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Modified objective function approach for multitime variational problems

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Abstract: The present paper is devoted to studying the modified objective function approach used for solving the considered multitime variational problem. In this method, a new multitime variational problem is constructed by modifying the objective function in the original considered multitime variational problem. Further, the equivalence between an optimal solution to the original multitime variational problem and its associated modified problem is established under both hypotheses of invexity and generalized invexity defined for a multitime functional. Thereafter, using the modified objective function method, we derive the saddle-point results for the considered multitime variational problem. Moreover, we provide some examples to illustrate the results established in the paper.

Key words: Multitime variational problem, modified objective function method, saddle-point criteria, optimality conditions, generalized invex functions

1. Introduction

Convexity has a dominant role in optimization theory. However, there exist optimization problems of various types for which the concept of convexity cannot be used in proving the fundamental results from optimization theory. Recently, many generalizations of convex functions have been proposed for the purpose of weakening the limitations of convexity. Among these generalizations, the notion of invexity was first introduced by Hanson [10]. The results developed by Hanson inspired a great deal of subsequent works, which have greatly expanded the role of invexity in optimization (see, for example, [2, 6, 7, 12]).

Variational problems come from calculus of variations. The relationship between optimization problems and calculus of variations was explored by Hanson [9]. Later, several researchers showed their interest in solving variational control problems. Craven [8] considered a multiobjective variational problem and established the Kuhn–Tucker type necessary optimality conditions for it under pseudoconvexity and quasiconvexity assumptions. Thereafter, Arana-Jimenez et al. [4] derived the various duality results for the considered multiobjective variational problem by using the introduced concept of pseudoinvexity. Some other contributions for variational control problems have been given in many works (see, for example, [3, 5, 11, 13, 15, 20] and others).

The term multitime was initially introduced in physics by Dirac in 1932 and was later used in mathematics (see, for example, [18, 21]). Multitime control theory is related to the partial derivatives of dynamical systems

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and their optimization over multitime is also known as the multidimensional control problems. Multitime control problems have been applied in various fields of science. Various operations research (O.R.) problems, for example, in applied science and technology ranging from economics (processes control), psychology (impulse control disorders), and medicine (bladder control) to engineering (robotics and automation) and biology (population ecosystems), lead to traditional control problems. However, such kinds of O.R. problems heavily rely on the temporal dependence of these applications. That is why multitime control problems have been intensively studied in the last few years both from theoretical and applied viewpoints. Methods for solving nonconvex multitime variational problems, as a type of variational problems, in our opinion, remain some unexplored questions for research. To the best of our knowledge, there are only a few papers devoted to methods that can be used for the characterization of solvability in nonconvex multitime variational problems.

Udriste and Tevy [22] extended the theory of single-time dynamic programming to the multitime case when the evolution is m-dimensional and the functional includes a path-independent curvilinear integral. They also described the use of multitime dynamic programming method in multitime optimal controls. Pitea and Postolache [17] considered curvilinear integral type multitime multiobjective variational problems and discussed Mond–Weir type duality under the assumption of (ρ, b)-quasiinvexity.

Recently, in [19], Postolache proved Mond–Weir–Zalmai type duality results for multitime multiobjective variational problems in which a vector of quotients of functionals of curvilinear integral type is minimized. Very recently, Pitea and Antczak [16] considered a new class of generalized nonconvex multitime multiobjective variational problems to investigate the sufficient optimality conditions for efficiency and proper efficiency of the considered vector optimization problem of such a type by using the introduced concept of univexity defined for functionals of curvilinear integral type.

In recent years, considerable attention has been given to devising new methods that solve the original mathematical programming problem and its duals by the help of some associated optimization problem. One of such methods is the modified objective function method, which was originally introduced by Antczak [1] for differentiable multiobjective programming problems. Antczak used this approach to obtain optimality conditions for (weak) Pareto optimality for the considered nonconvex multiobjective programming problem by constructing for it an equivalent vector minimization problem.

The aim of our paper is to explore optimality conditions by using the modified objective function method for a new class of nonconvex optimization problems, that is, multitime variational problems with invex functionals of curvilinear integral type. Hence, the modified objective function method, which was introduced by Antczak [1] for differentiable optimization problems, is extended to a new class of nonconvex extremum problems. In other words, this method is used for the first time for characterization of solvability of multitime variational problems. In this approach, for the original multitime variational problem, we construct at a fixed feasible point its associated multitime variational problem with the modified objective function. It turns out that such a construction of an associated multitime variational problem with the modified objective function makes it that an optimal solution to the original multitime variational problem is also an optimal solution to its associated modified multitime variational problem and vice versa. The equivalence between optimal solutions for the original multitime variational problem and in its associated modified multitime variational problem is established under invexity and generalized invexity hypotheses. Further, using the modified objective function approach, we establish the relationship between an optimal solution to the original considered multitime variational problem and a saddle-point of the Lagrange function in its associated modified multitime variational problem.

This paper is organized as follows: in Section 2, we recall some preliminary definitions, theorems, and lemmas that we use in proving the main results in the paper. Further, we introduce the definitions of invexity and psuedoinvexity for a multitime functional of curvilinear integral type. We also analyze the relationship between the introduced concepts of invexity and pseudoinvexity, presenting an example of such a multitime functional that is pseudoinvex but not invex with respect to the same function η. In Section 3, using the modified objective function method, we construct a new multitime variational problem by modifying the objective function of the considered multitime variational problem. Then we establish the relationship between an optimal solution for the original multitime variational problem and its associated modified multitime variational problem. Afterwards, in Section 4, we give the definition of the Lagrange function and its saddle-point in the associated modified multitime variational problem and establish the relationship between an optimal solution for the original multitime variational problem and a saddle-point in its associated modified multitime variational problem constructed in the used method. Finally, in Section 5, we conclude our paper.

