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Somos's theta-function identities of level 10

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Abstract: Somos discovered about 6277 theta-function identities of different levels using a computer and offered no proof for them, and these identities closely resemble Ramanujan's recordings. The purpose of this paper is to prove some of his theta-function identities of level 10 and to establish certain partition identities for them.

Key words: Theta-functions, modular equations, Dedekind η-function, colored partitions

1. Introduction

Throughout the paper, we use the standard q-series notation and f_k is defined as

$$f_k := (q^k;q^k)_\infty = \prod_{m=1}^{\infty}(1-q^{mk}), \qquad |q|<1.$$

and often we write

$$(a_1,a_2,...,a_n;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty...(a_n;q)_\infty.$$

Recall that the Ramanujan's theta-function $f(a,b)$ is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1.$$

Jacobi's triple product identity [1, p. 35] can be restated in Ramanujan's notation as follows:

$$f(a,b)=(-a,-b,ab;ab)_\infty.$$

The most important special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2f_4^2} \tag{1.1}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{1.2}$$

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Also, after Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty = \frac{f_2^2}{f_1 f_4} \qquad \textit{and} \qquad \chi(-q) := (q;q^2)_\infty = \frac{f_1}{f_2}. \tag{1.3}$$

Note that if $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind η-function for $Im(\tau) > 0$. The theta-function identity that relates $f(-q)$ to $f(-q^n)$ is called the theta-function identity of level n. Ramanujan recorded several identities that involve $f(-q), f(-q^2), f(-q^n)$, and $f(-q^{2n})$ in his second notebook [3] and 'lost' notebook [4]. For example [2, p. 206, Entry 53],

$$f_1^4 f_2^4 f_5^2 f_{10}^2 + 5 f_1^2 f_2^2 f_5^4 f_{10}^4 = f_2^6 f_5^6 + f_1^6 f_{10}^6.$$

Michael Somos recently used a computer to discover several new elegant modular equations in the spirit of Ramanujan and offered no proof for them. Somos's identities closely resemble Ramanujan's recordings of the above type. He has a large list of η-product identities and he runs PARI/GP scripts to look at each identity in $P-Q$ forms. Recently Yuttanan [6] proved certain Somos theta-function identities of different levels by employing Ramanujan's modular equations and deduced certain partition identities for them, and Vasuki and Veeresha [5] proved η-function identities of level 14 discovered by Somos. Somos discovered sixty new elegant η-function identities of level 10. The purpose of this paper is to prove some of these identities conjectured by Somos and to establish certain partition identities for them. Before proceeding to state and prove Somos's identities, we first recall certain modular equations and theta-function identities that we will need in the sequel. The Gauss ordinary hypergeometric series is defined by

$${}_2F_1(a,b;c;x) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k \qquad |x| < 1,$$

where $(a)_n := a(a+1)(a+2)...(a+n-1)$ for any positive integer n. Letting $F(x) := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$, then the relation between α and β induced by the following equation is called a modular equation of degree n:

$$n \frac{F(1-\alpha)}{F(\alpha)} = \frac{F(1-\beta)}{F(\beta)}.$$

Supposing that $y = \pi \frac{F(1-x)}{F(x)}$ and $z = F(x)$, then we have from [1, pp. 122–124, Entry 10(i) and Entry 12(v)],

$$\varphi(e^{-y}) = \sqrt{z} \tag{1.4}$$

and

$$\chi(e^{-y}) = 2^{1/6}\big(x(1-x)e^{-y}\big)^{-1/24}. \tag{1.5}$$

Also, we define the multiplier m by

$$m := \frac{F(\alpha)}{F(\beta)}.$$

On page 236 of his second notebook [3] and [1, pp. 280–288, Entry 13(ix) and (xiv)], Ramanujan recorded the following modular equations of degree 5. If β has degree 5 over α and m is the multiplier for degree 5, then

$$1 + 4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} = \frac{m}{2}\left(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\right), \tag{1.6}$$

$$1+4^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12}=\frac{5}{2m}\left(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right) \tag{1.7}$$

and if $P=\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$ and $Q=\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8}$, then

$$Q+\frac{1}{Q}+2\left(P-\frac{1}{P}\right)=0. \tag{1.8}$$

From (1.6) and (1.7), we deduce

$$\frac{m^2}{5}=\frac{1+4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}}{1+4^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12}}. \tag{1.9}$$

