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On the existence and uniqueness of solutions to dynamic equations

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Abstract: In this paper, we prove the well-known Cauchy–Peano theorem for existence of solutions to dynamic equations on time scales. Some simple examples are given to show that there may exist more than a single solution for dynamic initial value problems. Under some certain conditions, it is also shown that there exists only one solution.

Key words: Cauchy–Peano theorem, Picard–Lidelof theorem, dynamic equations, time scales

1. Introduction

Let D be some region in $\mathbb{R}\times\mathbb{R}$, $f : D \to \mathbb{R}$, $(a,\alpha) \in D$, and consider the differential initial value problem

$$\begin{cases} y' = f(t,y) \\ y(a) = \alpha. \end{cases} \tag{1}$$

The most important question in the theory of differential equations is the following.

(Q1) Under what conditions does there exist a solution of (1)?

The so-called Cauchy–Peano theorem, named after Giuseppe Peano and Augustin Louis Cauchy, is a fundamental theorem in the theory of ordinary differential equations, which delivers an answer for (Q1). Let us continue by presenting the theorem.

Cauchy–Peano Theorem ([9, Theorem 8.27])**.** *If the function f is continuous in D, then the initial value problem (1) has at least one solution defined in some neighborhood of a.*

In [10], Peano first published the theorem with an incorrect proof, and in [11], a new correct proof (for systems of equations) was presented by using successive approximations. The Cauchy–Peano theorem received much attention and now there are many different proofs (see for instance, [12]). The proof techniques can be collected into two groups. The first group uses approximation by sequences of function (such as Euler–Cauchy polygons or Tonelli sequences), while the second one uses fixed point theorems (mainly Schauder's fixed point theorem) for the corresponding integral equation.

The second most important question in the theory of differential equations is the following.

(Q2) Under what conditions does there exist a unique solution of (1)?

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The so-called Picard–Lindelöf theorem (also known as the Cauchy–Lipschitz theorem), which is named after Émile Picard, Ernst Lindelöf, Rudolf Lipschitz, and Augustin-Louis Cauchy, delivers an answer for (Q2). This fundamental theorem is presented below.

Picard–Lindelöf Theorem ([9, Theorem 8.13])**.** *If the function f is continuous in D and is Lipchitz continuous in its second component in D, i.e. $|f(t,y)-f(t,z)| \leq L|y-z|$ for all $(t,y),(t,z) \in D$, where $L>0$, then the initial value problem (1) has a unique solution defined in some neighborhood of a.*

In this paper, we are concerned with the dynamic initial value problem

$$\begin{cases} y^{\Delta} = f(t,y) \\ y(a) = \alpha, \end{cases} \tag{2}$$

where $\mathbb{T}$ is a time scale (a nonempty closed subset of reals), $a,b \in \mathbb{T}$ with $b>a$, $\alpha \in \mathbb{R}$ and $f:[a,b]\cap\mathbb{T}\times I \to \mathbb{R}$ for some interval $I \subset \mathbb{R}$.

As far as we know, [8, Theorem 3.1] is the first answer for (Q1). However, in [2, Example 1], a counterexample (for [8, Theorem 3.1]) was presented to show that the Picard–Lindelöf theorem is not straightforward for time scales. Later, this result was salvaged, particularly after assuming that $f(t,\cdot)$ is continuous for each $t \in [a,b]_{\mathbb{T}}$. An answer to (Q2) for (2) was presented in [3, Theorem 8.16] and [6, Theorem 2.1].

Here we introduce new techniques to provide answers to (Q1) and (Q2) for (2). More precisely, we prove the time scales generalization of the well-known Cauchy–Peano theorem. Our method follows the classical technique and neither requires Carathéodory conditions as in [2] nor applies the fixed point theorem as in [5], and hence it provides a new approximation technique for the solutions of the IVP (9). Then we present some examples similar to those due to Peano, where the initial value problems have more than one solution. We also give a uniqueness theorem without requiring the right-hand side function to be Lipschitzian. Finally, we combine the Cauchy–Peano theorem with the Lipschitz condition and give two proofs for the Picard–Lindelöf theorem. In the last section of the paper, we make our final comments to conclude the paper.

2. Preliminaries

2.1. Time scales essentials

A *time scale*, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here, and throughout this paper, a time scale is denoted by the symbol $\mathbb{T}$, and for an interval $J \subset \mathbb{R}$, $J_{\mathbb{T}}$ denotes the intersection of the usual interval with $\mathbb{T}$, i.e. $J_{\mathbb{T}} := J \cap \mathbb{T}$. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf(t,\infty)_{\mathbb{T}}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty,t)_{\mathbb{T}}$ and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined to be $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right-dense* if $\sigma(t) = t$ (i.e. $\mu(t) = 0$); otherwise, it is called *right-scattered*, and similarly *left-dense* and *left-scattered* points are defined with respect to the backward jump operator. The set $\mathbb{T}^{\kappa}$ is defined by $\mathbb{T}^{\kappa} := \mathbb{T}\backslash\{\sup\mathbb{T}\}$ if $\sup\mathbb{T}$ is finite and left-scattered; otherwise, $\mathbb{T}^{\kappa} := \mathbb{T}$. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, the Δ-derivative $f^{\Delta}(t)$ of f at the point t is defined to be the number, provided it exists, with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U,$$

where $f^{\sigma} := f \circ \sigma$ on $\mathbb{T}$. We mean the Δ-derivative of a function when we only say derivative unless otherwise specified. A function f is called *rd-continuous* provided that it is continuous at right-dense points in $\mathbb{T}$ and has a finite limit at left-dense points, and the *set of rd-continuous functions* is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. The set of functions $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ includes the functions whose derivative is in $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ too. For a function $f\in\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, the so-called *simple useful formula* holds:

$$f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t)\quad\text{for all } t\in\mathbb{T}^{\kappa}. \tag{3}$$

For $s,t\in\mathbb{T}$ and a function $f\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, the Δ-integral of f is defined by

$$\int_s^t f(\eta)\Delta\eta=F(t)-F(s)\quad\text{for } s,t\in\mathbb{T},$$

where $F\in\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is an antiderivative of f, i.e. $F^{\Delta}=f$ on $\mathbb{T}^{\kappa}$.

A function $f\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is called *regressive* if $1+\mu f\neq 0$ on $\mathbb{T}^{\kappa}$, and the *set of regressive functions* is denoted by $\mathcal{R}(\mathbb{T},\mathbb{R})$.

Letting $p\in\mathcal{R}(\mathbb{T},\mathbb{R})$, then the *exponential function* $\mathrm{e}_p(\cdot,s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the initial value problem

$$\begin{cases} y^{\Delta}(t)=p(t)y(t) & \text{for } t\in\mathbb{T}^{\kappa}\\ y(s)=1 \end{cases}$$

for some fixed $s\in\mathbb{T}$.

For $m\in\mathbb{N}_0$, the generalized monomial $\mathrm{h}_m:\mathbb{T}^2\to\mathbb{R}$ is defined recursively by

$$\mathrm{h}_m(t,s):=\int_s^t \mathrm{h}_{m-1}(\eta,s)\Delta\eta\quad\text{for } s,t\in\mathbb{T}\text{ and } m\in\mathbb{N} \tag{4}$$

with the convention that $\mathrm{h}_0(t,s):\equiv 1$ for $s,t\in\mathbb{T}$.

Readers are referred to [3] for further interesting details on time scale theory.