2. Notations and preliminaries

Let (T, h) and (M, g) be two Riemannian manifolds of dimensions m and n, respectively. Further, let Ω be the measurable set in T and $t = t^\alpha = (t^1, \ldots, t^m) \in \Omega$, $x = x^i = (x^1, \ldots, x^n)$ be the points in M. Consider $dv = dt^1 \ldots dt^m$ as the volume element on Ω and, moreover, a first order jet bundle associated to T, M is denoted by $J^1(T, M) = \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm}$.

Throughout this paper, we shall use the following inequalities and equalities for any two vectors $(x^1, \ldots, x^n)$, $(y^1, \ldots, y^n) \in M$:

(i) $x = y \Leftrightarrow x^i = y^i, \ \forall \ i = 1, \ldots, n;$

(ii) $x < y \Leftrightarrow x^i < y^i, \ \forall \ i = 1, \ldots, n;$

(iii) $x \leqq y \Leftrightarrow x^i \leqq y^i, \ \forall \ i = 1, \ldots, n;$

(iv) $x \leq y \Leftrightarrow x \leqq y$ and $x \neq y$.

In the paper, consider the following multitime variational problem:

$$\begin{aligned} \textbf{(MVP)} \quad \text{minimize} \quad & \int_\Omega f(\pi_x(t))\, dv \\ \text{subject to} \quad & g(\pi_x(t)) \leqq 0, \\ & h(\pi_x(t)) = 0, \\ & x(t)|_{\partial\Omega} = u(t), \ t \in \Omega, \end{aligned}$$

where $f : J^1(T, M) \to \mathbb{R}$, $g = (g^\alpha) : J^1(T, M) \to \mathbb{R}^m; \alpha = \{1, \ldots, m\}$ and $h = (h^\beta) : J^1(T, M) \to \mathbb{R}^q; \beta = \{1, \ldots, q\}$ are C^2-class functions, $\pi_x(t) = (t, x(t), x_\gamma(t))$, $x_\gamma(t) = \dfrac{\partial x(t)}{\partial t^\gamma}; \gamma = \{1, \ldots, m\}$ are the partial velocities and $x : \Omega \to M$.

Let S denote the feasible set of the considered multitime variational problem (MVP), i.e.

$$S = \{x \in M | g(\pi_x(t)) \leqq 0,\ h(\pi_x(t)) = 0, x(t)|_{\partial\Omega} = u(t),\ t \in \Omega\}.$$

Definition 2.1 *A point $\bar{x} \in S$ is said to be an optimal solution to the MVP if, for all $x \in S$,*

$$\int_\Omega f(\pi_x(t))\, dv \geqq \int_\Omega f(\pi_{\bar{x}}(t))\, dv.$$

To establish the various results in the subsequent parts of the paper, first we shall introduce the following definitions of invexity and pseudoinvexity for a multitime functional of curvilinear integral type.

Let D_γ be the total derivative and $\eta : J^1(T,M) \times J^1(T,M) \to \mathbb{R}^n$ be such a vector valued function for which the condition $\eta(\pi_x(t), \pi_x(t)) = 0$ is satisfied for all $x \in M$ and also on $\partial\Omega$. Also, let $(.)^\zeta$ denote the power of variables $(.)$, which is used in the sequel of the paper.

Definition 2.2 *A functional $\int_\Omega f(\pi_x(t))dv$ is said to be invex at $\bar{x} \in M$ on M with respect to η if, for all $x \in M$,*

$$\int_\Omega f(\pi_x(t))dv - \int_\Omega f(\pi_{\bar{x}}(t))dv \geqq \int_\Omega \Big[\eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big] dv.$$

Now we give an example of a nonconvex multitime functional to illustrate the concept of invexity introduced in the above definition.

Example 2.1 *Let $\Omega = [0,2] \times [0,2]$, $M = [0,1] \times [0,1]$. The functions $f : J^1(T,M) \to \mathbb{R}$ and $\eta : J^1(T,M) \times J^1(T,M) \to \mathbb{R}^2$ are defined as follows:*

$$f(\pi_x(t)) = x^2(t)e^{x^1(t)} - x^1(t)x^2(t) + \arctan(x^1(t))^2 - (x^2(t))^2,$$

$$\eta(\pi_x(t), \pi_{\bar{x}}(t)) = \begin{bmatrix} (x^1(t))^2 - (\bar{x}^1(t))^2 \\ -x^2(t) + \bar{x}^2(t) \end{bmatrix}.$$

Consider a point $\bar{x}(t) = (0,0)$. Now we show that the considered multitime functional is invex at $\bar{x}$ on M with respect to the function η defined above. Indeed, we have

$$\int_\Omega f(\pi_x(t))dv - \int_\Omega f(\pi_{\bar{x}}(t))dv - \int_\Omega \Big[\eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big] dv$$

$$= \int_\Omega \Big[x^2(t)e^{x^1(t)} - x^1(t)x^2(t) + \arctan(x^1(t))^2 + x^2(t) - (x^2(t))^2\Big] dv \geqq 0,\ \forall\, x \in M. \tag{1}$$

The fact that the inequality (1) is satisfied for all $x \in M$ is illustrated in Figure 1.
Hence, by Definition 2.2, $\int_\Omega f(\pi_x(t))dv$ is invex at $\bar{x}(t) = (0,0)$ with respect to η defined above.