Transcribing (1.8) and (1.9) into a theta-function by employing (1.4) and (1.5), we obtain

$$\frac{q\chi^3(q)}{\chi^3(q^5)}+\frac{\chi^3(q^5)}{\chi^3(q)}+\frac{4q}{\chi^2(q)\chi^2(q^5)}-\chi^2(q)\chi^2(q^5)=0 \tag{1.10}$$

and

$$\frac{\varphi^4(q)}{5\varphi^4(q^5)}=\frac{1+4\frac{q^2\chi^2(q)}{\chi^{10}(q^5)}}{1+4\frac{\chi^2(q^5)}{\chi^{10}(q)}}, \tag{1.11}$$

respectively. Also from (1.1)–(1.3) we observe that

$$\frac{\varphi(q)}{\varphi(q^5)}=\frac{\chi^2(q)}{\chi^2(q^5)}\frac{f_2}{f_{10}}. \tag{1.12}$$

Before concluding this section, for convenience we set

$$a:=a(q)=q^{-1/24}\chi(q) \qquad \text{and} \qquad b:=b(q)=q^{-5/24}\chi(q^5).$$

2. Somos's identities

Theorem 2.1 *We have*

$$f_1^8f_{10}^4+20qf_1f_2^3f_5^3f_{10}^5+4f_1^3f_2^5f_5f_{10}^3-5f_2^4f_5^8=0.$$

Proof On multiplying (1.10) throughout by $4a^{-5}b^{-3}(a^6-b^6)$, we obtain

$$\frac{4a^7}{b^3}+4b^8+\frac{16a^2}{b^2}-4b^2a^6-\frac{16b^4}{a^4}-\frac{4b^9}{a^5}=0,$$

which is equivalent to

$$b^8\left(\frac{a^5}{b}+4\right)\left(1+\frac{4a^2}{b^{10}}\right)-a^8\left(\frac{4b^2}{a^2}+\frac{b^7}{a^3}\right)\left(1+\frac{4b^2}{a^{10}}\right)=0.$$

Employing (1.11) in the above, we deduce

$$\frac{a^5}{b}+4-\frac{5a^8}{b^8}\left(\frac{4b^2}{a^2}+\frac{b^7}{a^3}\right)\frac{\varphi^4(q^5)}{\varphi^4(q)}=0.$$

Using (1.12) in the above, we see that

$$\frac{a^5}{b}+4-5q^{4/3}\left(\frac{4b^2}{a^2}+\frac{b^7}{a^3}\right)\left(\frac{f_{10}}{f_2}\right)^4=0.$$

Replacing q by $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_1^3f_2^5f_5f_{10}^3$, we obtain the required result. □

Theorem 2.2 *We have*

$$f_2^6f_5^6+qf_1^6f_{10}^6-f_1^4f_2^4f_5^2f_{10}^2-5qf_1^2f_2^2f_5^4f_{10}^4=0.$$

Proof Multiplying (1.10) throughout by $4a^{-10}b^{-22}(a^6-4ab+a^5b^5+b^6)(a^{12}-2a^6b^6+b^{12})$, we obtain

$$\frac{4a^{14}}{b^{22}}-\frac{72a^2}{b^{10}}+\frac{4b^2}{a^{10}}-\frac{64}{a^8b^8}+\frac{8a^6}{b^6}-4+\frac{32a^8}{b^{16}}+\frac{32}{a^4b^4}+\frac{128}{a^2b^{14}}-\frac{4a^{12}}{b^{12}}-\frac{64a^4}{b^20}=0,$$

which is equivalent to

$$\left(1-\frac{a^6}{b^6}\right)^2\left(1+\frac{4b^2}{a^{10}}\right)\left(1+\frac{4a^2}{b^{10}}\right)=5\left[\left(1+\frac{4a^2}{b^{10}}\right)-\frac{a^6}{b^6}\left(1+\frac{4b^2}{a^{10}}\right)\right]^2.$$

