^k f_k^2\left(x\right) d_q x + \int_0^k \left|D_{q^{-1}} f_k\left(x\right)\right|^2 d_q x$$
$$\geq \quad \left(2\varepsilon - 1\right)^2 \left\{ \begin{array}{c} \mu_{11,q^{-m}}\left(\xi\right) - \mu_{11,q^{-m}}\left(-\xi\right) \\ + \mu_{22,q^{-m}}\left(\xi\right) - \mu_{22,q^{-m}}\left(-\xi\right) \end{array} \right\}.$$

Hence, we obtain the assertion of the lemma for the functions $\mu_{11,q^{-m}}\left(-\xi\right)$ and $\mu_{22,q^{-m}}\left(-\xi\right)$ relying on their monotonicity. From (12), we get the assertion of the lemma for the function $\mu_{12,q^{-m}}\left(-\xi\right)$. $\qquad\square$

Now we recall the following theorems of Helly.

Theorem 2 ([10]) *Let $\left(w_n\right)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real nondecreasing functions on a finite interval $a \leq \lambda \leq b$. Then there exists a subsequence $\left(w_{n_k}\right)_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty} w_{n_k}\left(\lambda\right) = w\left(\lambda\right), \ a \leq \lambda \leq b.$$

Theorem 3 ([10]) *Assume that $\left(w_n\right)_{n\in\mathbb{N}}$ is a real, uniformly bounded sequence of nondecreasing functions on a finite interval $a \leq \lambda \leq b$, and suppose*

$$\lim_{n\to\infty} w_n\left(\lambda\right) = w\left(\lambda\right), \ a \leq \lambda \leq b.$$

If f is any continuous function on $a \leq \lambda \leq b$, then

$$\lim_{n\to\infty} \int_a^b f\left(\lambda\right) dw_n\left(\lambda\right) = \int_a^b f\left(\lambda\right) dw\left(\lambda\right).$$

Let ϱ be any nondecreasing function on $-\infty < \lambda < \infty$. Denote by $L_\varrho^2\left(\mathbb{R}\right)$ the Hilbert space of all functions $f: \mathbb{R} \to \mathbb{R}$ that are measurable with respect to the Lebesgue–Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^\infty f^2\left(\lambda\right) d\varrho\left(\lambda\right) < \infty,$$

with the inner product

$$(f,g)_\varrho := \int_{-\infty}^{\infty} f(\lambda) g(\lambda) \, d\varrho(\lambda).$$

The main results of this paper are the following three theorems.

Theorem 4 *Let $f(.) \in L_q^2(\mathbb{R})$. Then there exist monotonic functions $\mu_{11}(\lambda)$ and $\mu_{22}(\lambda)$, which are bounded over every finite interval, and a function $\mu_{12}(\lambda)$, which is of bounded variation over every finite interval with the property*

$$\int_{-\infty}^{\infty} f^2(x) \, d_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda) \, d\mu_{ij}(\lambda), \tag{15}$$

where

$$F_i(\lambda) = \lim_{n \to \infty} \int_{-q^{-n}}^{q^{-n}} f(x) \phi_i(x,\lambda) \, d_q x.$$

We note that the function $\mu = (\mu_{ij})_{i,j=1}^2$ $(\mu_{12} = \mu_{21})$ is called a *spectral function* for the equation (2).

Proof   Assume that the function $f_n(x)$ satisfies the following conditions:

1) $f_n(x)$ vanishes outside the interval $[-q^{-n}, q^{-n}]$, $q^{-n} < q^{-m}$.

2) The functions $f_n(x)$ and $D_q f_n(x)$ are q-regular at zero.

3) $f_n(x)$ satisfies the boundary conditions (4).

If we apply the Parseval equality to $f_n(x)$, we get

$$\int_{-q^{-n}}^{q^{-n}} f_n^2(x) \, d_q x = \sum_{k=-\infty}^{\infty} \frac{1}{z_k^2} \left\{ \int_{-q^{-m}}^{q^{-m}} f_n(x) y_k(x) \, d_q x \right\}^2. \tag{16}$$

Then, by integrating by parts, we obtain

$$\int_{-q^{-m}}^{q^{-m}} f_n(x) y_k(x) \, d_q x$$

$$= \frac{1}{\lambda_k} \int_{-q^{-m}}^{q^{-m}} f_n(x) \left[-\frac{1}{q} D_{q^{-1}} D_q y_k(x) + u(x) y_k(x) \right] d_q x$$

$$= \frac{1}{\lambda_k} \int_{-q^{-m}}^{q^{-m}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_n(x) + u(x) f_n(x) \right] y_k(x) \, d_q x.$$