2.2. Functional preliminaries

Definition 1 (Uniformly Bounded Functions)**.** *Let $\mathbb{T}$ be a time scale and $f_m:\mathbb{T}\to\mathbb{R}$ for each $m\in\mathbb{N}$. The sequence of functions $\{f_m\}_{m\in\mathbb{N}}$ is said to be **uniformly bounded** on $\mathbb{T}$ if there exists $M\in\mathbb{R}^+$ such that $|f_m(t)|\leq M$ for all $t\in\mathbb{T}$ and all $m\in\mathbb{N}$.*

Definition 2 (Equicontinuous Functions)**.** *Let $\mathbb{T}$ be a time scale and $f_m:\mathbb{T}\to\mathbb{R}$ for each $m\in\mathbb{N}$. The sequence of functions $\{f_m\}_{m\in\mathbb{N}}$ is said to be **equicontinuous** on $\mathbb{T}$ if for every $\varepsilon\in\mathbb{R}^+$, there exists $\delta\in\mathbb{R}^+$ such that $|f_m(t)-f_m(s)|<\varepsilon$ for all $s,t\in\mathbb{T}$ with $|t-s|<\delta$ and all $m\in\mathbb{N}$.*

Theorem 1 (Arzelà-Ascoli Theorem)**.** *Let $\mathbb{T}$ be a bounded time scale (i.e. compact subset of $\mathbb{R}$) and $f_m:\mathbb{T}\to\mathbb{R}$ for each $m\in\mathbb{N}$. Suppose that $\{f_m\}$ is uniformly bounded on $\mathbb{T}$ and is equicontinuous on $\mathbb{T}$. Then there exists a subsequence of $\{f_m\}$, which converges uniformly on $\mathbb{T}$.*

Lemma 1 (Grönwall's Inequality [3, Theorem 6.4])**.** *Let $\mathbb{T}$ be a time scale, $a\in\mathbb{T}$ and $y,p,f\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ such that $p(t)\geq 0$ for all $t\in\mathbb{T}$. Then*

$$y(t)\leq f(t)+\int_a^t y(\eta)p(\eta)\Delta\eta\quad\textit{for all } t\in\mathbb{T}$$

implies

$$y(t) \leq f(t) + \int_a^t \mathrm{e}_p\big(t, \sigma(\eta)\big) f(\eta) p(\eta) \Delta\eta \quad \textit{for all } t \in \mathbb{T}.$$

Lemma 2. *An alternative form for* h_2 *is given by*

$$\mathrm{h}_2(t, s) = \frac{1}{2}\left[(t-s)^2 - \int_s^t \mu(\eta)\Delta\eta\right] \quad \textit{for all } s, t \in \mathbb{T}.$$

Proof Define

$$f(t) := \mathrm{h}_2(t, s) - \frac{1}{2}\left[(t-s)^2 - \int_s^t \mu(\eta)\Delta\eta\right] \quad \text{for all } t \in \mathbb{T},$$

and then

$$f^{\Delta}(t) = (t-s) - \frac{1}{2}\big[(t-s) + \big(\sigma(t) - s\big) - \mu(t)\big] = 0 \quad \text{for all } t \in \mathbb{T}^{\kappa},$$

which implies that f is a constant function on $\mathbb{T}$. Thus, $f(t) = f(s) = 0$ for all $t \in \mathbb{T}$. This completes the proof.

2.3. Background for existence/uniqueness results

Definition 3 (Solution)**.** *A function* $\varphi : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *is said to be a* ***solution of the differential equation***

$$y^{\Delta} = f(t, y) \tag{5}$$

provided that $\varphi \in \mathrm{C}^1_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$ *and* $\varphi^{\Delta}(t) = f(t, \varphi(t))$ *for all* $t \in [a, b]^{\kappa}_{\mathbb{T}}$*.*

Definition 4 (Initial Value Problem)**.** *A function* $\varphi : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *is said to be a* ***solution of the initial value problem*** (2) *provided that* φ *is a solution of* (5) *with* $\varphi(a) = \alpha$*.*

Definition 5 (Cf. [7, Definition 3.1])**.** *Let* $f : \mathbb{T} \times I \to \mathbb{R}$*, where* $I \subset \mathbb{R}$ *is an interval, satisfy the following conditions:*

(P1) $f(\cdot, x)$ *is rd-continuous on* $\mathbb{T}$ *for each fixed* $x \in I$*.*

(P2) $f(t, \cdot)$ *is continuous on* I *for each fixed* $t \in \mathbb{T}$*.*

Then we say that f *is* ***rd-continuous*** *on* $\mathbb{T} \times \mathbb{R}$*.*

Lemma 3. *Let* $f : \mathbb{T} \times I \to \mathbb{R}$*, where* $I \subset \mathbb{R}$ *is an interval, be rd-continuous. Then* φ *is a solution of the initial value problem* (2) *on* $[a, b]_{\mathbb{T}}$ *if and only if* φ *is a solution of the integral equation*

$$y = \alpha + \int_a^{\cdot} f(\eta, y)\Delta\eta \quad \textit{on } [a, b]_{\mathbb{T}},$$

i.e.

$$\varphi(t) = \alpha + \int_a^t f\big(\eta, \varphi(\eta)\big)\Delta\eta \quad \textit{for all } t \in [a, b]_{\mathbb{T}}.$$

Proof The proof is clear and is omitted.

Definition 6 (Cf. [7, Definition 3.2]). *Let $f : \mathbb{T} \times I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, satisfy (P1) together with the following condition.*

(P3) $f(t, \cdot)$ is continuous on I uniformly for $t \in \mathbb{T}$, i.e. for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(t,x) - f(t,y)| < \varepsilon$ for all $x, y \in I$ with $|x - y| < \delta$ and all $t \in \mathbb{T}$.

Then we say that f is ***uniformly rd-continuous*** *on $\mathbb{T} \times I$.*

The following example shows that rd-continuity on a compact set does not imply uniformly rd-continuity.

Example 1. *Let $\mathbb{T} := \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0, 1, 2, \cdots\}$. Note that $0 < \mu(t) \leq 1$ for all $t \in \mathbb{T}^\kappa$. Define $f : \mathbb{T} \times [-1, 1]_\mathbb{R} \to \mathbb{R}$ by $f(t,x) := \frac{\mu(t)+x}{[\mu(t)]^2+x^2}$ for $(t,x) \in \mathbb{T} \times [-1,1]_\mathbb{R}$ (see Figure 1).*

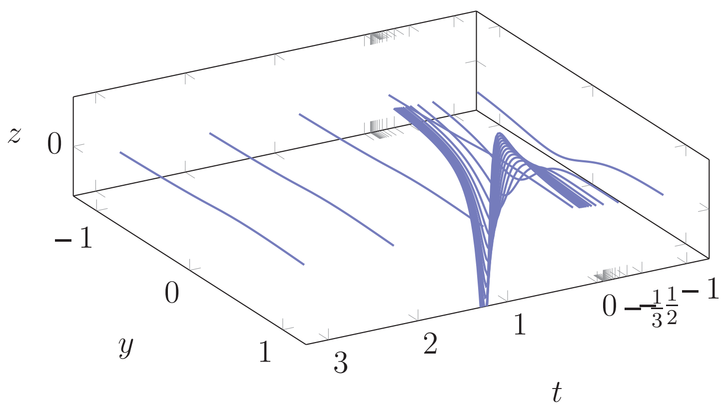

Figure 1. Graphic of $f : [-1, 3]_\mathbb{T} \times [-1, 1]_\mathbb{R} \to \mathbb{R}$.

Obviously, f is rd-continuous on $\mathbb{T} \times [-1,1]_\mathbb{R}$. With $x = \mu(t)$ and $y = -\mu(t)$ for $t \in \{-\frac{1}{n} : n \in \mathbb{N}\}$, we have $|f(t,x) - f(t,y)| = \frac{1}{2[\mu(t)]^2}|x - y|$, showing that f is not uniformly rd-continuous since $\frac{1}{2[\mu(t)]^2} \to \infty$ and $|x - y| = 2\mu(t) \to 0$ as $t \to 0^-$.

Example 2. *Let $\mathbb{T}$ be any time scale and define $f(t,y) := \operatorname{sgn}(y)p(t)|y|^\lambda$, where p is an rd-continuous function and $\lambda \in \mathbb{R}_0^+$. Then f is rd-continuous on $\mathbb{T} \times \mathbb{R}$ if $\lambda \in (1,\infty)_\mathbb{R}$ while it is uniformly rd-continuous on $\mathbb{T} \times \mathbb{R}$ if $\lambda \in (0,1]_\mathbb{R}$ and p is bounded on $\mathbb{T}$ (see [3, Theorem 1.60 (ii) and Theorem 1.65]).*

Note that if f is uniformly rd-continuous on $\mathbb{T} \times I$, then it is rd-continuous on $\mathbb{T} \times I$.