Using (1.11) in the above, we see that

$$\left(1-\frac{a^6}{b^6}\right)\frac{\varphi^2(q^5)}{\varphi^2(q)}-1+5\frac{a^6}{b^6}\frac{\varphi^4(q^5)}{\varphi^4(q)}=0.$$

Employing (1.12) in the above, we obtain

$$q^{2/3}\left(\frac{b^4}{a^4}-\frac{a^2}{b^2}\right)\left(\frac{f_{10}}{f_2}\right)^2-1+q^{4/3}\frac{5b^2}{a^2}\left(\frac{f_{10}}{f_2}\right)^4=0.$$

Replacing q by $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_1^4f_2^4f_5^2f_{10}^2$, we obtain the required result. □

Theorem 2.3 *We have*

$$f_1^{10}f_5^2f_{10}^2+10qf_1^3f_2^3f_5^5f_{10}^3-f_2^{10}f_5^4-15q^2f_1^4f_2^2f_{10}^8=0.$$

Proof On multiplying (1.10) throughout by $b^{-4}(a^6-4ab-a^5b^5-b^6)$, we obtain

$$a^{10}b^6+\frac{a^{12}}{b^4}-b^8-\frac{16a^2}{b^2}-2a^{11}b-8ab^3=0, \tag{2.1}$$

which is equivalent to

$$b^8\left(\frac{a^{10}}{b^2}-1\right)\left(1+\frac{4a^2}{b^{10}}\right)-a^8\left(2a^3b+\frac{3a^4}{b^4}\right)\left(1+\frac{4b^2}{a^{10}}\right)=0.$$

Employing (1.11) in the above, we see that

$$b^8\left(\frac{a^{10}}{b^2}-1\right)a^8-\left(10a^3b+\frac{15a^4}{b^4}\right)\frac{\varphi^4(q^5)}{\varphi^4(q)}=0.$$

Using (1.12) in the above, we obtain

$$\left(\frac{a^{10}}{b^2}-1\right)-q^{4/3}\left(10a^3b+\frac{15a^4}{b^4}\right)\left(\frac{f_{10}}{f_2}\right)^4=0.$$

Replacing q by $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_2^{10}f_5^4$, we obtain the required result. □

Theorem 2.4 *We have*

$$f_1^9f_5^2f_{10}^3+9qf_1^2f_2^3f_5^5f_{10}^4-f_2^7f_5^7-16q^2f_1^3f_2^2f_{10}^9=0.$$

Proof Multiplying (2.1) throughout by $a^{-2}b^{-16}(16a^{12}-40a^{11}b^5+25a^{10}b^{10}-256a^2b^2-160ab^7+20b^{12})$, we obtain

$$25a^{18}+\frac{121a^{20}}{b^{10}}-\frac{90a^{19}}{b^5}-\frac{360a^9}{b^3}-\frac{12a^{10}}{b^8}+\frac{16a^{22}}{b^{20}}-\frac{72a^{21}}{b^{15}}+\frac{864a^{11}}{b^{13}}$$
$$-\frac{512a^{12}}{b^{18}}+\frac{4608a}{b^{11}}+\frac{4096a^2}{b^{16}}-5a^8b^2-\frac{20b^4}{a^2}+\frac{1216}{b^6}=0,$$

which is equivalent to

$$\left[5a^9\left(1+\frac{4a^2}{b^{10}}\right)-\frac{a^{10}}{b^5}\left(9+\frac{16a}{b^5}\right)\left(1+\frac{4b^2}{a^{10}}\right)\right]^2-5a^8b^2\left(1+\frac{4b^2}{a^{10}}\right)\left(1+\frac{4a^2}{b^{10}}\right)=0.$$

