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On spacelike rectifying slant helices in Minkowski 3-space

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Abstract: In this paper, we study the position vector of a spacelike rectifying slant helix with non-lightlike principal normal vector field in E_1^3. First we find the general equations of the curvature and the torsion of spacelike rectifying slant helices. After that, we construct second-order linear differential equations. By their solutions, we determine families of spacelike rectifying slant helices that lie on cones.

Key words: Spacelike curve, rectifying curve, slant helix, cone

1. Introduction

In classical differential geometry, a general helix in the Euclidean 3-space is a curve that makes a constant angle with a fixed direction [10].

The notion of rectifying curve was introduced by Chen [3, 4]. Chen showed under which conditions the position vector of a unit speed curve lies in its rectifying plane. He also stated the importance of rectifying curves in physics.

On the other hand, the notion of slant helix was introduced by Izuyama and Takeuchi [6, 7]. They showed under which conditions a unit speed curve is a slant helix. In [8, 9], Kula et al. studied the spherical images under both tangent and binormal indicatrices of slant helices and obtained that the spherical images of a slant helix are spherical helices.

Later, Ali published two papers about characterizations of slant helices in Minkowski 3-space [1, 2].

The papers mentioned above led us to study the notion of spacelike rectifying slant helices. We begin by finding the equations of curvature and torsion of spacelike rectifying slant helices. After that, we construct second-order linear differential equations to determine position vectors of spacelike rectifying slant helices. By solving these equations for some special cases, we obtain unit speed families of rectifying slant helices that lie on cones.

2. Basic concepts

The Minkowski 3-space E_1^3 is the real vector space R^3 with the metric

$$g = dx_1^2 + dx_2^2 - dx_3^2,$$

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where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since g is an indefinite metric, the pseudo-norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$.

A vector $v \in E_1^3$ is called spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$, and lightlike (null) if $g(v,v) = 0$ and $v \neq 0$ [5].

Given a curve $\alpha : I \subset R \to {E_1}^3$, we say that the curve α is spacelike (resp. timelike, lightlike) if $\alpha'(s)$ is spacelike (resp. timelike, lightlike) at any $s \in I$ where $\alpha'(s) = d\alpha/ds$ [5].

A spacelike curve $\alpha : I \subset R \longrightarrow {E_1}^3$ is said to be parametrized by the pseudo arclength parameter s if $g(\alpha^{'}(s), \alpha^{'}(s)) = 1$. In this case, we say that α is a unit speed curve.

Let us take an arbitrary plane π in E_1^3. We call π a spacelike plane (resp. timelike plane, lightlike plane) if $g|\pi$ is positive definite (resp. nondegenerate of index 1, degenerate). Recall that when α is a unit speed spacelike curve that has at least four continuous derivatives with spacelike or timelike rectifying plane, the Frenet equations are as follows [5]:

$$\begin{aligned} t^{'} &= \kappa n, \\ n^{'} &= -\varepsilon_1 \kappa t + \tau b, \\ b^{'} &= -\varepsilon_1 \varepsilon_2 \tau n, \end{aligned} \tag{1}$$

where κ is the curvature, τ is the torsion, and $\{t, n, b\}$ is the Frenet frame of the curve α with $\varepsilon_1 = g(n,n) = \pm 1$, $\varepsilon_2 = g(b,b) = \pm 1$, $\varepsilon_1 \varepsilon_2 = -1$. We denote unit spacelike tangent vector field with t, unit non-lightlike principal normal vector field with n, and the unit non-lightlike binormal vector field with b.

As we know, n can be considered as the normal indicatrix curve of the curve α. If n is a non-lightlike curve, we know that $\varepsilon_3 = \mathrm{sgn}[g(n^{'}, n^{'})] = \pm 1$. Note that when n is a timelike curve, $\varepsilon_3 = -1$ in equation (1).

Definition 1 *A curve is called a slant helix if its principal normal makes a constant angle with a fixed direction in E_1^3 [7].*

Lemma 1 *Let α be a unit speed spacelike curve with timelike principal normal vector field in E_1^3. Then α is a slant helix if and only if the geodesic curvature of the spherical image of principal normal indicatrix (n) of α,*

$$\sigma = \frac{\kappa^2}{(\tau^2 + \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)^{'},$$

is constant everywhere $\tau^2 + \kappa^2$ does not vanish [2].

