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Fedja's proof of Deepti's inequality

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Abstract: The paper aims to present, in a systematic way, an elegant proof of Deepti's inequality. Both the inequality and various ideas concerning the issue were discussed on the Mathoverflow website by a number of users, but none have appeared in the literature thus far. In this work, suggestions pertaining to users 'Deepti' and 'fedja' are traced, whence the title. The results or the paper are new, and the proof is divided into a series of statements, many of which are of interest in themselves.

Key words: Peano kernel, quadrature formula, inequalities

1. Introduction

The following question was posted on the Mathoverflow website:

Is it true that, for all $q \in (0,1)$, $k \in \mathbb{N}_0$ and $x \in [0,1]$, the inequality

$$\frac{x^k}{(1-q)(1-q^3)\cdots(1-q^{2k-1})} \leqslant \prod_{j=1}^{\infty} \frac{1}{1-q^{2j-1}x} \tag{1}$$

holds?

The question was posted by a Mathoverflow user with the nickname 'Deepti', and for this reason we call (1) *Deepti's inequality*. This question drew the interest of a number of users who discussed it from different angles. A guideline (see https://mathoverflow.net/questions/269740/inequality-for-functions-on-0-1) of the solution to this problem as presented here was proposed by the user known to us only by his/her user name, 'fedja'. Being inspired by the intriguing nature of the suggested approach, the authors provide a complete solution in such a way that it becomes accessible to a wide range of readers including undergraduate students. To our knowledge, the inequality and its proof have not appeared anywhere in the literature before.

It has to be mentioned here that Deepti's inequality, being of interest in itself, has already found applications related to the theory of q-Bernstein operators; see [1], where its generalization has also been obtained. We are positive that this inequality will be used in future works and that the ideas and techniques presented in this article will be handy for many researchers.

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2. First glance

First of all, let us try to perform some preliminary analysis of (1). Denote the functions in the LHS and RHS by $u_k(x), k \in \mathbb{N}$ and $\varphi(x)$, respectively. Evidently, $\varphi(x)$ is monotone increasing on $[0,1]$ with $\varphi(0) = 1$ and $\varphi(1) = \prod_{j=1}^{\infty} \left(1 - q^{2j-1}\right)^{-1}$. Obviously, the last infinite product is convergent.

When $k = 0$, then $u_0(x) \equiv 1$ and there is nothing to prove. Furthermore, it is clear that (1) holds if $x = 0$ and $x = 1$ for all $k \in \mathbb{N}$.

Next, let us rewrite (1) in the following form:

$$\frac{x(1-qx)}{1-q} \cdot \frac{x(1-q^3x)}{1-q^3} \cdots \frac{x(1-q^{2k-1}x)}{1-q^{2k-1}} \leqslant \prod_{j=k+1}^{\infty} \frac{1}{1-q^{2j-1}x},$$

and note that, if $q \leqslant 1/2$, then each ratio in the LHS is a monotone increasing function in x on $[0,1]$. Its maximum value is attained at $x = 1$ and equals 1, justifying the last inequality along with (1) for $q \leqslant 1/2$. As a result, only the case $q > 1/2, x \in [0,1)$, and $k \geqslant 1$ is left to be examined.

We proceed with the next observation.

Observation 2.1 *If, for every* $k \in \mathbb{N}$, *the inequality*

$$u_k(x) \leqslant \varphi(x) \tag{2}$$

holds when $x \in [0, 1-q^{2k+1}]$, *then it holds for all* $x \in [0,1]$.