Employing (1.11) in the above, we see that

$$\frac{a^9}{b^5}-\frac{a^{10}}{b^{10}}\left(9+\frac{16a}{b^5}\right)\frac{\varphi^4(q^5)}{\varphi^4(q)}-\frac{a^4}{b^4}\frac{\varphi^2(q^5)}{\varphi^2(q)}=0.$$

Using (1.12) in the above, we obtain

$$\frac{a^9}{b^5}-q^{4/3}\left(\frac{9a^2}{b^2}+\frac{16a^3}{b^7}\right)\left(\frac{f_{10}}{f_2}\right)^4-q^{2/3}\left(\frac{f_{10}}{f_2}\right)^2=0.$$

Replacing q by $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_2^7f_5^7$, we obtain the required result. □

Theorem 2.5 *We have*

$$f_1^9f_5^2f_{10}^3+9qf_1f_2^6f_5^2f_{10}^5-f_2^7f_5^7-25q^2f_1^3f_2^2f_{10}^9=0.$$

Proof Multiplying (2.1) throughout by $a^{-6}b^{-16}(5a^{12}+10a^{11}b^5+5a^{10}b^{10}-44a^2b^2+40ab^7+4b^{12})$, we obtain

$$5a^{18}-\frac{10a^{20}}{b^{10}}-\frac{285a^{10}}{b^8}+\frac{5a^{22}}{b^{20}}-\frac{124a^{12}}{b^{18}}+\frac{704a^2}{b^{16}}-a^8b^2-\frac{4b^4}{a^2}-\frac{18a^9}{b^3}-\frac{72}{ab}$$
$$-\frac{72a^{11}}{b^{13}}-\frac{288a}{b^{11}}-\frac{340}{b^6}=0,$$

which is equivalent to

$$5\left[a^7\left(1+\frac{4a^2}{b^{10}}\right)-\frac{5a^9}{b^{10}}\left(1+\frac{4b^2}{a^{10}}\right)\right]^2=a^4b^2\left(1+\frac{9a}{b^5}\right)^2\left(1+\frac{4b^2}{a^{10}}\right)\left(1+\frac{4a^2}{b^{10}}\right).$$

Using (1.11) in the above, we see that

$$\frac{a^9}{b^5}-\frac{25a^{11}}{b^{15}}\frac{\varphi^4(q^5)}{\varphi^4(q)}=\frac{a^4}{b^4}\left(1+\frac{9a}{b^5}\right)\frac{\varphi^2(q^5)}{\varphi^2(q)}.$$

Employing (1.12) in the above, we obtain

$$\frac{a^9}{b^5}-25q^{4/3}\frac{a^3}{b^7}\left(\frac{f_{10}}{f_2}\right)^4=q^{2/3}\left(1+\frac{9a}{b^5}\right)\left(\frac{f_{10}}{f_2}\right)^2.$$

Replacing q by $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_2^7f_5^7$, we obtain the required result. □

Theorem 2.6 *We have*

$$f_1^8f_{10}^4+24qf_1f_2^3f_5^3f_{10}^5-f_2^4f_5^8-16qf_2^6f_{10}^6=0.$$

Proof On multiplying (2.1) throughout by $16a^{-12}b^{-8}(25a^{12}-10a^{11}b^5+a^{10}b^{10}+80a^2b^2-40ab^7-b^{12})$, we obtain

$$16a^8b^8+\frac{736a^{10}}{b^2}-192a^9b^3-\frac{32b^{10}}{a^2}+3168+\frac{16b^{12}}{b^{12}}+\frac{768b^7}{a^{11}}+\frac{4096b^2}{a^{10}}-\frac{576b^5}{a}$$
$$+\frac{400a^{12}}{b^{12}}-\frac{960a^{11}}{b^7}-\frac{3840a}{b^5}-\frac{5120a^2}{b^{10}}-\frac{20480}{a^8b^8}=0,$$

which is equivalent to

$$a^8b^8\left[5\left(1+\frac{4a^2}{b^{10}}\right)-\left(1+\frac{24a}{b^5}\right)\left(1+\frac{4b^2}{a^{10}}\right)\right]^2=1280\left(1+\frac{4a^2}{b^{10}}\right)\left(1+\frac{4b^2}{a^{10}}\right).$$