Lemma 2 *Let α be a unit speed spacelike curve with spacelike principal normal vector field in E_1^3. Then α is a slant helix if and only if the geodesic curvature of the spherical image of principal normal indicatrix (n) of α,*

$$\sigma = \frac{\kappa^2}{(-\varepsilon_3 \tau^2 + \varepsilon_3 \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)^{'},$$

is constant everywhere $\tau^2 - \kappa^2$ does not vanish [2].

Definition 2 *The curve α is called a rectifying curve when the position vector of it always lies in its rectifying plane [3, 4].*

Thus, for a rectifying curve, we can write

$$\alpha(s) = \lambda(s)\, t(s) + \mu(s)\, b(s).$$

Lemma 3 *Let α be a unit speed non-lightlike curve with spacelike or timelike principal normal vector field in E_1^3; then α is congruent to a rectifying curve if and only if*

$$\frac{\tau(s)}{\kappa(s)} = c_1 s + c_2$$

for some constants c_1 and c_2, with $c_1 \neq 0$ [5].

The angle between two vectors in E_1^3 is defined in [1].

Definition 3 *Let u and v be spacelike vectors in E_1^3 that span a spacelike vector subspace. Then there is a unique positive real number θ such that*

$$|g(u,v)| = \|u\|\,\|v\| \cos\theta.$$

θ is called the Lorentzian spacelike angle between u and v.

Definition 4 *Let u and v be spacelike vectors in E_1^3 that span a timelike vector subspace. Then there is a unique positive real number θ such that*

$$|g(u,v)| = \|u\|\,\|v\| \cosh\theta.$$

θ is called the Lorentzian timelike angle between u and v.

Definition 5 *Let u be a spacelike vector and v a positive timelike vector in E_1^3. Then there is a unique positive real number θ such that*

$$|g(u,v)| = \|u\|\,\|v\| \sinh\theta.$$

θ is called the Lorentzian timelike angle between u and v.

Definition 6 *Let u and v be positive (negative) timelike vectors in E_1^3. Then there is a unique positive real number θ such that*

$$|g(u,v)| = \|u\|\,\|v\| \cosh\theta.$$

θ is called the Lorentzian timelike angle between u and v.

3. Spacelike rectifying slant helices in E_1^3

In E_1^3, if the position vector of a unit speed spacelike slant helix always lies in its rectifying plane, we call it a spacelike rectifying slant helix. For a spacelike rectifying slant helix, we have the following theorems.

Theorem 1 *Let α be a unit speed spacelike curve that has timelike principal normal vector field in E_1^3; then α is a spacelike rectifying slant helix if and only if the curvature and torsion of the curve satisfy the equations below:*

$$\kappa(s) = \frac{c_3}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}}, \tau(s) = \frac{c_3\,(c_1 s + c_2)}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}},$$

where $c_1 \neq 0$, $c_2 \in R$, and $c_3 \in R^+$.

Proof Let α be a unit speed spacelike rectifying slant helix in E_1^3; then, by combining the equations in Lemma 1 and Lemma 3, we have

$$\sigma = \frac{c_1}{\kappa\left((c_1 s + c_2)^2 + 1\right)^{3/2}},$$

where $\sigma \neq 0$ is a constant. Thus, we can write κ as follows

$$\kappa(s) = \frac{c_3}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}},$$

and then

$$\tau(s) = \frac{c_3\left(c_1 s + c_2\right)}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}},$$

where $c_3 = |c_1/\sigma|$.

Conversely, it can be easily seen that the curvature functions mentioned above satisfy the equations in Lemma 1 and Lemma 3. Thus, α is a unit speed spacelike rectifying slant helix that has timelike principal normal in E_1^3. □

Since the proofs of Theorem 2 and Theorem 3 are similar to the proof above, we omit them.

Theorem 2 *Let α be a unit speed spacelike curve that has spacelike principal normal vector field and spacelike principal normal indicatrix in E_1^3; then α is a spacelike rectifying slant helix if and only if the curvature and torsion of the curve satisfy the equations below:*

$$\kappa(s) = \frac{c_3}{\left(1 - (c_1 s + c_2)^2\right)^{3/2}}, \tau(s) = \frac{c_3\left(c_1 s + c_2\right)}{\left(1 - (c_1 s + c_2)^2\right)^{3/2}},$$

where $c_1 \neq 0$, $c_2 \in R$, and $c_3 \in R^+$.