Proof The validity of (2) at $x = 1$ has already been stated. For $x \in [0,1)$, denote

$$u(x) := \max_{k \in \mathbb{N}} u_k(x). \tag{3}$$

As $\varphi(x)$ does not depend on k, inequality (1) is equivalent to $u(x) \leqslant \varphi(x)$, $x \in [0,1)$. It is not difficult to see that

$$u(x) = \begin{cases} u_1(x) & \text{for} \quad x \in [0, 1-q^3], \\ u_2(x) & \text{for} \quad x \in [1-q^3, 1-q^5], \\ \dots & \dots \quad \dots, \\ u_k(x) & \text{for} \quad x \in [1-q^{2k-1}, 1-q^{2k+1}], \\ \dots & \dots \quad \dots \end{cases} \tag{4}$$

and, therefore, due to the fact that $[0,1) = \cup_{k=1}^{\infty} [0, 1-q^{2k+1}]$, it is sufficient to prove that, for each $k \in \mathbb{N}$, there holds:

$$u_k(x) \leqslant \varphi(x) \ \text{ when } \ x \in [0, 1-q^{2k+1}].$$

□

Before the proof of (1) is presented, some auxiliary results will be given.

3. Some challenging calculus

In this section, a few results that will later contribute to proving the main theorem are presented. Each one can be viewed as a challenging problem of calculus, although typically absent from standard texts. It shall also be pointed out that they can be of interest outside of the context of this article.

Proposition 3.1 *Let* $f:(0,\infty)\to(0,\infty)$ *be a decreasing continuous involution; that is,* $f^{-1}=f$. *Then, for every* $S,T>0$, *the following inequality is valid:*

$$-ST+\int_0^T f(x)dx \leqslant \int_S^\infty f(x)dx. \tag{5}$$

Proof The inequality can be established geometrically using the symmetry of the curve $y=f(x)$ about the line $y=x$; see the Figure.

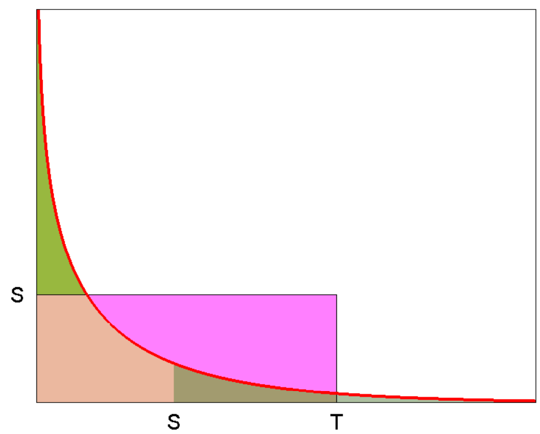

Figure. $\int_0^T f(t)dt \leqslant ST+\int_S^\infty f(t)dt$.

□

Example 3.1 *In the sequel, this proposition will be applied to the function* $f(x)=-\ln(1-e^{-x})$. *This is a continuous decreasing function on* $(0,\infty)$. *In addition,*

$$y=-\ln(1-e^{-x}) \Leftrightarrow e^{-x}+e^{-y}=1,$$

whence f *in an involution.*

By plain calculation, one obtains:

$$\rho(x):=f''(x)=\frac{1}{e^x+e^{-x}-2}, \quad x>0. \tag{6}$$

This function will be used repeatedly within this work.

Proposition 3.2 *Let* ρ *be given by* (6). *Then, for all* $s,t>0$, *the following inequality is valid:*

$$\rho(s+t)\leqslant e^{-s}\rho(t). \tag{7}$$

Proof Equivalently, one may prove that $1/\rho(s+t)\geqslant e^s/\rho(t)$; that is, $e^{-t}-2\leqslant e^{-2s-t}-2e^{-s}$. If $s=0$, then both sides are equal, while for $s>0$, the derivative of the right-hand side with respect to s is positive, which yields the statement. □