Definition 7. *Let $f : \mathbb{T} \times I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, satisfy (P1) together with the following condition.*

(P4) $f(t, \cdot)$ is Lipchitz continuous on I uniformly for $t \in \mathbb{T}$, i.e. there exists $L \in \mathbb{R}^+$ such that $|f(t,x) - f(t,y)| \leq L|x - y|$ for all $x, y \in I$ and all $t \in \mathbb{T}$.

Then we say that f is ***Lipschitz rd-continuous*** *on $\mathbb{T} \times I$.*

Note that if f is Lipschitz rd-continuous on $\mathbb{T} \times I$, then it is uniformly rd-continuous on $\mathbb{T} \times I$.

3. Cauchy–Peano existence theorem

For $h \in \mathbb{R}^+$, we define

$$R_h := \{(t,x):\ t \in [a,b]_{\mathbb{T}} \text{ and } x \in \mathbb{R} \text{ with } |x-\alpha| \leq h\}$$

and let $M_h \in \mathbb{R}^+$ satisfy

$$|f(t,x)| \leq M_h \quad \text{for all } (t,x) \in R_h. \tag{6}$$

Theorem 2 (Cauchy–Peano theorem)**.** *Let $f : [a,b]_{\mathbb{T}} \times I \to \mathbb{R}$ be uniformly rd-continuous on $R_{h_0} \subset [a,b]_{\mathbb{T}} \times I$ for some $h_0 \in \mathbb{R}^+$ and $M \in \mathbb{R}^+$ satisfy (6). Then the initial value problem (2) admits a solution on $[a,\sigma(\xi)]_{\mathbb{T}}$, where $\xi \in [a,b]_{\mathbb{T}}$ satisfies $(\xi-a)M \leq h_0$.*

Proof Since the case where $\xi = a$ is trivial, below we let $\xi > a$. It follows from [4, Lemma 2.7] that for each $\delta \in \mathbb{R}^+$ there is a partition of $\mathcal{P}_\delta :\ a =: t_0 < t_1 < \cdots < t_n := \xi$ of $[a,\xi]_{\mathbb{T}}$ such that, for each $k \in \{0,1,\cdots,n-1\}$, either $t_{k+1}-t_k < \delta$ or $t_{k+1}-t_k \geq \delta$ and $\sigma(t_k) = t_{k+1}$. Fix $m \in \mathbb{N}$ and consider the partition $\mathcal{P}_{\frac{1}{m}}$.

1. The approximating sequence. Recall that $y(t_0) = y(a) = \alpha$, and we recursively define the function $\varphi_m : [a,\xi]_{\mathbb{T}} \to \mathbb{R}$ by

$$\varphi_m(t) := \begin{cases} \alpha + \displaystyle\int_a^t f(\eta,\alpha)\Delta\eta, & t \in [a,t_1]_{\mathbb{T}} \\ \varphi_m(t_k) + \displaystyle\int_{t_k}^t f\big(\eta,\varphi_m(t_k)\big)\Delta\eta, & t \in (t_k,t_{k+1}]_{\mathbb{T}} \text{ and } k \in \{1,2,\cdots,n-1\}. \end{cases} \tag{7}$$

2. For each fixed $m \in \mathbb{N}$, φ_m is well defined on $[a,\xi]_{\mathbb{T}}$. Letting $t \in [t_0,t_1]_{\mathbb{T}}$, then we compute that

$$|\varphi_m(t)-\alpha| = \left|\int_{t_k}^t f(\eta,\alpha)\Delta\eta\right| \leq \int_{t_0}^t |f(\eta,\alpha)|\Delta\eta \leq M(\xi-a) \leq h.$$

Suppose now that φ_m is well defined on $[t_0,t_k]_{\mathbb{T}}$ for some $k \in \{1,2,\cdots,n-1\}$. We will prove that φ_m is also well defined on $[t_0,t_{k+1}]_{\mathbb{T}}$. To this end, let $t \in (t_k,t_{k+1}]_{\mathbb{T}}$, and then

$$\begin{aligned} |\varphi_m(t)-\alpha| \leq& \left|[\varphi_m(t)-\varphi_m(t_k)] + \sum_{\ell=0}^{k-1}[\varphi_m(t_{\ell+1})-\varphi_m(t_\ell)]\right| \\ \leq& \left|\int_{t_k}^t f\big(\eta,\varphi_m(t_k)\big)\Delta\eta + \sum_{\ell=0}^{k-1}\int_{t_\ell}^{t_{\ell+1}} f\big(\eta,t_\ell,\varphi_m(t_\ell)\big)\Delta\eta\right| \\ \leq& \sum_{\ell=0}^{k}\int_{t_\ell}^{t_{\ell+1}} \big|f\big(\eta,t_\ell,\varphi_m(t_\ell)\big)\big|\Delta\eta \\ \leq& M(\xi-a) \leq h, \end{aligned}$$

which proves that φ_m is well defined on $[t_0,t_{k+1}]_{\mathbb{T}}$ too. Thus, by mathematical induction, φ_m is well defined on $[a,\xi]_{\mathbb{T}}$.

3. The sequence $\{\varphi_m\}$ *is equicontinuous.* We claim that

$$|\varphi_m(t)-\varphi_m(s)| \le M|t-s| \quad \text{for all } s,t\in[a,\xi]_{\mathbb{T}}. \tag{8}$$

Without loss of generality, we let $s,t\in[a,\xi]_{\mathbb{T}}$ with $t>s$. We consider the following two possible cases.

- $s,t\in(t_i,t_{i+1}]_{\mathbb{T}}$ for some $i\in\{0,1,\cdots,n-1\}$. Then we have

$$\begin{aligned}|\varphi_m(t)-\varphi_m(s)| =&\left|\int_s^t f\big(\eta,\varphi_m(t_i)\big)\Delta\eta\right| \le \int_s^t \big|f\big(\eta,\varphi_m(t_i)\big)\big|\Delta\eta\\ \le& M|t-s|.\end{aligned}$$

- $s\in(t_i,t_{i+1}]_{\mathbb{T}}$ and $t\in(t_j,t_{j+1}]_{\mathbb{T}}$ for some $j\in\{1,2,\cdots,n-1\}$ and some $i\in\{0,1,\cdots,j-1\}$, which yields

$$\begin{aligned}|\varphi_m(t)-\varphi_m(s)| =&\left|[\varphi_m(t)-\varphi_m(t_j)]+\sum_{\ell=i}^{j-1}[\varphi_m(t_{\ell+1})-\varphi_m(t_\ell)]+[\varphi_m(t_i)-\varphi_m(s)]\right|\\ =&\left|\int_{t_j}^t f\big(\eta,\varphi_m(t_j)\big)\Delta\eta+\sum_{\ell=i}^{j-1}\int_{t_\ell}^{t_{\ell+1}} f\big(\eta,\varphi_m(t_\ell)\big)\Delta\eta+\int_{t_i}^s f\big(\eta,\varphi_m(t_i)\big)\Delta\eta\right|\\ \le& \int_{t_j}^t \big|f\big(\eta,\varphi_m(t_j)\big)\big|\Delta\eta+\sum_{\ell=i}^{j-1}\int_{t_\ell}^{t_{\ell+1}} \big|f\big(\eta,\varphi_m(t_\ell)\big)\big|\Delta\eta\\ &+\int_{t_i}^s \big|f\big(\eta,\varphi_m(t_i)\big)\big|\Delta\eta\\ \le& M\left[[t-t_j]+\sum_{\ell=i}^{j-1}[t_{\ell+1}-t_\ell]+[s-t_i]\right]=M|t-s|.\end{aligned}$$

This proves (8), which justifies the equicontinuity of the sequence of functions $\{\varphi_m\}$.

4. The sequence $\{\varphi_m\}$ *is uniformly bounded.* Clearly, we have

$$|\varphi_m(t)|\le|\varphi_m(t)-\alpha|+|\alpha|\le h+|\alpha| \quad \text{for all } t\in[a,\xi]_{\mathbb{T}} \text{ and } m\in\mathbb{N}.$$

Then, by the Ascoli–Arzela theorem, there exists a subsequence of $\{\varphi_m\}$ that converges uniformly on $[a,\xi]_{\mathbb{T}}$. For simplicity of notation, we may (and do) suppose that $\{\varphi_m\}$ itself is that uniformly converging subsequence.