Theorem 3 *Let α be a unit speed spacelike curve that has spacelike principal normal vector field and timelike principal normal indicatrix in E_1^3; then α is a spacelike rectifying slant helix if and only if the curvature and torsion of the curve satisfy the equations below:*

$$\kappa(s) = \frac{c_3}{\left((c_1 s + c_2)^2 - 1\right)^{3/2}}, \tau(s) = \frac{c_3\left(c_1 s + c_2\right)}{\left((c_1 s + c_2)^2 - 1\right)^{3/2}},$$

where $c_1 \neq 0$, $c_2 \in R$, and $c_3 \in R^+$.

Theorem 4 *Let α be a unit speed spacelike rectifying slant helix that has timelike principal normal vector field in E_1^3. Then the vector $h = \frac{n'}{\kappa}$ satisfies the linear vector differential equation of second order as follows:*

$$h''(s) - \frac{c_3^2}{\left((c_1 s + c_2)^2 + 1\right)^2} h(s) = 0. \tag{2}$$

Proof Let α be a unit speed spacelike rectifying slant helix and then we can write Frenet equations as follows:

$$\begin{aligned} t^{'} &= \kappa n, \\ n^{'} &= \kappa t + f\kappa b, \\ b^{'} &= f\kappa n, \end{aligned} \tag{3}$$

where $f(s) = c_1 s + c_2$. If we divide the second equation by κ, we have

$$\frac{n'}{\kappa} = t + fb. \tag{4}$$

By differentiating equation (4), we have

$$c_1 b = \left(\frac{n'}{\kappa}\right)' - \kappa(f^2+1)n. \tag{5}$$

By differentiating equation (5) and using equation (3), we have

$$\left(\frac{n'}{\kappa}\right)'' - \kappa(f^2+1)n' - \left[\left(\kappa(f^2+1)\right)' + c_1 f\kappa\right] n = 0, \tag{6}$$

and we know

$$\kappa(s) = \frac{c_3}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}}$$

and with the necessary calculations we easily see

$$\left(\kappa(f^2+1)\right)' + c_1 f\kappa = 0.$$

Thus, equation (6) becomes

$$\left(\frac{n'}{\kappa}\right)'' - \kappa(f^2+1)n' = 0. \tag{7}$$

Let us denote $\frac{n'}{\kappa} = h$. Then equation (7) becomes

$$h''(s) - \frac{c_3^2}{\left((c_1 s + c_2)^2 + 1\right)^2} h(s) = 0.$$

This completes the proof. □

Since the proofs of Theorem 5 and Theorem 6 are similar to the proof above, we again omit them.

Theorem 5 *Let α be a unit speed spacelike curve that has spacelike principal normal vector field and spacelike principal normal indicatrix in E_1^3. Then the vector $h = \frac{n'}{\kappa}$ satisfies the linear vector differential equation of second order as follows:*

$$h''(s) + \frac{c_3^2}{\left(1 - (c_1 s + c_2)^2\right)^2} h(s) = 0. \tag{8}$$

Theorem 6 *Let α be a unit speed spacelike curve that has spacelike principal normal vector field and timelike principal normal indicatrix in E_1^3. Then the vector $h = \frac{n'}{\kappa}$ satisfies the linear vector differential equation of second order as follows:*

$$h''(s) - \frac{c_3^2}{\left((c_1 s + c_2)^2 - 1\right)^2} h(s) = 0. \tag{9}$$

Now we give another theorem to determine c_3 in Theorem 1. Some parts of this theorem will be useful for us later on.

Theorem 7 *Let α be a unit speed spacelike rectifying slant helix whose timelike principal normal vector field makes a constant angle with a timelike vector v, and then the curvature and torsion of α satisfy the equations below:*

$$\kappa(s) = \frac{|c_1 \tanh(\theta)|}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}}, \tau(s) = \frac{|c_1 \tanh(\theta)|\,(c_1 s + c_2)}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}},$$

where $c_1 \neq 0$, $c_2 \in R$.