Definition 2.3 *A functional $\int_\Omega f(\pi_x(t))dv$ is said to be pseudoinvex at $\bar{x} \in M$ on M with respect to η if, for all $x \in M$,*

$$\int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big] dv \geqq 0$$

$$\Rightarrow \int_\Omega f(\pi_x(t))dv - \int_\Omega f(\pi_{\bar{x}}(t))dv \geqq 0,$$

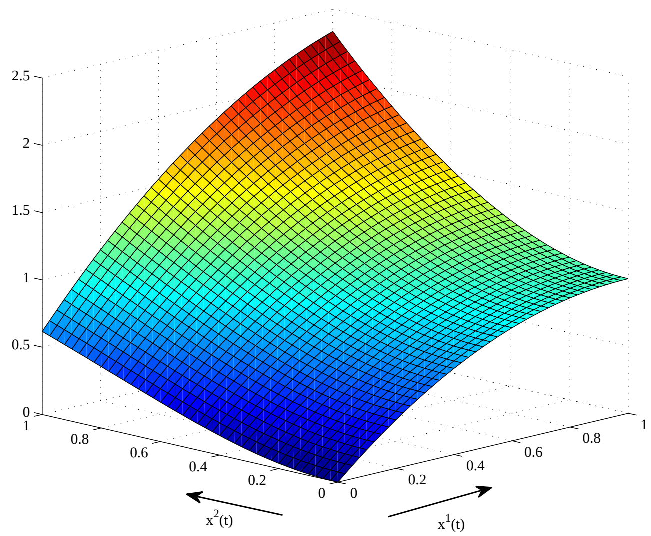

Figure 1. Graph of $\Big[x^2(t)e^{x^1(t)} - x^1(t)x^2(t) + \arctan(x^1(t))^2 + x^2(t) - (x^2(t))^2\Big]$.

and equivalently,

$$\int_\Omega f(\pi_x(t))dv < \int_\Omega f(\pi_{\bar{x}}(t))dv \Rightarrow \int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv < 0.$$

In order to illustrate the relationship between the concepts of pseudoinvexity and invexity, we now give an example of a multitime functional that is pseudoinvex but is not invex with respect to the same function η.

Example 2.2 *Let* $\Omega = [0,1]\times[0,1]$, $M = [-1,1]\times[-1,1]$. *Assume that the functions* $f : J^1(T,M) \to \mathbb{R}$ *and* $\eta : J^1(T,M)\times J^1(T,M) \to \mathbb{R}^2$ *are defined as follows:*

$$f(\pi_x(t)) = \log(1+(x^2(t))^2) - \frac{(x^2(t))^2}{5} - e^{x^1(t)},$$

$$\eta(\pi_x(t),\pi_{\bar{x}}(t)) = \begin{bmatrix} x^1(t) + (x^1(t))^2 - \bar{x}^1(t) - (\bar{x}^1(t))^2 \\ x^2(t) + (x^2(t))^2 - \bar{x}^2(t) - (\bar{x}^2(t))^2 \end{bmatrix}.$$

Consider a point $\bar{x}(t) = (1,0)$. *Now we show that the considered nonconvex multitime functional is pseudoinvex at* $\bar{x}$ *on* M *with respect to the function* η *defined above. Indeed, we have*

$$\int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$

$$= \int_\Omega -e(x^1(t) + (x^1(t))^2 - 2)dv \geqq 0, \forall\ x \in M. \tag{2}$$

The fact that the inequality (2) is satisfied for all $x \in M$ *is illustrated in Figure 2.*

Then (2) implies

$$\int_\Omega f(\pi_x(t))dv - \int_\Omega f(\pi_{\bar{x}}(t))dv = \int_\Omega \Big(\log(1+(x^2(t))^2) - \frac{(x^2(t))^2}{5} - e^{x^1(t)} + e\Big)dv \geqq 0, \forall\ x \in M.$$

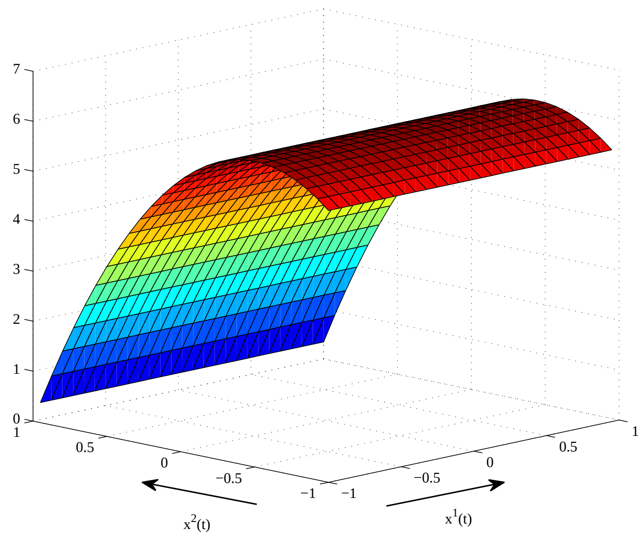

Figure 2. Graph of $\Big[-e(x^1(t)+(x^1(t))^2-2)\Big]$.