5. The error function converges uniformly to zero on $[a,\xi]_{\mathbb{T}}$. For $m\in\mathbb{N}$, we define $E_m:[a,\xi]_{\mathbb{T}}^{\kappa}\to\mathbb{R}$ by

$$E_m(t):=\begin{cases}\varphi_m^{\Delta}(t)-f\big(t,\varphi_m(t)\big), & t\in(t_k,t_{k+1})_{\mathbb{T}} \text{ for some } k\in\{0,1,\cdots,n-1\}\\ 0, & t=t_k \text{ for some } k\in\{0,1,\cdots,n\}.\end{cases}$$

It is obvious for each fixed $m\in\mathbb{N}$ that E_m is piecewise continuous on $[a,\xi]_{\mathbb{T}}^{\kappa}$. Let $\varepsilon\in\mathbb{R}^+$, and then it follows from (P3) that there exists $\delta\in\mathbb{R}^+$ such that $|f(t,x)-f(t,y)|<\varepsilon$ for all $(t,x),(t,y)\in R_h$ with $|x-y|<\delta$. By equicontinuity, we have $|\varphi_m(t)-\varphi_m(s)|<\delta$ for all $|t-s|<\frac{\delta}{M}$ and all $m\in\mathbb{N}$. Pick $m_0\in\mathbb{N}$ sufficiently large so that $m_0>\frac{M}{\delta}$. Clearly, for $m\in\{m_0,m_0+1,\cdots\}$, if $t_{i+1}-t_i\ge\frac{1}{m}$, then $(t_i,t_{i+1})_{\mathbb{T}}=(t_i,\sigma(t_i))_{\mathbb{T}}=\emptyset$, and on the other hand, if $t_{i+1} - t_i < \frac{1}{m}$, then $t \in (t_i, t_{i+1})_{\mathbb{T}}$ implies $|t - t_i| < \frac{1}{m} < \frac{\delta}{M}$. Therefore, we estimate that $|E_m(t)| = |f(t, \varphi_m(t_i)) - f(t, \varphi_m(t))| < \varepsilon$ for all $t \in (t_i, t_{i+1})_{\mathbb{T}}$ and all $m \in \{m_0, m_0+1, \cdots\}$, which proves $\lim_{m\to\infty} E_m = 0$ uniformly on $[a, \xi]^{\kappa}_{\mathbb{T}}$.

6. The associated integral equation on $[a,\xi]_{\mathbb{T}}$. Fix $m \in \mathbb{N}$ and $t \in [a,\xi]_{\mathbb{T}}$, and then there exists $k \in \{0, 1, \cdots, n\}$ such that $t \in [t_k, t_{k+1}]_{\mathbb{T}}$. Hence, we compute for $t \in [a,\xi]_{\mathbb{T}}$ that

$$\begin{aligned}\varphi_m(t) - \alpha =& [\varphi_m(t) - \varphi_m(t_k)] + \sum_{\ell=0}^{k-1} [\varphi_m(t_{\ell+1}) - \varphi_m(t_\ell)] \\ =& \int_{t_k}^{t} \varphi_m^{\Delta}(\eta)\Delta\eta + \sum_{\ell=0}^{k-1} \int_{t_\ell}^{t_{\ell+1}} \varphi_m^{\Delta}(\eta)\Delta\eta \\ =& \int_a^t \varphi_m^{\Delta}(\eta)\Delta\eta,\end{aligned}$$

which shows that φ_m satisfies the integral equation

$$\varphi_m(t) = \alpha + \int_a^t \left[f\big(\eta, \varphi_m(\eta)\big) + E_m(\eta)\right]\Delta\eta \quad \text{for all } t \in [a,\xi]_{\mathbb{T}}.$$

Since $\varphi_m \to \varphi$ and $E_m \to 0$ uniformly on $[a,\xi]_{\mathbb{T}}$ as $m \to \infty$, we obtain

$$\varphi(t) = \alpha + \int_a^t f\big(\eta, \varphi(\eta)\big)\Delta\eta \quad \text{for all } t \in [a,\xi]_{\mathbb{T}}, \tag{9}$$

which by Lemma 3 proves that φ is a solution that exists on $[a,\xi]_{\mathbb{T}}$.

7. Extending the solution to $[a,\sigma(\xi)]_{\mathbb{T}}$. By using the so-called simple useful formula in (3), we now define the function ψ by

$$\psi(t) := \begin{cases} \varphi(t), & t \in [a,\xi]_{\mathbb{T}} \\ \varphi(\xi) + \mu(\xi) f\big(\xi, \varphi(\xi)\big), & t = \sigma(\xi) \end{cases} \quad \text{for } t \in [a, \sigma(\xi)]_{\mathbb{T}}. \tag{10}$$

Therefore, ψ is the desired solution of (2), which exists on $[a,\sigma(\xi)]_{\mathbb{T}}$.

The proof is therefore completed.

Remark 1. *If a is right-dense, then we can pick $\xi \in (a,b]_{\mathbb{T}}$ such that $(\xi - a)M \leq h_0$, which yields $\sigma(\xi) \geq \xi > a$. On the other hand, if a is right-scattered, then either $\xi = a$ and $\sigma(\xi) > \xi$ or $\xi > a$. Combining the cases above, we always have $\sigma(\xi) > a$.*

Remark 2. *If the auxiliary solution φ exists uniquely on $[a,\xi]_{\mathbb{T}}$, then the extended solution ψ is also unique on $[a,\sigma(\xi)]_{\mathbb{T}}$.*

On a particular time scale, we will show below for linear equations that the approximating sequence converges to a single function, which is the unique solution of the equation on the entire time scale.

Example 3. *Let $\mathbb{T}$ be a bounded time scale such that μ is increasing on $\mathbb{T}^\kappa$ with $\mu(a) = 0$, where $a := \min\mathbb{T}$. Consider the linear equation*

$$\begin{cases} y^\Delta = p(t)y & \text{for } t \in \mathbb{T}^\kappa \\ y(a) = \alpha, \end{cases} \tag{11}$$

where $p \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and $\alpha \in \mathbb{R}$. For each $m \in \mathbb{N}$, there corresponds $s = s(m) \in \mathbb{T}^\kappa$ such that $\mu(s) < \frac{1}{m}$ and $\mu(\sigma(s)) \geq \frac{1}{m}$. Consider the partition $\mathcal{P}_{\frac{1}{m}}$: $a < s < \sigma(s) < \sigma^2(s) < \cdots < \sigma^n(s) = b$, where $b := \max\mathbb{T}$. Thus, the approximating sequence is

$$\begin{aligned} \varphi_m(t) &= \begin{cases} \alpha + \displaystyle\int_a^t p(\eta)\alpha\Delta\eta, & t \in [a, s]_\mathbb{T} \\ \varphi_m\big(\sigma^k(s)\big) + \displaystyle\int_{\sigma^k(s)}^t p(\eta)\varphi_m\big(\sigma^k(s)\big)\Delta\eta, & t \in (\sigma^k(s), \sigma^{k+1}(s)]_\mathbb{T} \text{ and } k \in [0, n)_\mathbb{Z} \end{cases} \\ &= \begin{cases} \alpha\left[1 + \displaystyle\int_a^t p(\eta)\Delta\eta\right], & t \in [a, s]_\mathbb{T} \\ \varphi_m\big(\sigma^k(s)\big)\left[1 + \displaystyle\int_{\sigma^k(s)}^t p(\eta)\Delta\eta\right], & t \in (\sigma^k(s), \sigma^{k+1}(s)]_\mathbb{T} \text{ and } k \in [0, n)_\mathbb{Z}. \end{cases} \end{aligned}$$