Proof Let α be a unit speed spacelike rectifying slant helix whose timelike principal normal vector field makes a constant angle with a timelike unit vector v. Then, from Definition 6,

$$g(n, v) = \cosh(\theta),$$

where $\theta \in R^+$. If we differentiate this equation with respect to pseudo arclength parameter s, we have

$$g(\kappa t + \tau b, v) = 0.$$

If we divide both parts of the equation by κ, we get

$$g(t + (c_1 s + c_2) b, v) = 0. \tag{10}$$

Then,

$$g(t, v) = -(c_1 s + c_2) g(b, v).$$

While $\{t, n, b\}$ is a orthonormal frame, we can write

$$v = \lambda_1 t + \lambda_2 n + \lambda_3 b$$

with $\lambda_1^2 - \lambda_2^2 + \lambda_3^2 = -1$. If we make the neccessary calculations, we have

$$\lambda_1 = \mp \frac{(c_1 s + c_2)\sinh(\theta)}{\sqrt{(c_1 s + c_2)^2 + 1}}, \quad \lambda_2 = \cosh(\theta), \quad \lambda_3 = \pm \frac{\sinh(\theta)}{\sqrt{(c_1 s + c_2)^2 + 1}}.$$

By differentiating equation (10), we have

$$\pm \frac{c_1 \sinh(\theta)}{\kappa \sqrt{(c_1 s + c_2)^2 + 1}} - \left(1 + (c_1 s + c_2)^2\right) \cosh(\theta) = 0.$$

Therefore,

$$\kappa(s) = \frac{|c_1 \tanh(\theta)|}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}}$$

and

$$\tau(s) = \frac{|c_1 \tanh(\theta)| \, (c_1 s + c_2)}{\left((c_1 s + c_2)^2 + 1\right)^{3/2}}.$$

□

Remark 1 *For a unit speed spacelike rectifying slant helix that has spacelike principal normal vector field that makes a constant angle with a unit vector* v *in* E_1^3*, we easily see:*

Case 1: *If* v *is a spacelike unit vector and* $\{n, v\}$ *spans a timelike subspace, then*

$$c_3 = |c_1 \tanh \theta| \, .$$

Case 2: *If* v *is a spacelike unit vector and* $\{n, v\}$ *spans a spacelike subspace, then*

$$c_3 = |c_1 \tan \theta| \, .$$

Case 3: *If* v *is a positive timelike unit vector, then*

$$c_3 = |c_1 \coth \theta| \, .$$

3.1. Position vector of spacelike rectifying slant helices

Now we will find the position vector of a spacelike rectifying slant helix by using the differential equation (2):

$$\begin{aligned} h_1(s) &= -\sqrt{\left((c_1 s + c_2)^2 + 1\right)} \sin\left[\operatorname{sech}(\theta) \arctan\left(c_1 s + c_2\right)\right], \\ h_2(s) &= \sqrt{\left((c_1 s + c_2)^2 + 1\right)} \cos\left[\operatorname{sech}(\theta) \arctan\left(c_1 s + c_2\right)\right], \\ h_3(s) &= 0, \end{aligned} \tag{11}$$

is a solution for equation (2) where $h = (h_1, h_2, h_3)$.

If α is a unit speed spacelike rectifying slant helix that has a timelike principal normal vector field that makes a constant angle θ with e_3, then from Definition 6 we can write

$$g(n, e_3) = \cosh(\theta).$$

Therefore, from equation (11), we can write

$$\begin{aligned} n_1(s) &= \textstyle\int \kappa(s) h_1(s) ds = \sinh(\theta) \cos\left[\operatorname{sech}(\theta) \arctan\left(c_1 s + c_2\right)\right], \\ n_2(s) &= \textstyle\int \kappa(s) h_2(s) ds = \sinh(\theta) \sin\left[\operatorname{sech}(\theta) \arctan\left(c_1 s + c_2\right)\right], \\ n_3(s) &= -\cosh(\theta), \end{aligned}$$

with

$$g(n', n') = \frac{c_1^2 \tanh^2(\theta)}{\left((c_1 s + c_2)^2 + 1\right)^2} > 0.$$