The fact that the above inequality is satisfied for all $x \in M$ *is illustrated in Figure 3.*

Hence, by Definition 2.3, the functional $\int_\Omega f(\pi_x(t))dv$ *is pseudoinvex at* $\bar{x}(t) = (1,0)$ *with respect to* η*. However, it is not difficult to show by Definition 2.2 that this functional is not invex at* $\bar{x}(t) = (1,0)$ *with respect to the same* η*. Indeed, we have*

$$\int_\Omega f(\pi_x(t))dv - \int_\Omega f(\pi_{\bar{x}}(t))dv - \int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$

$$= \int_\Omega \Big[\log(1+(x^2(t))^2) - \frac{(x^2(t))^2}{5} - e^{x^1(t)} + e(x^1(t)+(x^1(t))^2-1)\Big]dv \ngeqq 0, \ \forall\, x \in M.$$

The fact that the above inequality is not satisfied for all $x \in M$ *is illustrated in Figure 4.*

Now we give the following necessary optimality conditions for the considered MVP established by Mititelu et al. [14].

Theorem 2.1 *Let* $\bar{x}$ *be an optimal solution to (MVP). Then there exist* $\tau \in \mathbb{R}$ *and piecewise smooth multipliers* $\lambda^\alpha(t) \in \mathbb{R}^m$*,* $\mu^\beta(t) \in \mathbb{R}^q$ *which, for all* $t \in \Omega$*, satisfy the following conditions:*

$$\tau\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + \lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x}(\pi_{\bar{x}}(t)) + \mu^\beta(t)\frac{\partial h^\beta}{\partial x}(\pi_{\bar{x}}(t))$$
$$= D_\gamma\Big(\tau\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t)) + \lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t)) + \mu^\beta(t)\frac{\partial h^\beta}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big), \tag{3}$$

$$\lambda^\alpha(t) g^\alpha(\pi_x(t)) = 0, \ \text{for each } \alpha = \{1,\dots,m\}, \tag{4}$$

$$\tau \geq 0, \ \lambda^\alpha(t) \geqq 0. \tag{5}$$

Definition 2.4 *[14] An optimal solution* $\bar{x} \in S$ *in the problem (MVP) is called normal if* $\tau \neq 0$ *.*

According to this definition, without the loss of generality, in what follows we can take $\tau = 1$*.*

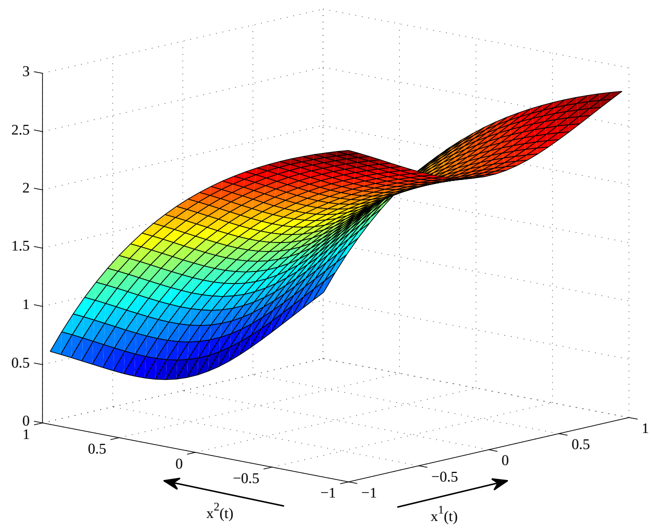

Figure 3. Graph of $\left[\log(1+(x^2(t))^2) - \dfrac{(x^2(t))^2}{5} - e^{x^1(t)} + e\right]$.

Remark 2.1 *We shall use the following property to prove the main results in the paper:*

$$\int_\Omega D_\gamma(\eta(\pi_x(t), \pi_{\bar{x}}(t)))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))dv = -\int_\Omega \eta(\pi_x(t), \pi_{\bar{x}}(t))\Big(D_\gamma \frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big)dv.$$

3. Modified multitime variational problem and optimality conditions

Let $\bar{x}$ be an arbitrary given feasible solution to the considered MVP. Then, in the used modified objective function approach, the multitime variational problem (MVP$_\eta(\bar{x})$) with the modified objective function corresponding to (MVP) is constructed as follows:

$$\textbf{(MVP}_\eta(\bar{x})\textbf{)} \quad \text{minimize} \quad \int_\Omega \Big[\eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$

$$\text{subject to} \quad g(\pi_x(t)) \leqq 0,$$

$$h(\pi_x(t)) = 0,$$

$$x(t)|_{\partial\Omega} = u(t),\ t \in \Omega,$$

where $f, g,$ and h are defined in the original MVP.

Remark 3.1 *Note that the feasible set of (MVP$_\eta(\bar{x})$) is the same as that of (MVP).*

Remark 3.2 *As it follows directly from Definition 2.1, a point $\hat{y} \in S$ is said to be an optimal solution to the multitime variational problem (MVP$_\eta(\bar{x})$) with the modified objective function if, for all $x \in S$,*

$$\int_\Omega \Big[\eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$\geqq \int_\Omega \Big[\eta(\pi_{\hat{y}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_{\hat{y}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv.$$