If $t \in (\sigma^k(s), \sigma^{k+1}(s)]_\mathbb{T}$ for $k \in \{0, 1, \cdots, n-1\}$, then $t = \sigma^{k+1}(s)$ and we obtain

$$\begin{aligned} \varphi_m\big(\sigma^{k+1}(s)\big) =& \varphi_m\big(\sigma^k(s)\big)\left[\int_{\sigma^k(s)}^{\sigma^{k+1}(s)} p(\eta)\Delta\eta\right] \\ =& \varphi_m\big(\sigma^k(s)\big)\big[1 + \mu\big(\sigma^k(s)\big)p(\sigma^k(s))\big], \end{aligned}$$

which yields by repeating the recursion that

$$\begin{aligned} \varphi_m\big(\sigma^{k+1}(s)\big) =& \varphi_m(s)\prod_{\ell=0}^{k}\big[1 + \mu\big(\sigma^\ell(s)\big)p\big(\sigma^\ell(s)\big)\big] \\ =& \alpha\left[1 + \int_a^s p(\eta)\Delta\eta\right]\mathrm{e}_p\big(\sigma^{k+1}(s), s\big). \end{aligned}$$

Thus, in general, we have

$$\varphi_m(t) = \begin{cases} \alpha\left[1 + \displaystyle\int_a^t p(\eta)\Delta\eta\right], & t \in [a, s]_\mathbb{T} \\ \alpha\left[1 + \displaystyle\int_a^s p(\eta)\Delta\eta\right]\mathrm{e}_p(t, s), & t \in (s, b]_\mathbb{T}. \end{cases}$$

As $m \to \infty$ implies $s \to a$, we see that

$$\lim_{m\to\infty}\varphi_m(t) = \alpha\mathrm{e}_p(t, a) \quad \text{for } t \in [a, b]_\mathbb{T},$$

which is (known to be) the unique solution of (11).

The existence interval (which is the entire time scale) of the solution of the equation in Example 3 follows from application of Corollary 1 several times in Section 7, while the uniqueness of the solution follows from Corollary 2 in Section 7.

4. Some examples similar to Peano's

It is known that the solutions of the IVP (2) always exist and this is unique on isolated time scales. The following example demonstrates that the uniqueness of solutions may be lost even if there is a single right-dense point in the time scale.

Example 4. *Consider the time scale* $\mathbb{T} := \{\pm q^n : n \in \mathbb{Z}\} \cup \{0\}$, *where* $q \in (1, \infty)_{\mathbb{R}}$. *We see that* $\mathbb{T}$ *has the single right-dense point* 0. *For some fixed* $\lambda \in \mathbb{N}_0$, *consider the initial value problem*

$$\begin{cases} y^{\Delta} = t^{\lambda} \sqrt{|y|} & \text{for } t \in [-1, 1]_{\mathbb{T}}^{\kappa} \\ y(-1) = 0. \end{cases}$$

In Figure 2, the graphic of a prototype of the function $z = f(t, y)$ *for* $(t, y) \in \mathbb{T} \times \mathbb{R}$ *is given.*

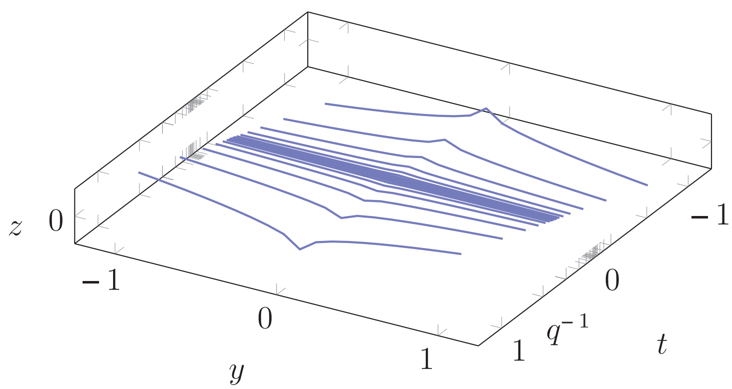

Figure 2. Graphic of $f : [-1, 1]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ with $\lambda = 1$.

Clearly, $\varphi(t) :\equiv 0$ *for* $t \in [-1, 1]_{\mathbb{T}}$ *is a solution. On the other hand, consider the function*

$$\psi(t) := \begin{cases} 0, & t \in [-1, 0]_{\mathbb{T}} \\ \left(\dfrac{q-1}{q^{2(\lambda+1)} - 1}\right)^2 t^{\lambda+1}, & t \in [0, 1]_{\mathbb{T}}. \end{cases}$$

Obviously, both ψ *and* ψ^{Δ} *are continuous on* $[-1, 1]_{\mathbb{T}}$. *By [3, Theorem 1.24 (i)] and the fact that* $\sigma(t) = qt$ *for* $t \in [0, 1]_{\mathbb{T}}$, *we compute that*

$$\begin{aligned} \psi^{\Delta}(t) =& \left(\frac{q-1}{q^{2(\lambda+1)} - 1}\right)^2 \sum_{\nu=0}^{2\lambda+1} \big(\sigma(t)\big)^{\nu} t^{2\lambda+1-\nu} = \left(\frac{q-1}{q^{2(\lambda+1)} - 1}\right)^2 \sum_{\nu=0}^{2\lambda+1} (qt)^{\nu} t^{2\lambda+1-\nu} \\ =& \left(\frac{q-1}{q^{2(\lambda+1)} - 1}\right)^2 \left(\sum_{\nu=0}^{2\lambda+1} q^{\nu}\right) t^{2\lambda+1} = \left(\frac{q-1}{q^{2(\lambda+1)} - 1}\right)^2 \left(\frac{q^{2(\lambda+1)} - 1}{q-1}\right) t^{2\lambda+1} \\ =& \frac{q-1}{q^{2(\lambda+1)} - 1} t^{2\lambda+1} = t^{\lambda} \sqrt{|\psi(t)|} \end{aligned}$$

for $t \in [0, 1]_{\mathbb{T}}^{\kappa}$. *Hence,* ψ *is also a solution of the initial value problem.*

For the continuous case, it is known that if $f(t,\cdot)$ is nonincreasing on $\mathbb{R}$ for each fixed $t \in [a,b]_{\mathbb{R}}$, then (2) can have at most one solution on $[a,b]_{\mathbb{R}}$ (see [1, Theorem 10.2]). This, combined with the Cauchy–Peano theorem, provides existence and uniqueness. Now we will illustrate with the following example that the nonincreasing nature of f on the right-dense points of the time scale is not sufficient for guaranteeing uniqueness of solutions under the conditions of the Cauchy–Peano theorem.

Example 5. *Let $\mathbb{T} := (-\infty,0]_{\mathbb{R}} \cup \{q^n : n \in \mathbb{Z}\}$, where $q \in (1,\infty)_{\mathbb{R}}$, and define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(t,y) := \begin{cases} -y, & t \le 0 \text{ and } y \in \mathbb{R} \\ \operatorname{sgn}(y)\big(\sqrt{|y|}+1\big)\dfrac{t}{q-1} - y, & t \ge 0 \text{ and } y \in \mathbb{R} \text{ with } |y| \ge \big(\frac{t}{q-1}\big)^2 \\ \operatorname{sgn}(y)\sqrt{|y|}, & t \ge 0 \text{ and } y \in \mathbb{R} \text{ with } |y| \le \big(\frac{t}{q-1}\big)^2, \end{cases}$$

whose graphic is given in Figure 3.

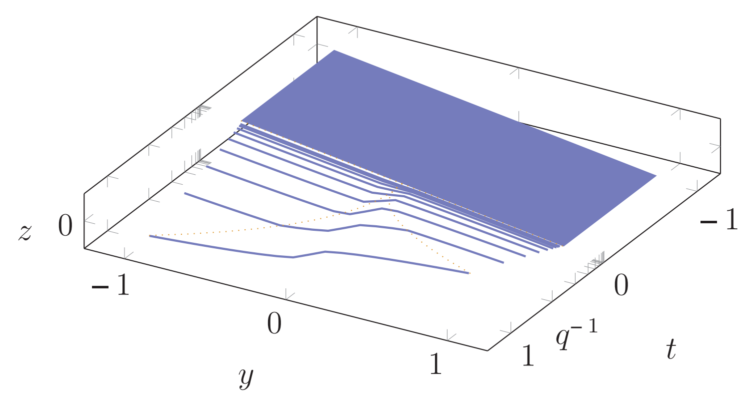

Figure 3. Graphic of $f : [-1,1]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$.