If we use Frenet equations for spacelike unit speed curves, we have

$$\begin{array}{l}\alpha_1(s) = \int(\int \kappa(s) n_1(s) ds) ds, \\ \alpha_2(s) = \int(\int \kappa(s) n_2(s) ds) ds, \\ \alpha_3(s) = \int(\int \kappa(s) n_3(s) ds) ds,\end{array}$$

and so,

$$\begin{aligned} \alpha_1(s) =& \frac{\cosh(\theta)}{c_1}\sqrt{(c_1 s + c_2)^2 + 1}\cos\left[\operatorname{sech}(\theta)\arctan(c_1 s + c_2)\right], \\ \alpha_2(s) =& \frac{\cosh(\theta)}{c_1}\sqrt{(c_1 s + c_2)^2 + 1}\sin\left[\operatorname{sech}(\theta)\arctan(c_1 s + c_2)\right], \\ \alpha_3(s) =& -\frac{\sinh(\theta)}{c_1}\sqrt{(c_1 s + c_2)^2 + 1}, \end{aligned} \tag{12}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

With the help of the lemma below, we reach our goal.

Lemma 4 *Let α be a curve with the equation (12) in E_1^3 where $\theta \in R^+$, $c_1 \neq 0$, and $c_2 \in R$. Then α is a unit speed spacelike rectifying slant helix that lies on the cone*

$$z^2 = \tanh^2(\theta)\left(x^2 + y^2\right).$$

Proof With direct calculations, we have $g(\alpha', \alpha') = 1$, $g(n, n) = -1$, $g(n', n') > 0$, and the curvature functions of α as

$$\kappa(s) = \frac{|c_1 \tanh(\theta)|}{((c_1 s + c_2)^2 + 1)^{3/2}}, \tau(s) = \frac{|c_1 \tanh(\theta)|\,(c_1 s + c_2)}{((c_1 s + c_2)^2 + 1)^{3/2}},$$

with

$$\frac{\kappa^2(s)}{(\kappa^2(s) + \tau^2(s))^{3/2}}\left(\frac{\tau(s)}{\kappa(s)}\right)' = \coth(\theta)$$

and

$$\frac{\tau(s)}{\kappa(s)} = c_1 s + c_2.$$

Thus, α is a unit speed spacelike rectifying slant helix. We also have

$$\tanh^2(\theta)\left({\alpha_1}^2(s) + {\alpha_2}^2(s)\right) - {\alpha_3}^2(s) = 0,$$

and then α lies on the cone above. □

By using equations (8) and (9), we have the following remark and lemmas. Since the proofs are similar to the proof above, we omit them.

Remark 2 *For a unit speed spacelike rectifying slant helix that has a spacelike principal normal vector field that makes a constant angle with a unit vector v in E_1^3, we have:*

Case 1: *If* $v = e_2$ *and* $g(n', n') > 0$, *then*

$$\begin{aligned} \beta_1(s) &= -\frac{\cosh(\theta)}{c_1}\sqrt{1-(c_1 s + c_2)^2}\sinh\left[\operatorname{sech}(\theta)\operatorname{arctanh}(c_1 s + c_2)\right], \\ \beta_2(s) &= -\frac{\sinh(\theta)}{c_1}\sqrt{1-(c_1 s + c_2)^2}, \\ \beta_3(s) &= -\frac{\cosh(\theta)}{c_1}\sqrt{1-(c_1 s + c_2)^2}\cosh\left[\operatorname{sech}(\theta)\operatorname{arctanh}(c_1 s + c_2)\right], \end{aligned} \tag{13}$$

where $\beta = (\beta_1, \beta_2, \beta_3)$.

Case 2: *If* $v = e_1$ *and* $g(n', n') < 0$, *then*

$$\begin{aligned} \gamma_1(s) &= -\frac{1}{c_1}\sqrt{(c_1 s + c_2)^2 - 1}\sin(\theta), \\ \gamma_2(s) &= -\frac{\cos(\theta)}{c_1}\sqrt{(c_1 s + c_2)^2 - 1}\cosh\left[\sec(\theta)\operatorname{arccoth}(c_1 s + c_2)\right], \\ \gamma_3(s) &= \frac{\cos(\theta)}{c_1}\sqrt{(c_1 s + c_2)^2 - 1}\sinh\left[\sec(\theta)\operatorname{arccoth}(c_1 s + c_2)\right], \end{aligned} \tag{14}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$.