Proposition 3.3 *Let* $a < b < c$, $K(x) \geqslant 0$ *for* $x \in [a,c]$, *and* f/g *be an increasing function on* $[a,c]$ *with* $f(x) \neq 0$ *for* $x \in [a,b]$. *If there is a number* $\alpha > 0$ *such that*

$$\int_a^b K(x)f(x)\,dx \geqslant \alpha \int_b^c K(x)f(x)\,dx, \tag{8}$$

then

$$\int_a^b K(x)g(x)dx \geqslant \alpha \int_b^c K(x)g(x)dx. \tag{9}$$

Proof It is clear that

$$\frac{f(x)}{g(x)} \leqslant \frac{f(b)}{g(b)} \leqslant \frac{f(y)}{g(y)} \quad \text{for all} \quad \text{a} \leqslant x \leqslant b \leqslant y \leqslant c.$$

Therefore,

$$\begin{aligned}\int_a^b K(x)g(x)dx &= \int_a^b K(x)f(x)\,\frac{g(x)}{f(x)}\,dx \geqslant \frac{g(b)}{f(b)}\int_a^b K(x)f(x)\,dx\\ &\geqslant \alpha\frac{g(b)}{f(b)}\int_b^c K(x)f(x)\,dx = \alpha\frac{g(b)}{f(b)}\int_b^c K(x)g(x)\,\frac{f(x)}{g(x)}\,dx\\ &\geqslant \alpha\frac{g(b)}{f(b)}\,\frac{f(b)}{g(b)}\int_b^c K(x)g(x)\,dx = \alpha\int_b^c K(x)g(x)\,dx.\end{aligned}$$

□

The next claim is based on the classical Peano theorem regarding error estimate in quadrature formulae. Here, we only supply a simplified version needed for Section 4, while the theorem itself can be found, for example, in [2, Theorem 3.2.3, page 123]. Let $\int_a^b f(x)dx$ be approximated by the midpoint rule, i.e., by

$$I(f) = (b-a)f\left(\frac{a+b}{2}\right).$$

The error of this approximation is

$$R(f) = \int_a^b f(x)dx - I(f).$$

For $f \in C^2[a,b]$, the error estimate can be derived from the Peano theorem. The result is below.

Proposition 3.4 *Let* $f \in C^2[a,b]$ *and* $\int_a^b f(x)dx$ *be approximated by the midpoint rule. Then the error* $R(f)$ *equals:*

$$R(f) = \int_a^b K_{a,b}(x)f''(x)dx,$$

where

$$K_{a,b}(x) = \begin{cases} \frac{(x-a)^2}{2} & \text{if} \quad a \leqslant x \leqslant (a+b)/2,\\ \frac{(b-x)^2}{2} & \text{if} \quad (a+b)/2 \leqslant x \leqslant b.\end{cases} \tag{10}$$

The proof is omitted because this is a direct consequence of Peano's theorem.

Now consider the approximation of $\int_a^b f(x)dx$ by the midpoint Riemann sums with step $h = (b-a)/k$; that is, set

$$Q(f) = h \sum_{j=1}^{k} f\left(a + \frac{2j-1}{2} \cdot h\right). \tag{11}$$

In the case $b = \infty$, for any step $h > 0$, we take

$$Q(f) = h \sum_{j=1}^{\infty} f\left(a + \frac{2j-1}{2} \cdot h\right). \tag{12}$$

Denote the errors of approximation with (11) and (12) by $E_{a,b}$ and $E_{a,\infty}$, respectively. To be exact, set

$$E_{a,b} := \int_a^b f(x)dx - Q(f) \tag{13}$$

and likewise for $b = \infty$. Then,

$$E_{a,b} = \sum_{j=1}^{k} E_{a+(j-1)h, a+jh}.$$

Denote by $K(x)$ the h-periodic extension of $K_{0,h}(x)$ – see (10) – on $\mathbb{R}$. In other words, one has $K(x+h) = K(x)$ for all $x \in \mathbb{R}$ and

$$K_{0,h}(x) = \begin{cases} \frac{x^2}{2} & \text{if} \quad 0 \leqslant x \leqslant h/2, \\ \frac{(h-x)^2}{2} & \text{if} \quad h/2 \leqslant x \leqslant h. \end{cases} \tag{14}$$