Using (1.11) in the above, we see that

$$\frac{\varphi^4(q)}{\varphi^4(q^5)}-\frac{24a}{b^5}-1-\frac{16}{a^4b^4}\frac{\varphi^2(q)}{\varphi^2(q^5)}=0.$$

Employing (1.12) in the above, we obtain

$$q^{-4/3}\frac{a^8}{b^8}\left(\frac{f_2}{f_{10}}\right)^4-\frac{24a}{b^5}-1-\frac{16q^{-2/3}}{b^8}\left(\frac{f_2}{f_{10}}\right)^2=0.$$

Replacing q by $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_2^4 f_5^8$, we obtain the required result. □

Theorem 2.7 *We have*

$$f_1^9 f_{10}^3 + 25q f_1^2 f_2^3 f_5^3 f_{10}^4 - f_2^7 f_5^5 - 16q f_1 f_2^6 f_{10}^5 = 0.$$

Proof On multiplying (2.1) throughout by $a^{-2}b^{-20}(80a^{12} + 256a^2b^2 - 40a^{11}b^5 - 160ab^7 + 5a^{10}b^{10} + 4b^{12})$, we obtain

$$5a^{18} + \frac{165a^{20}}{b^{10}} - 50\frac{a^{19}}{b^5} - \frac{168a^9}{b^3} + \frac{80a^{22}}{b^{20}} - \frac{200a^{21}}{b^{15}} - \frac{672a^{11}}{b^{13}} + \frac{740a^{10}}{b^8}$$
$$+\frac{960}{b^6} - a^8b^2 - \frac{4b^4}{a^2} + \frac{128}{ab} + \frac{512a}{b^{11}} - \frac{1024a^{12}}{b^{18}} - \frac{4096a^2}{b^{16}} = 0,$$

which is equivalent to

$$5\left[a^9\left(1+\frac{4a^2}{b^{10}}\right) - \frac{5a^{10}}{b^5}\left(1+\frac{4b^2}{a^{10}}\right)\right]^2 = a^8b^2\left(1-\frac{16a}{b^5}\right)^2\left(1+\frac{4a^2}{b^{10}}\right)\left(1+\frac{4b^2}{a^{10}}\right).$$

Employing (1.11) in the above, we see that

$$\frac{a^9}{b^5} - \frac{25a^{10}}{b^{10}}\frac{\varphi^4(q^5)}{\varphi^4(q)} = \frac{a^4}{b^4}\left(1-\frac{16a}{b^5}\right)\frac{\varphi^2(q^5)}{\varphi^2(q)}.$$

Using (1.12) in the above, we deduce

$$\frac{a^9}{b^5} - 25q^{4/3}\frac{a^2}{b^2}\left(\frac{f_{10}}{f_2}\right)^4 = q^{2/3}\left(1-\frac{16a}{b^5}\right)\left(\frac{f_{10}}{f_2}\right)^2.$$

Replacing q by $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_2^7 f_5^5$, we obtain the required result. □

Theorem 2.8 *We have*

$$f_1^6 f_2 f_5^6 + 25q^2 f_1^4 f_2 f_{10}^8 + 6q f_1^7 f_5 f_{10}^5 - f_2^9 f_5^4 = 0.$$

Proof Multiplying (2.1) throughout by $(19a^{12} - 10a^{11}b^5 + a^{10}b^{10} + 80a^2b^2 - 40ab^7 + 5b^{12})$, after simplification we obtain

$$a^{20}b^{16}\left(b^2-\frac{6a}{b^3}\right)^2\left(1+\frac{4b^2}{a^{10}}\right)\left(1+\frac{4a^2}{b^{10}}\right) = 5\left[b^{12}\left(1+\frac{4a^2}{b^{10}}\right) - 5a^{12}\left(1+\frac{4b^2}{a^{10}}\right)\right]^2.$$