Case 3: *If* $v = e_3$ *and* $g(n', n') > 0$, *then*

$$\begin{aligned} \varphi_1(s) &= -\frac{\sinh(\theta)}{c_1}\sqrt{1-(c_1 s + c_2)^2}\cos\left[\operatorname{csch}(\theta)\operatorname{arctanh}(c_1 s + c_2)\right], \\ \varphi_2(s) &= -\frac{\sinh(\theta)}{c_1}\sqrt{1-(c_1 s + c_2)^2}\sin\left[\operatorname{csch}(\theta)\operatorname{arctanh}(c_1 s + c_2)\right], \\ \varphi_3(s) &= \frac{\cosh(\theta)}{c_1}\sqrt{1-(c_1 s + c_2)^2}, \end{aligned} \tag{15}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$.

Lemma 5 *Let* β *be a curve with the equation (13) in* E_1^3 *where* $\theta \in R^+$, $c_1 \neq 0$, *and* $c_2 \in R$. *Then* β *is a unit speed spacelike rectifying slant helix that lies on the cone*

$$z^2 = \coth^2(\theta)y^2 + x^2.$$

Lemma 6 *Let* γ *be a curve with the equation (14) in* E_1^3 *where* $\theta \in R^+$, $c_1 \neq 0$, *and* $c_2 \in R$. *Then* γ *is a unit speed spacelike rectifying slant helix that lies on cone*

$$y^2 = \cot^2(\theta)x^2 + z^2.$$

Lemma 7 *Let* φ *be a curve with the equation (15) in* E_1^3 *where* $\theta \in R^+$, $c_1 \neq 0$, *and* $c_2 \in R$. *Then* α *is a unit speed spacelike rectifying slant helix that lies on the cone*

$$z^2 = \coth^2(\theta)\left(x^2 + y^2\right).$$

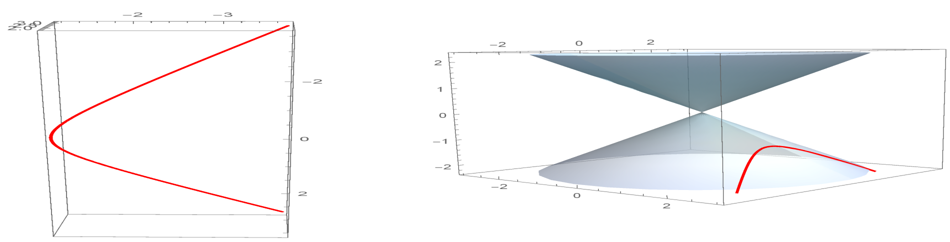
Example 1 *If we take* $c_1 = 1$, $c_2 = 0$, *and* $\theta = 1$ *in (12), then we have*

$$\alpha(s) = \sqrt{s^2+1}\Big(\cosh(1)\cos\left[\operatorname{sech}(1)\arctan(s)\right], \\ \cosh(1)\sin\left[\operatorname{sech}(1)\arctan(s)\right], \\ -\sinh(1)\Big),$$

$$\kappa(s) = \frac{\tanh(1)}{(s^2+1)^{3/2}}, \quad \tau(s) = \frac{s\tanh(1)}{(s^2+1)^{3/2}},$$

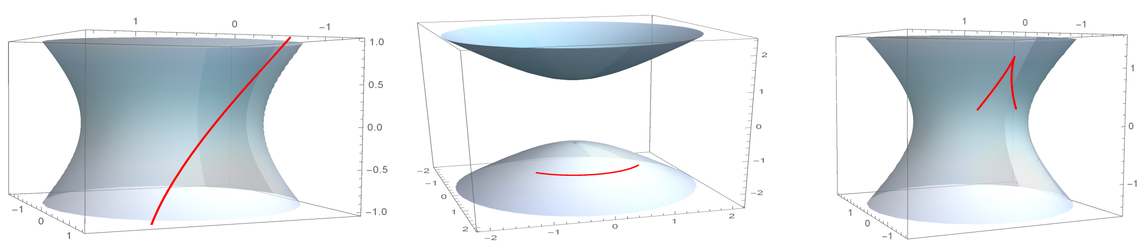
$$\tanh^2(1)\left(x^2+y^2\right) = z^2.$$

We can see the curve α *in Figures 1a and 1b. The indicatrices of it are given in Figures 2a–2c.*



(a) (b)

Figure 1. (a) Spacelike rectifying slant helix α. (b) Spacelike rectifying slant helix α lies on the cone, $\tanh^2(1)\left(x^2+y^2\right) = z^2$.



(a) (b) (c)

Figure 2. Tangent indicatrix of the curve α. (b) Normal indicatrix of the curve α. (c) Binormal indicatrix of the curve α.