Peano's theorem implies that

$$E_{a,b} = \int_a^b K(x-a) f''(x) dx. \tag{15}$$

Proposition 3.5 *Let* $f(x) = -\ln(1-e^{-x})$, $x > 0$. *Then, for all* $s, a > 0$ *and any step size* h, *one obtains the following:*

$$E_{s+a,\infty} \leqslant e^{-s} E_{a,\infty}. \tag{16}$$

Proof By virtue of (15), we can write:

$$E_{s+a,\infty} = \int_{s+a}^{\infty} K(x-a-s)\rho(x)dx,$$

where ρ is given by (6). An application of (7) yields:

$$E_{s+a,\infty} = \int_a^{\infty} \rho(x+s)K(x-a)dx \leqslant e^{-s} \int_a^{\infty} \rho(x)K(x-a)dx = e^{-s}E_{a,\infty}.$$

□

Proposition 3.6 *For every $h > 0$, the following inequality is true:*

$$\int_0^h \frac{K(x)}{x^2} dx \geqslant 8 \int_h^{3h} \frac{K(x)}{x^2} dx. \quad (17)$$

Proof By direct calculations, using (14), one arrives at:

$$\int_0^h \frac{K(x)}{x^2} dx = \frac{1}{2} \int_0^{h/2} dx + \frac{1}{2} \int_{h/2}^h \frac{(h-x)^2}{x^2} dx = \frac{h}{4} \left[1 + \int_0^1 \frac{(1-x)^2}{(x+1)^2} dx \right].$$

Likewise, one gets:

$$\int_h^{3h} \frac{K(x)}{x^2} dx = \frac{h}{4} \left\{ \int_0^1 (1-x)^2 \left[\frac{1}{(3-x)^2} + \frac{1}{(3+x)^2} + \frac{1}{(5-x)^2} + \frac{1}{(5+x)^2} \right] dx \right\}.$$

The obtained integrals can be estimated as follows:

$$1 + \int_0^1 \frac{(1-x)^2}{(x+1)^2} dx \geqslant 1 + \frac{1}{4} \int_0^1 (1-x)^2 dx = \frac{13}{12},$$

while, for the second one, we may use the fact that both functions $\frac{1}{(3-x)^2} + \frac{1}{(3+x)^2}$ and $\frac{1}{(5-x)^2} + \frac{1}{(5+x)^2}$ are increasing on $[0,1]$ yielding:

$$\int_0^1 (1-x)^2 \left[\frac{1}{(3-x)^2} + \frac{1}{(3+x)^2} + \frac{1}{(5-x)^2} + \frac{1}{(5+x)^2} \right]$$
$$\leqslant \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{36} \right) \int_0^1 (1-x)^2 dx = \frac{29}{216}.$$

Since

$$\frac{13}{12} \geqslant 8 \cdot \frac{29}{216} = \frac{29}{27},$$

inequality (17) is justified. □

Corollary 3.2 *For $f(x) = -\ln(1-e^{-x})$ and any step $h > 0$, there holds:*

$$E_{0,h} \geqslant 8E_{h,3h}.$$

Proof First, let us notice that the mapping

$$x \mapsto \frac{1}{x^2 f''(x)} = \frac{e^x + e^{-x} - 2}{x^2} = \frac{1/x^2}{\rho(x)}$$

is increasing in x for all $x > 0$. This can be observed from the fact that Taylor's expansion of

$$\frac{e^x + e^{-x} - 2}{x^2} = \sum_{n=0}^{\infty} \frac{2x^{2n}}{(2n+2)!}$$

has all nonnegative coefficients. Now Propositions 3.3 and 3.6 imply that

$$E_{0,h} = \int_0^h K(x)\rho(x) \geqslant 8 \int_h^{3h} K(x)\rho(x) dx = 8E_{h,3h}.$$

□

4. Proof of Deepti's inequality

Let us formulate the main outcome of this paper.