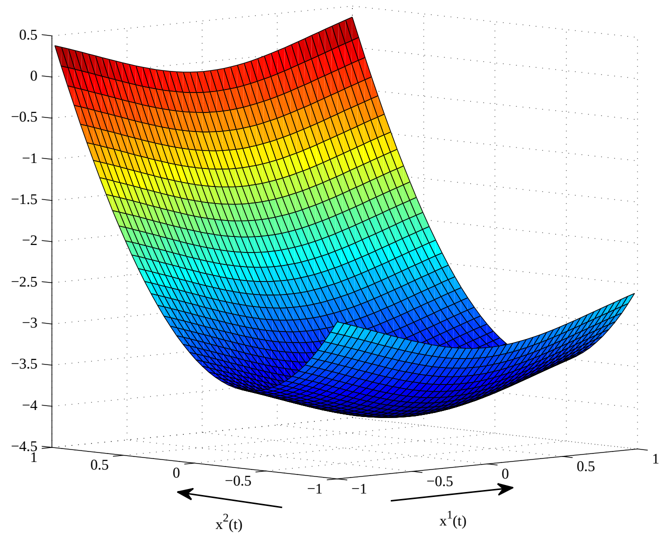

Figure 4. Graph of $\left[\log(1+(x^2(t))^2) - \dfrac{(x^2(t))^2}{5} - e^{x^1(t)} + e(x^1(t) + (x^1(t))^2 - 1)\right]$.

Now we establish the equivalence between optimal solutions to (MVP) and $(\mathrm{MVP}_\eta(\bar{x}))$ under invexity assumptions.

Theorem 3.1 *Let $\bar{x}$ be a normal optimal solution to (MVP) at which the necessary optimality conditions (3)–(5) are satisfied with piecewise smooth multipliers $\lambda^\alpha(t), \mu^\beta(t)$. Assume that $\int_\Omega \lambda^\alpha(t) g^\alpha(\pi_x(t))dv$ and $\int_\Omega \mu^\beta(t) h^\beta(\pi_x(t))dv$ are invex at $\bar{x}$ on S with respect to η. Then $\bar{x}$ is an optimal solution to $(MVP_\eta(\bar{x}))$.*

Proof Since $\bar{x}$ is a normal optimal solution to (MVP), therefore, the conditions (3)–(5) are satisfied at $\bar{x}$ with piecewise smooth multipliers $\lambda^\alpha(t), \mu^\beta(t)$. Suppose, contrary to the result, that $\bar{x}$ is not an optimal solution to $(\mathrm{MVP}_\eta(\bar{x}))$. Then there exists a point $y \in S$ such that

$$\int_\Omega \left[\eta(\pi_y(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_y(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\right]dv$$
$$< \int_\Omega \left[\eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\right]dv.$$

Since $\eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t)) = 0$, therefore, the above inequality reduces to

$$\int_\Omega \left[\eta(\pi_y(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_y(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\right]dv < 0. \tag{6}$$

From the feasibility of y and (5), we have

$$\lambda^\alpha(t) g^\alpha(\pi_y(t)) \leqq 0.$$

Using (4), the above inequality yields

$$\int_\Omega \lambda^\alpha(t) g^\alpha(\pi_y(t))dv - \int_\Omega \lambda^\alpha(t) g^\alpha(\pi_{\bar{x}}(t))dv \leqq 0. \tag{7}$$

Since $\int_\Omega \lambda^\alpha(t)g^\alpha(\pi_x(t))dv$ is invex at $\bar{x}$ on S with respect to η, therefore, by Definition 2.2, it follows that

$$\int_\Omega \lambda^\alpha(t)g^\alpha(\pi_y(t))dv - \int_\Omega \lambda^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))dv$$
$$\geqq \int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_y(t),\pi_{\bar{x}}(t))\lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv,$$

which in turn, by using (7), implies that

$$\int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_y(t),\pi_{\bar{x}}(t))\lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv \leqq 0. \tag{8}$$

Again, using the definition of invexity for the functional $\int_\Omega \mu^\beta(t)h^\beta(\pi_x(t))dv$, we have

$$\int_\Omega \mu^\beta(t)h^\beta(\pi_y(t))dv - \int_\Omega \mu^\beta(t)h^\beta(\pi_{\bar{x}}(t))dv$$
$$\geqq \int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\mu^\beta(t)\frac{\partial h^\beta}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_y(t),\pi_{\bar{x}}(t))\mu^\beta(t)\frac{\partial h^\beta}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv.$$

Now, by the feasibility of y and $\bar{x}$, the above inequality reduces to

$$\int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\mu^\beta(t)\frac{\partial h^\beta}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_y(t),\pi_{\bar{x}}(t))\mu^\beta(t)\frac{\partial h^\beta}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv \leqq 0. \tag{9}$$

Combining (6), (8), and (9), we obtain

$$\int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\Big\{\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + \lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x}(\pi_{\bar{x}}(t)) + \mu^\beta(t)\frac{\partial h^\beta}{\partial x}(\pi_{\bar{x}}(t))\Big\}$$
$$+ D_\gamma\eta(\pi_y(t),\pi_{\bar{x}}(t))\Big\{\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t)) + \lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t)) + \mu^\beta(t)\frac{\partial h^\beta}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big\}\Big]dv < 0.$$

Using Remark 2.1, the above inequality can be rewritten as

$$\int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\Big\{\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + \lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x}(\pi_{\bar{x}}(t)) + \mu^\beta(t)\frac{\partial h^\beta}{\partial x}(\pi_{\bar{x}}(t))\Big\}$$
$$- \eta(\pi_y(t),\pi_{\bar{x}}(t))D_\gamma\Big\{\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t)) + \lambda^\alpha(t)\frac{\partial g^\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t)) + \mu^\beta(t)\frac{\partial h^\beta}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big\}\Big]dv < 0,$$

which contradicts (3). Thus, $\bar{x}$ is an optimal solution (MVP$_\eta(\bar{x})$). This completes the proof. □

Now we give an example of a nonconvex multitime variational problem to illustrate the result established in Theorem 3.1.