Thus, we have

$$\sum_{|\lambda_k| \geq s} \frac{1}{z_k^2} \left\{ \int_{-q^{-m}}^{q^{-m}} f_n(x) y_k(x) \, d_q x \right\}^2$$

$$\leq \frac{1}{s^2} \sum_{|\lambda_k| \geq s} \frac{1}{z_k^2} \left\{ \int_{-q^{-m}}^{q^{-m}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_n(x) + u(x) f_n(x) \right] y_k(x) d_q x \right\}^2$$

$$\leq \frac{1}{s^2} \sum_{k=-\infty}^{\infty} \frac{1}{z_k^2} \left\{ \int_{-q^{-m}}^{q^{-m}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_n(x) + u(x) f_n(x) \right] y_k(x) d_q x \right\}^2$$

$$= \frac{1}{s^2} \int_{-q^{-n}}^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_n(x) + u(x) f_n(x) \right]^2 d_q x.$$

Using (16), we obtain

$$\left| \int_{-q^{-n}}^{q^{-n}} f_n(x) y_k(x) d_q x - \sum_{-\kappa \leq \lambda_k \leq \kappa} \frac{1}{z_k^2} \left\{ \int_{-q^{-m}}^{q^{-m}} f_n(x) y_k(x) d_q x \right\}^2 \right|$$

$$\leq \frac{1}{s^2} \int_{-q^{-n}}^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_n(x) + u(x) f_n(x) \right]^2 d_q x.$$

Furthermore, we have

$$\sum_{-\kappa \leq \lambda_k \leq \kappa} \left\{ \int_{-q^{-m}}^{q^{-m}} f_n(x) y_k(x) d_q x \right\}^2$$

$$= \sum_{-\kappa \leq \lambda_k \leq \kappa} \left\{ \int_{-q^{-m}}^{q^{-m}} f_n(x) \{ c_k \phi_1(x, \lambda_k) + d_k \phi_2(x, \lambda_k) \} d_q x \right\}^2$$

$$= \int_{-\kappa}^{\kappa} \sum_{i,j=1}^{2} F_{in}(\lambda) F_{jn}(\lambda) d\mu_{ij,q^{-m}}(\lambda),$$

where

$$F_{in}(\lambda) = \int_{-q^{-m}}^{q^{-m}} f_n(x) \phi_i(x, \lambda) d_q x \quad (i = 1, 2).$$

Consequently, we get

$$\left| \int_{-q^{-n}}^{q^{-n}} f_n^2(x) d_q x - \int_{-\kappa}^{\kappa} \sum_{i,j=1}^{2} F_{in}(\lambda) F_{jn}(\lambda) d\mu_{ij,q^{-m}}(\lambda) \right|$$

$$\leq \frac{1}{\kappa^2} \int_{-q^{-n}}^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_n(x) + u(x) f_n(x) \right]^2 d_q x. \tag{17}$$

By Lemma 1 and Theorems 2 and 3, we can find sequences $\{-q^{-m_k}\}$ and $\{q^{-m_k}\}$ such that the function $\mu_{ij,q^{-m}}(\lambda)$ converges to a monotone function $\mu_{ij}(\lambda)$. Passing to the limit with respect to $\{-q^{-m_k}\}$ and

$\{q^{-m_k}\}$ in (17), we have

$$\left| \int_{-q^{-n}}^{q^{-n}} f_n^2(x) \, d_q x - \int_{-\kappa}^{\kappa} \sum_{i,j=1}^{2} F_{in}(\lambda) F_{jn}(\lambda) \, d\mu_{ij}(\lambda) \right|$$

$$\leq \frac{1}{\kappa^2} \int_{-q^{-n}}^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} D_q f_n(x) + u(x) f_n(x) \right]^2 d_q x.$$

As $\kappa \to \infty$, we get

$$\int_{-q^{-n}}^{q^{-n}} f_n^2(x) \, d_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{in}(\lambda) F_{jn}(\lambda) \, d\mu_{ij}(\lambda).$$

Let $f(.) \in L_q^2(\mathbb{R})$. Choose functions $\{f_\eta(x)\}$ satisfying conditions 1–3 and such that

$$\lim_{\eta \to \infty} \int_{-\infty}^{\infty} (f(x) - f_\eta(x))^2 \, d_q x = 0.$$

Let

$$F_{i\eta}(\lambda) = \int_{-\infty}^{\infty} f_\eta(x) \phi_i(x, \lambda) \, d_q x \quad (i = 1, 2).$$