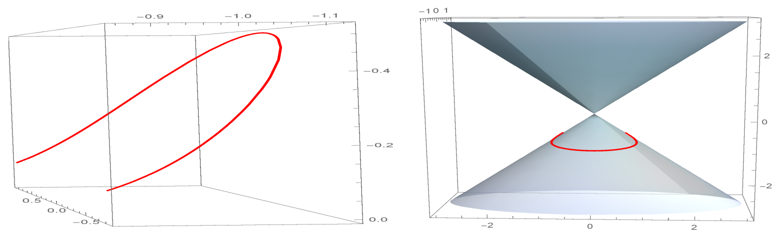
Example 2 *If we take* $c_1 = 1$, $c_2 = 0$, *and* $\theta = 1/2$ *in (13), then we have*

$$\beta(s) = -\sqrt{1-s^2}\Big(\cosh(1/2)\sinh\left[\operatorname{sech}(1/2)\operatorname{arctanh}(s)\right], \\ \sinh(1/2), \\ \cosh(1/2)\cosh\left[\operatorname{sech}(1/2)\operatorname{arctanh}(s)\right]\Big),$$

$$\kappa(s) = \frac{\tanh(1/2)}{(1-s^2)^{3/2}}, \quad \tau(s) = \frac{s\tanh(1/2)}{(1-s^2)^{3/2}},$$

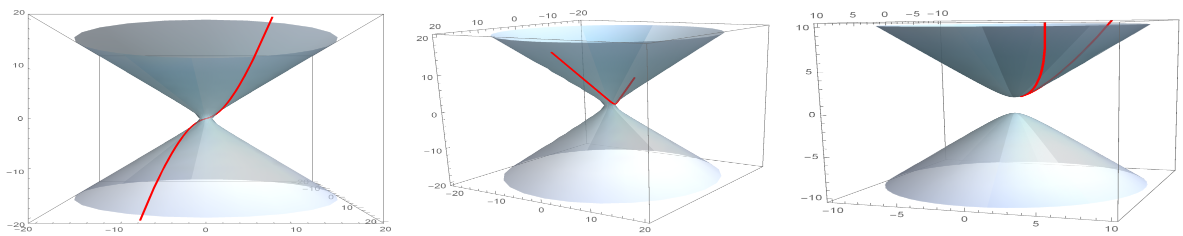
$$z^2 = \coth^2(1/2)\, y^2 + x^2.$$

We can see the curve β in Figures 3a and 3b. The indicatrices of it are given in Figures 4a–4c.



(a) (b)

Figure 3. Spacelike rectifying slant helix β. (b) Spacelike rectifying slant helix β lies on the cone, $z^2 = \coth^2(1/2)\, y^2 + x^2$.



(a) (b) (c)

Figure 4. (a) Tangent indicatrix of the curve β. (b) Normal indicatrix of the curve β. (c) Binormal indicatrix of the curve β.

Example 3 *If we take $c_1 = 1$, $c_2 = 0$, and $\theta = \pi/3$ in (14), then we have*

$$\gamma(s) = \sqrt{s^2-1}\Big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\cosh\left[2\,\mathrm{arccoth}\,(s)\right], \frac{1}{2}\sinh\left[2\,\mathrm{arccoth}\,(s)\right]\Big),$$

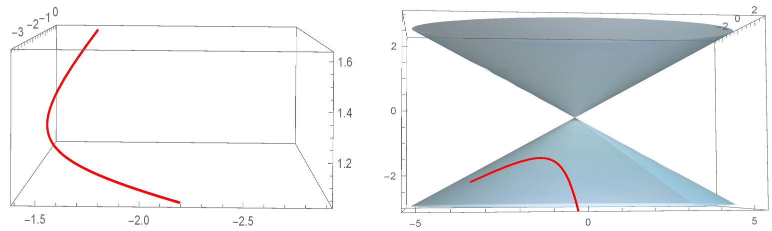
$$\kappa(s) = \frac{\sqrt{3}}{(s^2-1)^{3/2}}, \quad \tau(s) = \frac{\sqrt{3}s}{(s^2-1)^{3/2}},$$

$$y^2 = \frac{1}{3}x^2 + z^2.$$

We can see the curve γ in Figures 5a and 5b. The indicatrices of it are given in Figures 6a–6c.