Theorem 4.1 *For all* $k \in \mathbb{N}_0, q \in (0,1),$ *and all* $x \in [0,1],$ *the following inequality, called Deepti's inequality, holds:*

$$x^k \prod_{j=1}^{k} \frac{1}{1-q^{2j-1}} \leqslant \prod_{j=1}^{\infty} \frac{1}{1-q^{2j-1}x}. \tag{18}$$

Proof Summarizing the previous discussion, to achieve a complete proof of this theorem, we need to prove that, in the notation of Observation 2.1, one has:

$$u_k(x) \leqslant \varphi(x) \text{ for } q \geqslant 1/2,\ x \in [0, 1-q^{2k+1}],\ k \in \mathbb{N}. \tag{19}$$

Taking the logarithms of both sides, we can rewrite (18) in the form:

$$-k \ln\left(\frac{1}{x}\right) - \sum_{j=1}^{k} \ln(1-q^{2j-1}) \leqslant -\sum_{j=1}^{\infty} \ln(1-q^{2j-1}x). \tag{20}$$

Let us choose

$$S = \ln\left(\frac{1}{x}\right),\ h = \ln\left(\frac{1}{q^2}\right), \text{ and } T = kh.$$

The conditions $q \geqslant 1/2$ and $x \leqslant 1 - q^{2k+1}$ can be now expressed as

$$h \leqslant h_0 = \ln 4 \text{ and } e^{-S} \leqslant 1 - e^{-T-h/2}, \tag{21}$$

respectively. It can be observed that, in terms of $f(x) = -\ln(1-e^{-x})$, inequality (20) can be stated as follows:

$$-kS + \sum_{j=1}^{k} f\left(\frac{2j-1}{2} \cdot h\right) \leqslant \sum_{j=1}^{\infty} f\left(S + \frac{2j-1}{2} \cdot h\right),$$

or, after multiplying both sides by h, as:

$$-ST + h\sum_{j=1}^{k} f\left(\frac{2j-1}{2} \cdot h\right) \leqslant h\sum_{j=1}^{\infty} f\left(S + \frac{2j-1}{2} \cdot h\right).$$

The sums on both sides of the last inequality are the midpoint sums with step h for $\int_0^T f(x)dx$ and $\int_S^\infty f(x)dx$, respectively. We refer to formulae (11) and (12). Hence, with the help of (13), the needed inequality becomes:

$$-ST + \int_0^T f(x)dx - E_{0,T} \leqslant \int_S^\infty f(x)dx - E_{S,\infty},$$

subject to conditions (21). By virtue of inequality (5), it suffices to prove that, for $h \leqslant h_0 = \ln 4$,

$$E_{0,T} \geqslant E_{S,\infty}$$

whenever (21) holds. By Corollary 3.2, it can be concluded that

$$E_{0,h} \geqslant 8E_{h,3h} \geqslant e^{3h/2}E_{h,3h} \text{ for } h \leqslant \ln 4$$

and, consequently, with the help of (16) that

$$E_{h,\infty} = E_{h,3h} + E_{3h,\infty} \leqslant e^{-3h/2}E_{0,h} + e^{-2h}E_{h,\infty} \leqslant e^{-3h/2}E_{0,\infty}.$$

What is more, applying (16) once again, one derives:

$$E_{T,\infty} \leqslant e^{-T+h}E_{h,\infty} \leqslant e^{-T-h/2}E_{0,\infty}.$$

As a result, bearing in mind (21), we arrive at:

$$E_{0,T} = E_{0,\infty} - E_{T,\infty} \geqslant \left(1 - e^{-T-h/2}\right) E_{0,\infty} \geqslant e^{-S}E_{0,\infty} \geqslant E_{S,\infty}$$

whenever step size $h \leqslant \ln 4$; that is, $q \geqslant 1/2$.

The proof is complete. □

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