Then we have

$$\int_{-\infty}^{\infty} f_\eta^2(x) \, d_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{i\eta}(\lambda) F_{j\eta}(\lambda) \, d\mu_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{\infty} (f_{\eta_1}(x) - f_{\eta_2}(x))^2 \, d_q x \to 0 \text{ as } \eta_1, \eta_2 \to \infty,$$

we have

$$\int_{-\infty}^{\infty} \sum_{i=1}^{2} (F_{i\eta_1}(\lambda) F_{j\eta_1}(\lambda) - F_{i\eta_2}(\lambda) F_{j\eta_2}(\lambda)) \, d\mu_{ij}(\lambda)$$

$$= \int_{-\infty}^{\infty} (f_{\eta_1}(x) - f_{\eta_2}(x))^2 \, d_q x \to 0$$

as $\eta_1, \eta_2 \to \infty$. Therefore, there is a limit function F that satisfies

$$\int_{-\infty}^{\infty} f^2(x) \, d_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda) \, d\mu_{ij}(\lambda),$$

by the completeness of the space $L_\mu^2(\mathbb{R})$.

Now we will show that the sequence (K_η) defined by

$$K_\eta(\lambda) = \int_{-q^{-\eta}}^{q^{-\eta}} \{f_1(x) \phi_1(x, \lambda) + f_2(x) \phi_2(x, \lambda)\} \, d_q x$$

converges as $\eta \to \infty$ to F in the metric of space $L_\mu^2(\mathbb{R})$. Let g be another function in $L_q^2(\mathbb{R})$. By a similar argument, $G(\lambda)$ be defined by g.

It is obvious that

$$\int_0^\infty (f(x) - g(x))^2 \, d_q x$$

$$= \int_{-\infty}^\infty \sum_{i,j=1}^2 \left\{ (F_i(\lambda) - G_i(\lambda))(F_j(\lambda) - G_j(\lambda)) \right\} d\mu_{ij}(\lambda).$$

Let

$$g(x) = \begin{cases} f(x), & x \in [-q^{-\eta}, q^{-\eta}] \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\int_{-\infty}^\infty \sum_{i,j=1}^2 \left\{ (F_i(\lambda) - K_{\eta i}(\lambda))(F_j(\lambda) - K_{\eta j}(\lambda)) \right\} d\mu_{ij}(\lambda)$$

$$= \int_{-\infty}^{-q^{-\eta}} f^2(x) \, d_q x + \int_{q^{-\eta}}^\infty f^2(x) \, d_q x \to 0 \ (\eta \to \infty),$$

which proves that (K_η) converges to F in $L_\mu^2(\mathbb{R})$ as $\eta \to \infty$. □

Theorem 5 *Suppose that the functions $f(.)$ and $g(.)$ are in $L_q^2(\mathbb{R})$, and $F(\lambda)$ and $G(\lambda)$ are their Fourier transforms. Then we have*

$$\int_{-\infty}^\infty f(x) g(x) \, d_q x = \int_{-\infty}^\infty \sum_{i,j=1}^2 F_i(\lambda) G_j(\lambda) \, d\mu_{ij}(\lambda),$$

which is called the generalized Parseval equality.

Proof It is clear that $F \mp G$ are transforms of $f \mp g$. Therefore, we have

$$\int_{-\infty}^\infty (f(x) + g(x))^2 \, d_q x$$

$$= \int_{-\infty}^\infty \sum_{i,j=1}^2 (F_i(\lambda) + G_i(\lambda))(F_j(\lambda) + G_j(\lambda)) \, d\mu_{ij}(\lambda),$$

$$\int_{-\infty}^\infty (f(x) - g(x))^2 \, d_q x$$

$$= \int_{-\infty}^\infty \sum_{i,j=1}^2 (F_i(\lambda) - G_i(\lambda))(F_j(\lambda) - G_j(\lambda)) \, d\mu_{ij}(\lambda).$$

Subtracting one of these equalities from the other one, we get the desired result. □

Theorem 6 *Let $f(.) \in L_q^2(\mathbb{R})$. Then the integrals*

$$\int_{-\infty}^{\infty} F_i(\lambda)\,\phi_j(x,\lambda)\,d\mu_{ij}(\lambda) \ \ (i,j=1,2)$$

converge in $L_q^2(\mathbb{R})$. Consequently, we have

$$f(x) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\,\phi_j(x,\lambda)\,d\mu_{ij}(\lambda),$$

which is called the expansion theorem.