Clearly, f is continuous on $\mathbb{T} \times \mathbb{R}$ and $f(t,\cdot)$ is decreasing on $\mathbb{R}$ for each fixed $t \in (-\infty,0]_{\mathbb{T}}$. For the initial value problem

$$\begin{cases} y^{\Delta} = f(t,y) & \text{for } t \in [-1,1]_{\mathbb{T}}^{\kappa} \\ y(-1) = 0, \end{cases} \tag{12}$$

it is obvious that $\varphi(t) :\equiv 0$ for $t \in [-1,1]_{\mathbb{T}}$ is a solution. As in Example 4, we can show that ψ defined by

$$\psi(t) := \begin{cases} 0, & t \in [-1,0]_{\mathbb{T}} \\ \left(\dfrac{q-1}{q^2-1}\right)^2 t^2, & t \in [0,1]_{\mathbb{T}} \end{cases}$$

is another solution of the initial value problem (12).

5. Peano's uniqueness theorem

In this section, we will provide a uniqueness result that can be regarded as time scales generalization of the monotonicity condition mentioned in Section 4 (see [9, Theorem 8.36]).

Theorem 3. *Assume that f is rd-continuous and satisfies*

$$(x-y)[f(t,x)-f(t,y)] \le -\frac{1}{2}\mu(t)[f(t,x)-f(t,y)]^2 \quad \text{for all } (t,x),(t,y) \in R_{h_0} \tag{13}$$

for some $h_0 \in \mathbb{R}^+$. Then (2) can admit at most one solution on any subinterval $[a,\zeta]_{\mathbb{T}}$ of $[a,b]_{\mathbb{T}}$ whose graph lies in R_{h_0}.

Proof Suppose the contrary, that (2) admits two different solutions φ and ψ on $[a,\zeta]_{\mathbb{T}}$ whose graphs are in R_{h_0}. Define $\omega(t) := \varphi(t) - \psi(t)$ for $t \in [a,\zeta]_{\mathbb{T}}$. We compute for $t \in [a,\zeta]_{\mathbb{T}}^{\kappa}$ that

$$\begin{aligned}(\omega^2)^{\Delta}(t) =& 2\omega(t)\omega^{\Delta}(t) + \mu(t)\big(\omega^{\Delta}(t)\big)^2 \\ =& 2[\varphi(t)-\psi(t)]\big[\varphi^{\Delta}(t)-\psi^{\Delta}(t)\big] + \mu(t)\big[\varphi^{\Delta}(t)-\psi^{\Delta}(t)\big]^2 \\ =& 2[\varphi(t)-\psi(t)]\big[f\big(t,\varphi(t)\big)-f\big(t,\psi(t)\big)\big] \\ &+ \mu(t)\big[f\big(t,\varphi(t)\big)-f\big(t,\psi(t)\big)\big]^2 \le 0,\end{aligned}$$

which yields $\omega(t) \equiv 0$ for $t \in [a,\zeta]_{\mathbb{T}}$. Therefore, the proof is completed.

Remark 3. *This result can be extended to any even power of ω by using the formula*

$$(\omega^n)^{\Delta} = \sum_{\ell=1}^{n} \binom{n}{\ell} \omega^{n-\ell}\mu^{\ell-1}(\omega^{\Delta})^{\ell}.$$

Example 6 (Cf. [1, Lecture 10, Problem 10.8]). *Consider the time scale $\mathbb{P}_{\alpha,\beta} := \cup_{k\in\mathbb{Z}}[k(\alpha+\beta), k(\alpha+\beta)+\alpha]_{\mathbb{R}}$, where $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}_0^+$ such that*

$$\beta \le \sqrt{2}\alpha. \tag{14}$$

Consider the initial value problem

$$\begin{cases} y^{\Delta} = f(t,y) & \text{for } t \in [0,l]_{\mathbb{P}_{\alpha,\beta}}^{\kappa} \\ y(0) = 0, \end{cases} \tag{15}$$

where $l \in \mathbb{R}^+$ and the function f is defined by

$$f(t,y) := \begin{cases} 2\mathrm{h}_1(t,0), & t \in [0,l]_{\mathbb{P}_{\alpha,\beta}} \text{ and } y \le 0 \\ 2\left(1 - \dfrac{y}{\mathrm{h}_2(t,0)}\right)\mathrm{h}_1(t,0), & t \in (0,l]_{\mathbb{P}_{\alpha,\beta}} \text{ and } 0 < y \le 2\mathrm{h}_1(t,0) \\ -2\mathrm{h}_1(t,0), & t \in [0,l]_{\mathbb{P}_{\alpha,\beta}} \text{ and } y \ge 2\mathrm{h}_1(t,0). \end{cases} \tag{16}$$

Clearly, f is rd-continuous on $[0,l]_{\mathbb{P}_{\alpha,\beta}} \times \mathbb{R}$ (see Figure 4).

Letting $\varphi(t) := \frac{2}{3}\mathrm{h}_2(t,0)$ for $t \in [0,l]_{\mathbb{P}_{\alpha,\beta}}$, compute that

$$\varphi^{\Delta}(t) = \frac{2}{3}\mathrm{h}_1(t,0) \quad \text{for } t \in [0,l]_{\mathbb{P}_{\alpha,\beta}}^{\kappa}$$

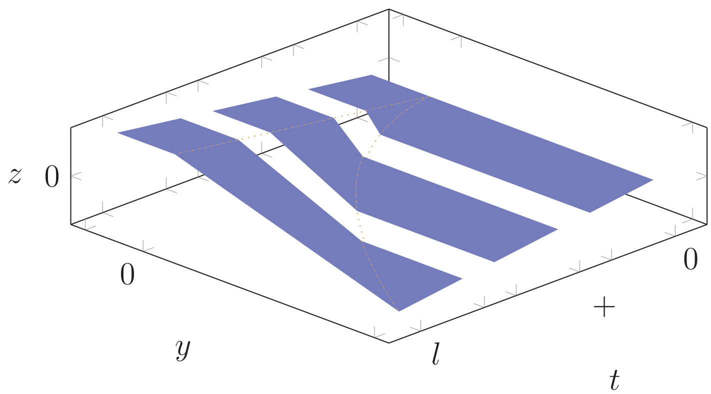

Figure 4. Graphic of $f:[0,l]_{\mathbb{P}_{\alpha,\beta}}\times\mathbb{R}\to\mathbb{R}$.

and

$$f(t,\varphi(t))=2\left(1-\frac{2}{3}\right)\mathrm{h}_1(t,0)=\frac{2}{3}\mathrm{h}_1(t,0)\quad \textit{for } t\in[0,l]_{\mathbb{P}_{\alpha,\beta}}.$$

Thus, φ *satisfies* (15)*. Further, we compute that*

$$(x-y)[f(t,x)-f(t,y)]=\begin{cases}0, & t\in[0,l]_{\mathbb{P}_{\alpha,\beta}} \textit{ and } y\leq 0\\ -2\dfrac{\mathrm{h}_1(t,0)}{\mathrm{h}_2(t,0)}(x-y)^2, & t\in(0,l]_{\mathbb{P}_{\alpha,\beta}} \textit{ and } 0<y\leq 2\mathrm{h}_2(t,0)\\ 0, & t\in[0,l]_{\mathbb{P}_{\alpha,\beta}} \textit{ and } y\geq 2\mathrm{h}_2(t,0)\end{cases}$$

and

$$-\frac{1}{2}\mu(t)[f(t,x)-f(t,y)]^2=\begin{cases}0, & t\in[0,l]_{\mathbb{P}_{\alpha,\beta}} \textit{ and } y\leq 0\\ -2\mu(t)\left(\dfrac{\mathrm{h}_1(t,0)}{\mathrm{h}_2(t,0)}\right)^2(x-y)^2, & t\in(0,l]_{\mathbb{P}_{\alpha,\beta}} \textit{ and } 0<y\leq 2\mathrm{h}_2(t,0)\\ 0, & t\in[0,l]_{\mathbb{P}_{\alpha,\beta}} \textit{ and } y\geq 2\mathrm{h}_2(t,0),\end{cases}$$

i.e. (13) *holds if and only if*

$$\mathrm{h}_2(t,0)\geq\mu(t)\mathrm{h}_1(t,0)\quad \textit{for all } t\in[0,l]_{\mathbb{P}_{\alpha,\beta}},$$

which, by Lemma 2*, is equivalent to*

$$t^2-2t\mu(t)-\int_0^t\mu(\eta)\Delta\eta\geq 0\quad \textit{for all } t\in[0,l]_{\mathbb{P}_{\alpha,\beta}}.$$