Example 3.1 *Let* $\Omega = [0,1]\times[0,1]$, $M = [0,1]\times[0,1]$*, and* $\alpha = 1,2$; $\beta = 1$*. Consider the following multitime* *variational problem:*

$$\text{(MVP1) minimize } \int_\Omega \left[\sin x^1(t) + x^2(t) + e^{x^1(t)x^2(t)}\right] dv$$

$$\text{subject to } g^1(\pi_x(t)) = (x^1(t))^2 - x^1(t) \leqq 0,$$

$$g^2(\pi_x(t)) = -x^2(t) \leqq 0,$$

$$h^1(\pi_x(t)) = -x^1(t) + x^2(t) = 0,$$

$$x(t)|_{\partial\Omega} = u(t).$$

The feasible set of (MVP1) is given by $S = \{x \in M : 0 \leqq x^1(t) \leqq 1, x^2(t) \geqq 0, x^1(t) = x^2(t),\ x(t)|_{\partial\Omega} = u(t),\ t \in \Omega\}$. *Consider* $\bar{x}(t) = (0,0) \in S$. *Let* $\eta : J^1(T,M) \times J^1(T,M) \to \mathbb{R}^2$ *be defined as*

$$\eta(\pi_x(t), \pi_{\bar{x}}(t)) = \begin{cases} \begin{bmatrix} x^1(t) - \bar{x}^1(t) \\ 2(x^2(t) - \bar{x}^2(t)) \end{bmatrix}, & t \in \Omega \\ 0, & t \in \partial\Omega. \end{cases}$$

Therefore, the multitime variational problem $(MVP1_\eta(\bar{x}))$ *constructed in the modified objective function method is given as follows:*

$$(MVP1_\eta(\bar{x})) \quad \text{minimize} \quad \int_\Omega \Big(x^1(t) + 2x^2(t)\Big) dv$$

$$\text{subject to } g^1(\pi_x(t)) = (x^1(t))^2 - x^1(t) \leqq 0,$$

$$g^2(\pi_x(t)) = -x^2(t) \leqq 0,$$

$$h^1(\pi_x(t)) = -x^1(t) + x^2(t) = 0,$$

$$x(t)|_{\partial\Omega} = u(t).$$

Note that $(MVP1_\eta(\bar{x}))$ *has a simpler form in comparison to the original variational problem considered in this example. Clearly,* $\bar{x}(t) = (0,0)$ *is an optimal solution to (MVP1). By the necessary optimality conditions (3)–(5), it follows that* $\lambda^\alpha(t) = (2,0), \mu^\beta(t) = -1$. *Further, it is not difficult to show, by Definition 2.2, that the functionals* $\int_\Omega \lambda^\alpha(t) g^\alpha(\pi_x(t))dv$ *and* $\int_\Omega \mu^\beta(t) h^\beta(\pi_x(t))dv$ *are invex at* $\bar{x}(t) = (0,0)$ *on* S *with respect to* η *given above. Since all hypotheses of Theorem 3.1 are fulfilled,* $\bar{x}(t) = (0,0)$ *is, therefore, an optimal solution to the modified multitime variational problem* $(MVP1_\eta(\bar{x}))$.

In the next theorem, we prove the equivalence between optimal solutions to $(\text{MVP}_\eta(\bar{x}))$ and (MVP) under weaker hypotheses.

Theorem 3.2 *Let* $\bar{x}$ *be an optimal solution to* $(MVP_\eta(\bar{x}))$. *Assume that the objective function* $\int_\Omega f(\pi_x(t))dv$ *is pseudoinvex at* $\bar{x}$ *on* S *with respect to* η. *Then* $\bar{x}$ *is also an optimal solution to (MVP).*

Proof Suppose, contrary to the result, that $\bar{x}$ is not an optimal solution to (MVP). Then there exists a point $y \in S$ such that

$$\int_\Omega f(\pi_y(t))dv < \int_\Omega f(\pi_{\bar{x}}(t))dv. \tag{10}$$

Since $\int_\Omega f(\pi_x(t))dv$ is pseudoinvex at $\bar{x}$ on S with respect to η, therefore, by Definition 2.3, (10) implies that

$$\int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_y(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv < 0.$$

Since $\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t)) = 0$, the inequality above implies that

$$\int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_y(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$< \int_\Omega \Big[\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv,$$

which contradicts the assumption that $\bar{x}$ is an optimal solution to (MVP$_\eta(\bar{x})$). Hence, $\bar{x}$ is an optimal solution to (MVP). This completes the proof. □

4. Saddle-point criteria for a multitime variational problem constructed in the modified objective function method

In this section, under generalized invexity hypotheses, we establish the equivalence between an optimal solution to the considered MVP and a saddle-point of the Lagrange function in the associated multitime variational problem (MVP$_\eta(\bar{x})$) constructed in the modified objective function method.

Motivated by Antczak [1], we define the Lagrange function and present the definition of its saddle-point in the multitime variational problem (MVP$_\eta(\bar{x})$) constructed in the modified objective function method.