Proof Take any function $f_s \in L_q^2(\mathbb{R})$ and any positive number s, and set

$$f_s(x) = \int_{-s}^{s} \sum_{i,j=1}^{2} F_i(\lambda)\,\phi_j(x,\lambda)\,d\mu_{ij}(\lambda).$$

Let $g(.) \in L_q^2(\mathbb{R})$ be a vector function that equals zero outside the finite interval $[-q^{-\tau}, q^{-\tau}]$ where $q^{-\tau} < q^{-m}$. Thus, we obtain

$$\int_{-q^{-\tau}}^{q^{-\tau}} f_s(x)\,g(x)\,d_q x =$$

$$\int_{-q^{-\tau}}^{q^{-\tau}} \left(\int_{-s}^{s} \sum_{i,j=1}^{2} F_i(\lambda)\,\phi_j(x,\lambda)\,d\mu_{ij}(\lambda) \right) g(x)\,d_q x$$

$$= \int_{-s}^{s} \sum_{i,j=1}^{2} F_i(\lambda) \left\{ \int_{-q^{-\tau}}^{q^{-\tau}} g(x)\,\phi_j(x,\lambda)\,d_q x \right\} d\mu_{ij}(\lambda)$$

$$= \int_{-s}^{s} \sum_{i,j=1}^{2} F_i(\lambda)\,G_j(\lambda)\,d\mu_{ij}(\lambda). \tag{18}$$

From Theorem 5, we get

$$\int_{-\infty}^{\infty} f(x)\,g(x)\,d_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\,G_j(\lambda)\,d\mu_{ij}(\lambda). \tag{19}$$

By (18) and (19), we have

$$\int_{-\infty}^{\infty} (f(x) - f_s(x))\,g(x)\,d_q x = \int_{|\lambda|>s} \sum_{i,j=1}^{2} F_i(\lambda)\,G_j(\lambda)\,d\mu_{ij}(\lambda).$$

Using the Cauchy–Schwarz inequality, we obtain

$$\int_{-\infty}^{\infty} \left(f\left(x \right) - f_s \left(x \right) \right) g\left(x \right) d_q x$$

$$\leq \sum_{i,j=1}^{2} \sqrt{\int_{|\lambda|>s} F_i^2 \left(\lambda \right) d\mu_{ij} \left(\lambda \right)} \sqrt{\int_{|\lambda|>s} G_j^2 \left(\lambda \right) d\mu_{ij} \left(\lambda \right)}.$$

If we apply this inequality to the function

$$g\left(x \right) = \left\{ \begin{array}{ll} f\left(x \right) - f_s \left(x \right), & x \in \left[-q^{-s}, q^{-s} \right] \\ 0, & \text{otherwise}, \end{array} \right.$$

then we get

$$\int_{-\infty}^{\infty} \left(f\left(x \right) - f_s \left(x \right) \right)^2 d_q x \leq \sum_{i,j=1}^{2} \int_{|\lambda|>s} F_i \left(\lambda \right) F_j \left(\lambda \right) d\mu_{ij} \left(\lambda \right).$$

Letting $s \to \infty$ yields the expansion result. □

References

[1] Annaby MH, Mansour ZS. q-Fractional Calculus and Equations. Lecture Notes in Mathematics, Vol 2056. Heidelberg, Germany: Springer-Verlag, 2012.

[2] Annaby MH, Mansour ZS, Soliman IA. q-Titchmarsh-Weyl theory: series expansion. Nagoya Math J 2012; 205: 67-118.

[3] Ernst T. The History of q-Calculus and a New Method. U. U. D. M. Report. Uppsala, Sweden: Department of Mathematics of Uppsala University, 2000.

[4] Euler L. Introductio in Analysin Infinitorum, Vol. 1. Lausanne, Switzerland: Bousquet, 1748 (in Latin).

[5] Guseinov GS. Eigenfunction expansions for a Sturm-Liouville problem on time scales. Int J Difference Equ 2007; 2: 93-104.

[6] Guseinov GS. An expansion theorem for a Sturm-Liouville operator on semi-unbounded time scales. Adv Dyn Syst Appl 2008; 3: 147-160.

[7] Hahn W. Beiträge zur Theorie der Heineschen Reihen. Math Nachr 1949; 2: 340-379 (in German).

[8] Jackson FH. On q-definite integrals. Quart J Pure Appl Math 1910; 41: 193-203.

[9] Kac V, Cheung P. Quantum Calculus. New York, NY, USA: Springer-Verlag, 2002.

[10] Kolmogorov AN, Fomin SV, Introductory Real Analysis. Translated by R.A. Silverman. New York, NY, USA: Dover Publications, 1970.

[11] Levitan BM, Sargsjan IS. Sturm-Liouville and Dirac Operators. Mathematics and its Applications (Soviet Series). Dordrecht, the Netherlands: Kluwer Academic Publishers Group, 1991.

[12] Swamy PN. Deformed Heisenberg algebra: origin of q-calculus. Physica A 2003; 328: 1-2 . 145–153.

[13] Titchmarsh EC. Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I. 2nd ed. Oxford, UK: Clarendon Press, 1962.