Employing (1.11) in the above, we see that

$$\frac{a^4}{b^4}\left(a^6b^2 - \frac{6a^7}{b^3}\right)\frac{\varphi^2(q^5)}{\varphi^2(q)} = 1 - \frac{25a^{12}}{b^{12}}\frac{\varphi^4(q^5)}{\varphi^4(q)}.$$

Using (1.12) in the above, we obtain

$$q^{2/3}\left(a^6b^2 - \frac{6a^7}{b^3}\right)\left(\frac{f_{10}}{f_2}\right)^2 = 1 - \frac{25q^{4/3}a^4}{b^4}\left(\frac{f_{10}}{f_2}\right)^4.$$

Replacing q by $-q$ in the above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_2^9 f_5^4$, we obtain the required result. □

Theorem 2.9 *We have*

$$f_1^{10} f_{10}^3 + 15 f_1 f_2^7 f_5^5 + 25q f_1^3 f_2^3 f_5^3 f_{10}^4 - 16 f_2^{10} f_5^2 f_{10} = 0.$$

Proof Multiplying (2.1) throughout by $b^{-24}(16a^{12} - 8a^{11}b^5 + a^{10}b^{10} - 256a^2b^2 - 32ab^7 - 76b^{12})$, after simplification we obtain

$$\left[\left(\frac{a^{10}}{b^2} - 16\right)\left(1 + \frac{4a^2}{b^{10}}\right) - \frac{5a^{11}}{b^7}\left(1 + \frac{4b^2}{a^{10}}\right)\right]^2 = \frac{45a^{10}}{b^2}\left(1 + \frac{4b^2}{a^{10}}\right)\left(1 + \frac{4a^2}{b^{10}}\right).$$

Employing (1.11) in the above, we see that

$$\frac{a^{10}}{b^2} - 16 + \frac{15a^5}{b}\frac{\varphi^2(q^5)}{\varphi^2(q)} - \frac{25a^{11}}{b^7}\frac{\varphi^4(q^5)}{\varphi^4(q)} = 0.$$

Using (1.12) in the above, we obtain

$$\frac{a^{10}}{b^2} - 16 + 15q^{2/3}ab^3\left(\frac{f_{10}}{f_2}\right)^2 - 25q^{4/3}a^3b\left(\frac{f_{10}}{f_2}\right)^4 = 0.$$

Replacing q by $-q$ above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_2^{10} f_5^2 f_{10}$, we obtain the required result. □

Theorem 2.10 *We have*

$$f_1^{10} f_{10}^2 + 25q f_1^3 f_2^3 f_5^3 f_{10}^3 - f_2^{10} f_5^2 + 15q f_1^2 f_2^6 f_{10}^4 = 0.$$

Proof Multiplying (2.1) throughout by $b^{-6}(16a^{12} - 8a^{11}b^5 + a^{10}b^{10} + 44a^2b^2 - 32ab^7 - b^{12})$, we obtain

$$32a^{22} + \frac{16a^{24}}{b^{10}} - \frac{212a^{14}}{b^8} - \frac{40a^{23}}{b^5} - \frac{120a^{13}}{b^3} - \frac{704a^4}{b^6} + \frac{160a^3}{b} - 2a^{10}b^{12}$$
$$+228a^2b^4 + 40ab^9 - 30a^{11}b^7 + 139a^{12}b^2 - 10a^{21}b^5 + a^{20}b^{10} + b^{14} = 0,$$

which is equivalent to

$$\left[b^5(a^2 - b^2)\left(1 + \frac{4a^2}{b^{10}}\right) - 5a^{11}\left(1 + \frac{4b^2}{a^{10}}\right)\right]^2 - 45a^{12}b^2\left(1 + \frac{4b^2}{a^{10}}\right)\left(1 + \frac{4a^2}{b^{10}}\right) = 0.$$

Using (1.11) in the above, we see that

$$\frac{a^{10}}{b^2} + \frac{25a^{11}}{b^7}\frac{\varphi^4(q^5)}{\varphi^4(q)} - 1 + \frac{15a^6}{b^6}\frac{\varphi^2(q^5)}{\varphi^2(q)} = 0.$$