Letting $t\in[k(\alpha+\beta),k(\alpha+\beta)+\alpha)_{\mathbb{R}}$ *and* $k\in\mathbb{N}_0$*, then*

$$\begin{aligned}t^2-2t\mu(t)-\int_0^t\mu(\eta)\Delta\eta=&t^2-k\beta^2\geq\big(k(\alpha+\beta)\big)^2-k\beta^2\\ =&k^2\alpha^2+2k^2\alpha\beta+k(k-1)\beta^2\geq 0.\end{aligned}$$

Letting $t=k(\alpha+\beta)+\alpha$ *and* $k\in\mathbb{N}_0$*, then*

$$t^2-2t\mu(t)-\int_0^t\mu(\eta)\Delta\eta=(k+1)^2\alpha^2+2(k^2-1)\alpha\beta+k(k-3)\beta^2,\tag{17}$$

which is nonnegative provided that $k=0$ *and* $\beta\leq 2\alpha$, *or* $k=1$ *and* $\beta\leq\sqrt{2}\alpha\approx 1.414\alpha$, *or* $k=2$ *and* $\beta\leq\frac{3}{2}(\sqrt{3}+1)\alpha\approx 4.098\alpha$, *or* $k=3,4,\cdots$. *By* (14), *we see that* (16) *holds for all* $k\in\mathbb{N}_0$. *Therefore,* $\varphi:=\frac{2}{3}\mathrm{h}_2(\cdot,0)$ *is the unique solution of* (15) *on* $[0,l]_{\mathbb{P}_{\alpha,\beta}}$ *(see Remark 5 for when* (14) *may not hold).*

6. Picard–Lindelöf existence and uniqueness theorem

Theorem 4 (Picard–Lindeöf Theorem)**.** *Let* $f:[a,b]_{\mathbb{T}}\times I\to\mathbb{R}$ *be Lipschitz rd-continuous on* $R_{h_0}\subset[a,b]_{\mathbb{T}}\times I$ *for some* $h_0\in\mathbb{R}^+$ *and* $M\in\mathbb{R}^+$ *satisfy* (6)*. Then the initial value problem* (2) *admits a unique solution on* $[a,\sigma(\xi)]_{\mathbb{T}}$, *where* $\xi\in[a,b]_{\mathbb{T}}$ *satisfies* $(\xi-a)M\leq h_0$.

Proof [Proof by Cauchy–Peano Theorem] It follows from the Cauchy–Peano theorem that there exists at least one solution on $[a,\xi]_{\mathbb{T}}$. Suppose that there exist two different solutions φ and ψ of (2). We define $\omega(t):=\sup_{\zeta\in[a,t]_{\mathbb{T}}}|\varphi(\zeta)-\psi(\zeta)|$ for $t\in[a,\xi]_{\mathbb{T}}$. Then, for all $t\in[a,\xi]_{\mathbb{T}}$, we see that

$$\begin{aligned}\omega(t)&=\sup_{\zeta\in[a,t]_{\mathbb{T}}}\left|\int_a^{\zeta}\big[f\big(\eta,\varphi(\eta)\big)-f\big(\eta,\psi(\eta)\big)\big]\Delta\eta\right|\\&\leq\sup_{\zeta\in[a,t]_{\mathbb{T}}}\int_a^{\zeta}\big|f\big(\eta,\varphi(\eta)\big)-f\big(\eta,\psi(\eta)\big)\big|\Delta\eta\\&=\int_a^{t}\big|f\big(\eta,\varphi(\eta)\big)-f\big(\eta,\psi(\eta)\big)\big|\Delta\eta\\&<L\int_a^t|\varphi(\eta)-\psi(\eta)|\Delta\eta\\&\leq L\int_a^t\sup_{\zeta\in[a,\eta]_{\mathbb{T}}}|\varphi(\zeta)-\psi(\zeta)|\Delta\eta,\end{aligned}$$

which yields

$$\omega(t)\leq L\int_a^t\omega(\eta)\Delta\eta.$$

This shows by an application of the Grönwall inequality that $\omega(t)\leq 0$ for all $t\in[a,\xi]_{\mathbb{T}}$. Since ω is nonnegative on $[a,\xi]_{\mathbb{T}}$, we see that $\omega(t)\equiv 0$ for $t\in[a,\xi]_{\mathbb{T}}$. Therefore, (2), which has at least one solution, can at the same time have at most one solution. The proof is therefore completed.

Proof [Direct Proof of Theorem 4] We will only give the proof of existence since uniqueness will follow verbatim with steps given above (cf. [6, Theorem 2.1]).

1. Picard iterates. We define recursively the sequence of functions $\{\varphi_m\}_{m\in\mathbb{N}}$ by

$$\varphi_m(t):=\alpha+\int_a^t f\big(\eta,\varphi_{m-1}(\eta)\big)\Delta\eta\quad\text{for } t\in[a,\xi]_{\mathbb{T}}\text{ and } m\in\mathbb{N},\tag{18}$$

where $\varphi_0(t):\equiv\alpha$ for $t\in[a,\xi]_{\mathbb{T}}$.

2. For each fixed $m\in\mathbb{N}$, φ_m *is well defined on* $[a,\xi]_{\mathbb{T}}$. It is easy to show for each fixed $m\in\mathbb{N}$ that $(t,\varphi_m(t))\in\mathbb{R}_h$ for all $t\in[a,\xi]_{\mathbb{T}}$.

3. Uniform convergence of Picard iterates. By induction, we can show that

$$|\varphi_m(t)-\varphi_{m-1}(t)|\leq ML^{m-1}\mathrm{h}_m(t,a)\quad\text{for } t\in[a,\xi]_{\mathbb{T}} \text{ and } m\in\mathbb{N},$$

where h_m is defined by (4). Now we prove that the $\{\varphi_m\}_{m\in\mathbb{N}}$ converges uniformly. For $t\in[a,\xi]$ and $m\in\mathbb{N}$, we have

$$\begin{aligned}\left|\sum_{\ell=0}^{m-1}\varphi_{\ell+1}(t)-\varphi_\ell(t)\right|\leq&\sum_{\ell=0}^{m-1}|\varphi_{\ell+1}(t)-\varphi_\ell(t)|\leq\frac{M}{L}\sum_{\ell=0}^{m-1}L^\ell\mathrm{h}_\ell(t,a)\\ \leq&\frac{M}{L}\sum_{\ell=0}^{\infty}L^\ell\mathrm{h}_\ell(t,a)=\frac{M}{L}[\mathrm{e}_L(t,a)-1]\\ \leq&\frac{M}{L}[\mathrm{e}_L(b,a)-1],\end{aligned}$$

which proves by Weierstrass M-test that the series $\sum_{\ell=0}^{\infty}[\varphi_{\ell+1}-\varphi_\ell]$ converges uniformly on $[a,\xi]_{\mathbb{T}}$. The partial sum of this series is $\varphi_m=\sum_{\ell=0}^{m-1}[\varphi_{\ell+1}-\varphi_\ell]$ on $[a,\xi]_{\mathbb{T}}$, i.e. there exists a function φ such that $(t,\varphi(t))\in R_{h_0}$ for all $t\in[a,\xi]_{\mathbb{T}}$ and $\lim_{m\to\infty}\varphi_m(t)=\varphi(t)$ uniformly for $t\in[a,\xi]_{\mathbb{T}}$. It follows from (P4) that $\lim_{m\to\infty}f(t,\varphi_m(t))=f(t,\varphi(t))$ uniformly for $t\in[a,\xi]_{\mathbb{T}}$ since $|f(t,\varphi_m(t))-f(t,\varphi(t))|\leq L|\varphi_m(t)-\varphi(t)|\to0$ uniformly for $t\in[a,\xi]_{\mathbb{T}}$ as $m\to\infty$.

4. The associated integral equation. Letting $m\to\infty$ in (18) and using [4, Theorem 3.11], we arrive at (9), which completes the proof by Lemma 3.