Definition 4.1 *The Lagrange function in the multitime variational problem (MVP$_\eta(\bar{x})$) is denoted by L_η and defined as*

$$L_\eta(x,\lambda^\alpha(t),\mu^\beta(t)) = \int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$+ \int_\Omega \lambda^\alpha(t) g^\alpha(\pi_x(t))dv + \int_\Omega \mu^\beta(t) h^\beta(\pi_x(t))dv.$$

Definition 4.2 *A point $(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)) \in S\times\mathbb{R}^m_+\times\mathbb{R}^q$ is said to be a saddle-point for the Lagrange function in the multitime variational problem (MVP$_\eta(\bar{x})$), if the following inequalities hold:*

(i) $L_\eta(\bar{x},\lambda^\alpha(t),\mu^\beta(t)) \leqq L_\eta(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)),\ \forall\ \lambda^\alpha(t)\in\mathbb{R}^m_+,\ \mu^\beta(t)\in\mathbb{R}^q,$

(ii) $L_\eta(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)) \leqq L_\eta(x,\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)),\ \forall\ x\in S.$

Theorem 4.1 *Let $\bar{x}\in S$. Assume that the objective function $\int_\Omega f(\pi_x(t))dv$ is pseudoinvex at $\bar{x}$ on S with respect to η. If $(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))$ is a saddle-point for the Lagrange function in (MVP$_\eta(\bar{x})$), then $\bar{x}$ is an optimal solution to (MVP).*

Proof Since $(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))$ is a saddle-point of the Lagrange function in (MVP$_\eta(\bar{x})$), by condition (i) in Definition 4.2, we have

$$L_\eta(\bar{x},\lambda^\alpha(t),\mu^\beta(t)) \leqq L_\eta(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)),\forall\ \lambda(t)\in\mathbb{R}^m_+,\ \mu(t)\in\mathbb{R}^q.$$

By the definition of the Lagrange function L_η, it follows that

$$\int_\Omega \Big[\eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big] dv + \int_\Omega \lambda^\alpha(t) g^\alpha(\pi_{\bar{x}}(t)) dv + \int_\Omega \mu^\beta(t) h^\beta(\pi_{\bar{x}}(t)) dv$$
$$\leqq \int_\Omega \Big[\eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big] dv + \int_\Omega \bar{\lambda}^\alpha(t) g^\alpha(\pi_{\bar{x}}(t)) dv + \int_\Omega \bar{\mu}^\beta(t) h^\beta(\pi_{\bar{x}}(t)) dv.$$

Since $\eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t)) = 0$, the above relation gives that the following inequality

$$\int_\Omega \lambda^\alpha(t) g^\alpha(\pi_{\bar{x}}(t)) dv + \int_\Omega \mu^\beta(t) h^\beta(\pi_{\bar{x}}(t)) dv \leqq \int_\Omega \bar{\lambda}^\alpha(t) g^\alpha(\pi_{\bar{x}}(t)) dv + \int_\Omega \bar{\mu}^\beta(t) h^\beta(\pi_{\bar{x}}(t)) dv$$

holds for all $\lambda^\alpha(t) \in \mathbb{R}^m_+,\ \mu^\beta(t) \in \mathbb{R}^q$.
Therefore, if we set $\lambda^\alpha(t) = 0$ and using the feasibility of $\bar{x}$ in the inequality above, then we get

$$\int_\Omega \bar{\lambda}^\alpha(t) g^\alpha(\pi_{\bar{x}}(t)) dv + \int_\Omega \bar{\mu}^\beta(t) h^\beta(\pi_{\bar{x}}(t)) dv \geqq 0. \tag{11}$$

Now suppose, contrary to the result, that $\bar{x}$ is not an optimal solution to (MVP). Then there exists a point $y \in S$ such that

$$\int_\Omega f(\pi_y(t)) dv < \int_\Omega f(\pi_{\bar{x}}(t)) dv.$$

By the pseudoinvexity of $\int_\Omega f(\pi_x(t))dv$ at $\bar{x}$ on S with respect to η, the inequality above implies

$$\int_\Omega \Big[\eta(\pi_y(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_y(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big] dv < 0.$$

Since $\eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t)) = 0$, the above inequality yields

$$\int_\Omega \Big[\eta(\pi_y(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_y(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big] dv$$
$$< \int_\Omega \Big[\eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_{\bar{x}}(t), \pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big] dv. \tag{12}$$

Since $y \in S$, therefore, we have that the following inequality

$$\int_\Omega \bar{\lambda}^\alpha(t) g^\alpha(\pi_y(t)) dv + \int_\Omega \bar{\mu}^\beta(t) h^\beta(\pi_y(t)) dv \leqq 0 \tag{13}$$

holds for all $\bar{\lambda}^\alpha(t) \in \mathbb{R}^m_+,\ \bar{\mu}^\beta(t) \in \mathbb{R}^q$.
From (11) and (13), it follows that

$$\int_\Omega \bar{\lambda}^\alpha(t) g^\alpha(\pi_y(t)) dv + \int_\Omega \bar{\mu}^\beta(t) h^\beta(\pi_y(t)) dv$$
$$\leqq \int_\Omega \bar{\lambda}^\alpha(t) g^\alpha(\pi_{\bar{x}}(t)) dv + \int_\Omega \bar{\mu}^\beta(t) h^\beta(\pi_{\bar{x}}(t)) dv. \tag{14}$$

Combining (12) and (14), we get

$$\int_\Omega \Big[\eta(\pi_y(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_y(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$+\int_\Omega \bar{\lambda}^\alpha(t) g^\alpha(\pi_y(t))dv + \int_\Omega \bar{\mu}^\beta(t) h^\beta(\pi_y(t))dv$$
$$< \int_\Omega \Big[\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$+\int_\Omega \bar{\lambda}^\alpha(t) g^\alpha(\pi_{\bar{x}}(t))dv + \int_\Omega \bar{\mu}^\beta(t) h^\beta(\pi_{\bar{x}}(t))dv.$$

Thus, by the definition of the Lagrange function for the modified multitime variational problem (MVP$_\eta(\bar{x})$), it follows that the following inequality

$$L_\eta(y,\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)) < L_\eta(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))$$

holds, which contradicts the inequality (ii) in Definition 4.2. Therefore, $\bar{x}$ is an optimal solution to (MVP). This completes the proof. □

Now we give an example of a nonconvex multitime variational problem to illustrate the result obtained in Theorem 4.1.