Employing (1.12) in the above, we obtain

$$\frac{a^{10}}{b^2} + 25q^{4/3}a^3b\left(\frac{f_{10}}{f_2}\right)^4 - 1 + 15q^{2/3}\frac{a^2}{b^2}\left(\frac{f_{10}}{f_2}\right)^2 = 0.$$

Replacing q by $-q$ above, expressing $a(-q)$ and $b(-q)$ in terms of f_n by employing (1.3), and then multiplying throughout by $f_2^{10}f_5^2$, we obtain the required result. □

Remark *Now we shall provide a technique to identify the multiplier, which we have used in each of the proofs. To obtain a multiplier in a and b, we first divide Somos's identity to be proved by any one of the terms present in the identity itself. After rearranging the terms, we employ (1.3) and then replace q by $-q$. Further, we employ (1.12) and (1.11) consecutively. Finally, we factorize the resulting identity to obtain a known modular equation (1.10) and a polynomial in a and b and a^kb^l, which is the multiplier itself, and this multiplier is not unique. If we change the dividing term in Somos's identity, we obtain a new multiplier in a and b. Since proofs of Somos's identities are monotonous, we have proved only some of his identities and the remaining identities can be proved by the same technique.*

3. Application to partitions

Somos's identities that we proved in Section 2 have applications in color partitions. In this section, we choose to demonstrate this by giving partition-theoretic interpretations for Theorem 2.1 and 2.2. For simplicity, we define

$$(q^{r\pm};q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty, \qquad (r<s); r,s\in\mathbb{N}.$$

For example, $(q^{2\pm};q^8)_\infty$ means $(q^2,q^6;q^8)_\infty$, which is $(q^2;q^8)_\infty\ (q^6;q^8)_\infty$.

Definition 3.1 *A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called "colored partitions".*

For example, if 2 is allowed to have two colors, say b (blue) and g (green), then all the colored partitions of 4 are $4, 3+1, 2_g+2_b, 2_b+2_g, 2_g+2_g, 2_b+1+1, 2_g+1+1, 1+1+1+1$. Also,

$$\frac{1}{(q^a;q^b)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to $a \pmod b$ and have k colors.

Theorem 3.1 *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm1,\pm3 \pmod{10}$ with seven colors and $\pm2,\pm4 \pmod{10}$ with four colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm1,\pm3,+5 \pmod{10}$ with five colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm1,\pm3 \pmod{10}$ with eight colors and $\pm2,\pm4 \pmod{10}$ with four colors and $+5 \pmod{10}$ with one color. Then for any positive integer $n\geq 1$, the following equality holds true:*

$$20p_1(n-1)+4p_2(n)-5p_3(n)=0.$$

Proof Rewriting Theorem 2.1 subject to the common base q^{10}, we obtain

$$1+\frac{20q}{(q_7^{1\pm},q_4^{2\pm},q_7^{3\pm},q_4^{4\pm};q^{10})_\infty}+\frac{4}{(q_5^{1\pm},q_5^{3\pm},q_5^{5+};q^{10})_\infty}$$

$$-\frac{5}{(q_8^{1\pm},q_4^{2\pm},q_8^{3\pm},q_4^{4\pm},q_1^{5+};q^{10})_\infty}=0.$$

The quotients of the above identity represent the generating functions for $p_1(n), p_2(n)$, and $p_3(n)$, respectively. Hence, the above identity is equivalent to

$$1 + 20q\sum_{n=0}^{\infty} p_1(n)q^n + 4\sum_{n=0}^{\infty} p_2(n)q^n - 5\sum_{n=0}^{\infty} p_3(n)q^n = 0,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. On equating the coefficients of q^n in the above, we obtain the desired result. □

Example Table 1 verifies the case for $n = 2$ in the above theorem.

Table 1.