Remark 4. *After reading Remark 2, one may have the impression that the Lipschitz rd-continuity of f in the Picard–Lindelöf theorem or the condition of Theorem 3 can be restricted to right-dense points in $[a,b]_{\mathbb{T}}$ only since the solution can be extended to right-scattered points uniquely. However, this is not true, as we can see from Example 5.*

Example 7. *Consider the simple dynamic equation*

$$\begin{cases}y^\Delta=py & \text{for } t\in[a,b]_{\mathbb{T}}^\kappa\\ y(a)=\alpha,\end{cases}$$

where $a\in\mathbb{T}^\kappa$, $\alpha\in\mathbb{R}$, and $p\in\mathbb{R}$. Here, $f(t,y):=py$ for $(t,y)\in[a,b]_{\mathbb{T}}^\kappa\times\mathbb{R}$, which satisfies all conditions of Theorem 4. Using (18) to define Picard iterates $\{\varphi_m\}_{m\in\mathbb{N}}$, we write that

$$\varphi_m(t):=\alpha+\int_a^t[p\varphi_{m-1}(\eta)]\Delta\eta\quad\text{for } t\in[a,b]_{\mathbb{T}} \text{ and } m\in\mathbb{N},$$

where $\varphi_0(t):\equiv\alpha$ for $t\in[a,b]_{\mathbb{T}}$. We compute for $t\in[a,b]_{\mathbb{T}}$ that

$$\begin{aligned}\varphi_1(t)=&\alpha+\int_t^a[p\alpha]\Delta\eta\\ =&\alpha[1+p\mathrm{h}_1(t,a)]\end{aligned}$$

and

$$\begin{aligned}\varphi_2(t) =&\alpha + \int_a^t \big[p\alpha[1 + p\mathrm{h}_1(\eta, a)]\big]\Delta\eta\\ =&\alpha + \int_a^t [p\alpha]\Delta\eta + \int_a^t p^2\alpha\mathrm{h}_1(\eta, a)\Delta\eta\\ =&\alpha\big[1 + p\mathrm{h}_1(t, a) + p^2\mathrm{h}_2(t, a)\big].\end{aligned}$$

Hence, in general, we find

$$\varphi_m(t) = \alpha\sum_{k=0}^{m} p^k\mathrm{h}_k(t, a) \quad \text{for } t \in [a, b]_{\mathbb{T}}.$$

Letting $m \to \infty$*, we see that the unique solution is*

$$\varphi(t) = \alpha\sum_{k=0}^{\infty} p^k\mathrm{h}_k(t, a) \quad \text{for } t \in [a, b]_{\mathbb{T}}.$$

On the other hand, we know that $\psi(t) := \alpha\mathrm{e}_p(t, a)$ *for* $t \in [a, b]_{\mathbb{T}}$ *is a solution. Thus, due to the uniqueness by Theorem 4, we obtain the Taylor series expansion*

$$\mathrm{e}_p(t, a) = \sum_{k=0}^{\infty} p^k\mathrm{h}_k(a, t) \quad \text{for } t \in [a, b]_{\mathbb{T}}.$$

7. Final discussion

From the main results of the paper, we can deduce the following corollaries.

Corollary 1. *Assume that a solution of (2) exists on* $[a, \zeta]_{\mathbb{T}}$*, where* $\zeta \in [a, b)_{\mathbb{T}}$ *is right-scattered. Then the solution can be extended to* $[a, \sigma(\zeta)]_{\mathbb{T}}$ *naturally as in (10). Furthermore, if the solution is unique on* $[a, \zeta]_{\mathbb{T}}$*, then the solution, which is extended to* $[a, \sigma(\zeta)]_{\mathbb{T}}$*, is also unique.*

Corollary 2. *Assume that* f *satisfies the condition (13) in Theorem 3 except possibly at a finite number of right-scattered points in* $\mathbb{T}$ *(i.e. (13) can be assumed to hold in some neighborhood of each right-dense point in* $\mathbb{T}$*). Then (2) can admit at most one solution.*

Now we make our final comments on the condition (14) in Example 6.

Remark 5. *For any* $h \in \mathbb{R}^+$*, the number of points in* $[0, l]_{\mathbb{P}_{\alpha,\beta}}$ *whose graininess is greater than* $\sqrt{2}\alpha$ *is finite. Thus, (14) holds for all points* $t \in [a, b]_{\mathbb{P}_{\alpha,\beta}}$ *for which* $\mu(t) \leq \sqrt{2}\alpha$ *holds. Therefore, for any* $\alpha \in \mathbb{R}^+$ *and* $\beta \in \mathbb{R}_0^+$*, the IVP (15) admits the unique solution* $\varphi := \frac{2}{3}\mathrm{h}_2(\cdot, a)$ *on* $[0, l]_{\mathbb{P}_{\alpha,\beta}}$*. On the other hand, it is obvious that* f *defined by (16) is not Lipschitz rd-continuous since*

$$|f(t, x) - f(t, y)| = 2\frac{\mathrm{h}_1(t, 0)}{\mathrm{h}_2(t, 0)}|x - y| \quad \text{for all } t \in (0, l]_{\mathbb{P}_{\alpha,\beta}} \text{ and } 0 < x, y \leq 2\mathrm{h}_2(t, 0)$$

and

$$\lim_{t\to 0^+}\frac{2\mathrm{h}_1(t, 0)}{\mathrm{h}_2(t, 0)} = \lim_{t\to 0^+}\frac{2t}{\frac{t^2}{2}} = \infty.$$

Moreover, it is easy to show that the Picard iterates for (15) *are*

$$\varphi_m(t) := (-1)^{m-1} 2\mathrm{h}_2(t,0) \quad \textit{for } t \in [0,l]_{\mathbb{P}_{\alpha,\beta}} \textit{ and } m \in \mathbb{N},$$

showing that they do not converge as $m \to \infty$*.*

Let us now make some remarks about the proof of the Cauchy–Peano theorem. In [8, Theorem 3.1], an existence result using the induction principle was given. However, this technique does not tell much about the structure of the solution.

The so-called Tonelli sequence

$$\varphi_m(t) := \begin{cases} \alpha, & t \in [a,t_1]_{\mathbb{R}} \\ \alpha + \displaystyle\int_a^{t-d_m} f\big(\eta, \varphi_m(\eta)\big)\mathrm{d}\eta, & t \in (t_k, t_{k+1}]_{\mathbb{R}} \text{ and } k \in \{1,2,\cdots,m-1\}, \end{cases}$$

where $d_m := \frac{b-a}{m}$ and $t_k := a + d_m k$ for $k = 1,2,\cdots,m-1$, significantly simplifies the proof of the usual Cauchy–Peano theorem by avoiding the requirement of the Arzelà–Ascoli theorem (see [1, Theorem 9.1] and [9, Theorem 8.27]). However, this technique cannot be easily adapted for arbitrary time scales because of the varying graininess function.

On the other hand, replacing in (7) the first component of f under the integral by the lower limit of the integrals leads us to the well-known Euler–Cauchy polygons:

$$\varphi_m(t) := \begin{cases} \alpha + f(a,\alpha)(t-\alpha), & t \in [a,t_1]_{\mathbb{T}} \\ \varphi_m(t_k) + f\big(t_k, \varphi_m(t_k)\big)(t-t_k), & t \in (t_k, t_{k+1}]_{\mathbb{T}} \text{ and } k \in \{1,2,\cdots,n-1\}, \end{cases}$$

which brings another handicap. If we try to proceed with Euler–Cauchy polygons, then we require f to be uniformly continuous on R_{h_0} to show that $\lim_{m\to\infty} E_m(t) = 0$ uniformly for $t \in [a,b]_{\mathbb{T}}$. However, under this assumption, we cannot even apply the existence theorem for the simple dynamic equation

$$\begin{cases} y^{\Delta} = p(t)y & \text{for } t \in [a,b]_{\mathbb{T}}^{\kappa} \\ y(a) = \alpha, \end{cases}$$

where $p \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$, whose solution is known to exist on the entire interval $[a,b]_{\mathbb{T}}$ (see [3, Theorem 8.24]).

The results obtained in this paper can be easily extended to systems of equations and to Banach spaces as in [6].

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