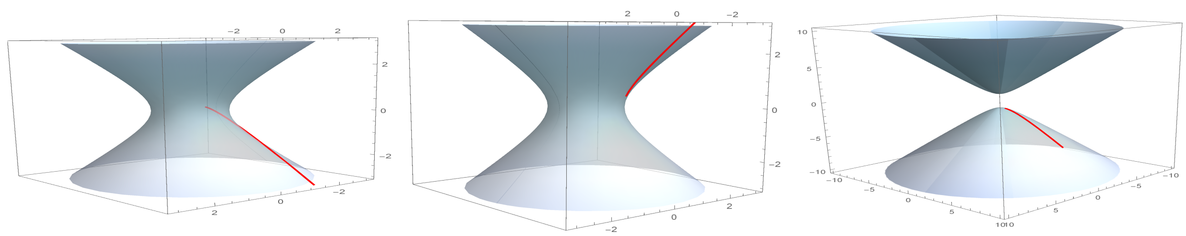
Example 4 *If we take $c_1 = 1, c_2 = 0$, and $\theta = 1/2$ in (15), then we have*

$$\begin{aligned}\varphi(s) = \sqrt{1-s^2}\Big(& -\sinh(1/2)\cos\left[\mathrm{csch}\,(1/2)\,\mathrm{arctanh}\,(s)\right], \\ & -\sinh(1/2)\sin\left[\mathrm{csch}\,(1/2)\,\mathrm{arctanh}\,(s)\right], \\ & \cosh(1/2)\Big),\end{aligned}$$



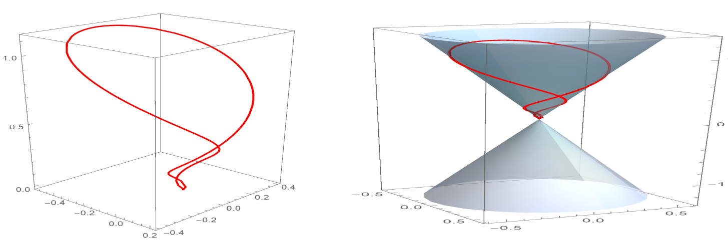
(a) (b)

Figure 5. (a) Spacelike rectifying slant helix γ. (b) Spacelike rectifying slant helix γ lies on the cone, $y^2 = (1/3)\,x^2 + z^2$.



(a) (b) (c)

Figure 6. (a) Tangent indicatrix of the curve γ. (b) Normal indicatrix of the curve γ. (c) Binormal indicatrix of the curve γ.



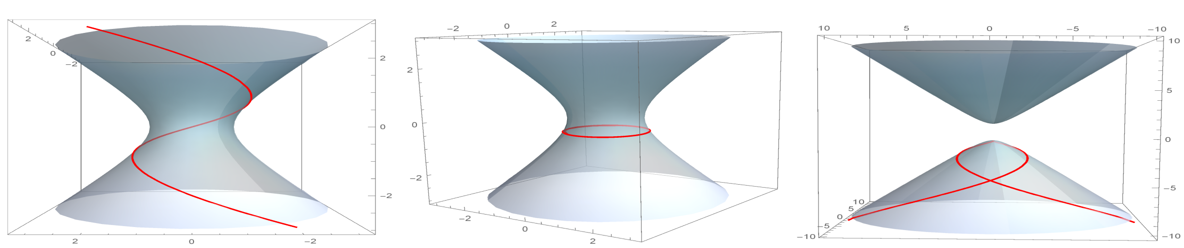
(a) (b)

Figure 7. (a) Spacelike rectifying slant helix φ. (b) Spacelike rectifying slant helix φ lies on the cone, $z^2 = \coth^2(1/2)\,(x^2 + y^2)$.

$$\kappa(s) = \frac{\coth(1/2)}{(1-s^2)^{3/2}}, \quad \tau(s) = \frac{s\coth(1/2)}{(1-s^2)^{3/2}},$$

$$z^2 = \coth^2(1/2)\,(x^2 + y^2).$$

We can see the curve φ in Figures 7a and 7b. The indicatrices of it are given in Figures 8a–8c.



(a) (b) (c)

Figure 8. (a) Tangent indicatrix of the curve φ. (b) Normal indicatrix of the curve φ. (c) Binormal indicatrix of the curve φ.

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