$p_1(1) = 7$	$1_r, 1_w, 1_g, 1_b, 1_y, 1_o, 1_{bl}$.
$p_2(2) = 15$	$1_r + 1_r, 1_w + 1_w, 1_g + 1_g, 1_b + 1_b, 1_{bl} + 1_{bl}, 1_r + 1_w, 1_r + 1_g, 1_r + 1_b,$ $1_r + 1_{bl}, 1_w + 1_g, 1_w + 1_b, 1_w + 1_{bl}, 1_g + 1_b, 1_g + 1_{bl}, 1_b + 1_{bl}$.
$p_3(2) = 40$	$1_r + 1_r, 1_w + 1_w, 1_g + 1_g, 1_b + 1_b, 1_{bl} + 1_{bl}, 1_y + 1_y, 1_o + 1_o, 1_m + 1_m,$ $1_r + 1_w, 1_r + 1_g, 1_r + 1_b, 1_r + 1_{bl}, 1_r + 1_y, 1_r + 1_o, 1_r + 1_m, 1_w + 1_g,$ $1_w + 1_{bl}, 1_w + 1_b, 1_w + 1_y, 1_w + 1_o, 1_w + 1_m, 1_g + 1_b, 1_g + 1_{bl}, 1_g + 1_y,$ $1_g + 1_o, 1_g + 1_m, 1_b + 1_{bl}, 1_b + 1_y, 1_b + 1_o, 1_b + 1_m, 1_{bl} + 1_y, 1_{bl} + 1_o,$ $1_{bl} + 1_m, 1_y + 1_o, 1_y + 1_m, 1_o + 1_m, 2_r, 2_w, 2_g, 2_b$.

Theorem 3.2 *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm3 \pmod{10}$ with six colors and $\pm2, \pm4 \pmod{10}$ with two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm2, \pm4 \pmod{10}$ with two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm3 \pmod{10}$ with two colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm2, \pm3, \pm4 \pmod{10}$ with four colors. Then, for any positive integer $n \geq 1$, the following equality holds true:*

$$p_1(n) + p_2(n-1) - p_3(n) - 5p_4(n-1) = 0.$$

Proof On rewriting Theorem 2.2 subject to the common base q^{10}, we obtain

$$\frac{1}{(q_6^{1\pm}, q_2^{2\pm}, q_6^{3\pm}, q_2^{4\pm}; q^{10})_\infty} + \frac{q}{(q_2^{2\pm}, q_2^{4\pm}; q^{10})_\infty} - \frac{1}{(q_2^{1\pm}, q_2^{3\pm}; q^{10})_\infty}$$

$$- \frac{5q}{(q_4^{1\pm}, q_4^{2\pm}, q_4^{3\pm}, q_4^{4\pm}; q^{10})_\infty} = 0.$$

The four quotients of the above identity represent the generating functions for $p_1(n), p_2(n), p_3(n)$, and $p_4(n)$, respectively. Hence, the above identity is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q\sum_{n=0}^{\infty} p_2(n)q^n - \sum_{n=0}^{\infty} p_3(n)q^n - 5q\sum_{n=0}^{\infty} p_4(n)q^n = 0,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. On equating the coefficients of q^n in the above, we obtain the desired result. □

Example Table 2 verifies the case for $n = 2$ in the above theorem.

Table 2.

$p_1(2) = 23$	$1_r + 1_r, 1_w + 1_w, 1_b + 1_b, 1_g + 1_g, 1_y + 1_y, 1_o + 1_o, 1_r + 1_w, 1_r + 1_b, 1_r + 1_g, 1_r + 1_y, 1_r + 1_o, 1_w + 1_b, 1_w + 1_g, 1_w + 1_y, 1_w + 1_o, 1_b + 1_g, 1_b + 1_y, 1_b + 1_o, 1_g + 1_y, 1_g + 1_o, 1_y + 1_o, 2_r, 2_w.$
$p_2(1) = 0$	
$p_3(2) = 3$	$1_r + 1_r, 1_w + 1_w, 1_r + 1_w.$
$p_4(1) = 4$	$1_r, 1_w, 1_b, 1_g.$

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