Example 4.1 *Let* $\Omega = [-2,2]\times[-2,2]$, $M = [-1,1]\times[-1,1]$ *and* $\alpha = 1,\ 2;\ \beta = 1$. *Consider the following multitime variational problem:*

$$\text{(MVP2)}\quad \text{minimize} \quad \int_\Omega ((x^1(t))^2 x^2(t) + \arctan(1+x^1(t)) + \cos x^1(t))dv$$
$$\text{subject to}\quad g^1(\pi_x(t)) = (x^1(t))^2 - 1 \leqq 0,$$
$$g^2(\pi_x(t)) = -(1+x^1(t)) \leqq 0,$$
$$h^1(\pi_x(t)) = x^1(t) - x^2(t) = 0,$$
$$x(t)|_{\partial\Omega} = u(t).$$

The feasible set of (MVP2) is given by $S = \{x \in M : -1 \leqq x^1(t) \leqq 1, x^1(t) = x^2(t),\ x(t)|_{\partial\Omega} = u(t),\ t\in\Omega\}$. *Consider a point* $\bar{x}(t) = (-1,-1) \in S$. *Let* $\eta : J^1(T,M)\times J^1(T,M) \to \mathbb{R}^2$ *be defined as*

$$\eta(\pi_x(t),\pi_{\bar{x}}(t)) = \begin{cases} \begin{bmatrix} (x^1(t))^3 - (\bar{x}^1(t))^3 \\ x^2(t) - \bar{x}^2(t)\end{bmatrix}, & t\in\Omega \\ 0, & t \in \partial\Omega. \end{cases}$$

Now we prove that the objective function in the considered multitime variational problem (MVP2) is pseudoinvex at $\bar{x}$ *on* S *with respect to* η *defined above. Note that the following relations hold:*

$$\int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$= \int_\Omega \Big[((x^1(t))^3+1)(3+\sin(1)) + (1+x^2(t))\Big]dv \geqq 0,\ \forall\ x(t)\in S. \tag{15}$$

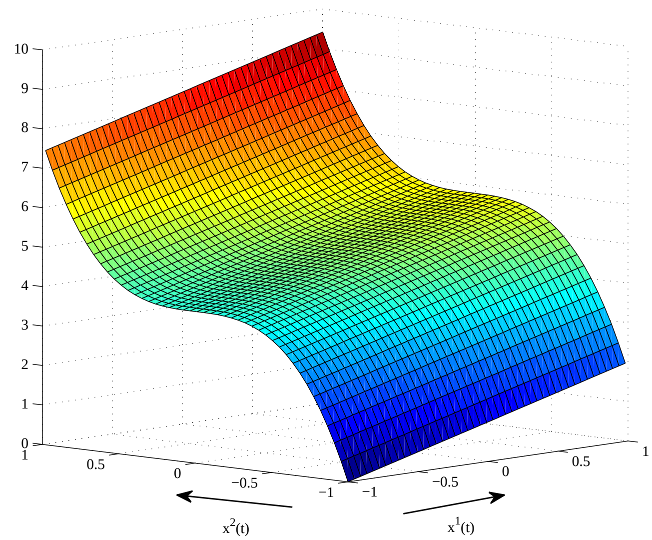

Figure 5. Graph of $\Big[((x^1(t))^3+1)(3+\sin(1))+(1+x^2(t))\Big]$.

This fact is illustrated in Figure 5.

Further, note that the following relations hold:

$$\int_\Omega f(\pi_x(t))dv - \int_\Omega f(\pi_{\bar{x}}(t))dv = \int_\Omega \Big[(x^1(t))^2x^2(t) + \arctan(1+x^1(t)) + \cos x^1(t) + 1 - \cos(1)\Big]dv \geqq 0, \forall\ x(t) \in S.$$

This fact is illustrated in Figure 6.

Therefore, as it follows from (15), the following inequality,

$$\int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma \eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv \geqq 0,$$

holds for all $x \in S$. Hence,

$$\int_\Omega f(\pi_x(t))dv - \int_\Omega f(\pi_{\bar{x}}(t))dv \geqq 0,\ \ \forall\ x(t) \in S,$$

which shows that $\int_\Omega f(\pi_x(t))dv$ is pseudoinvex at $\bar{x}(t) = (-1,-1)$ on S with respect to η given above.
Now, for the considered multitime variational problem (MVP2), its associated modified multitime variational problem $(MVP2_\eta(\bar{x}))$ is given as follows:

$$(MVP2_\eta(\bar{x}))\quad \textit{minimize}\ \int_\Omega \Big[((x^1(t))^3+1)(3+\sin(1))+(1+x^2(t))\Big]dv$$

$$\textit{subject to}\ g^1(\pi_x(t)) = (x^1(t))^2 - 1 \leqq 0,$$

$$g^2(\pi_x(t)) = -(1+x^1(t)) \leqq 0,$$

$$h^1(\pi_x(t)) = x^1(t) - x^2(t) = 0,$$

$$x(t)|_{\partial\Omega} = u(t).$$

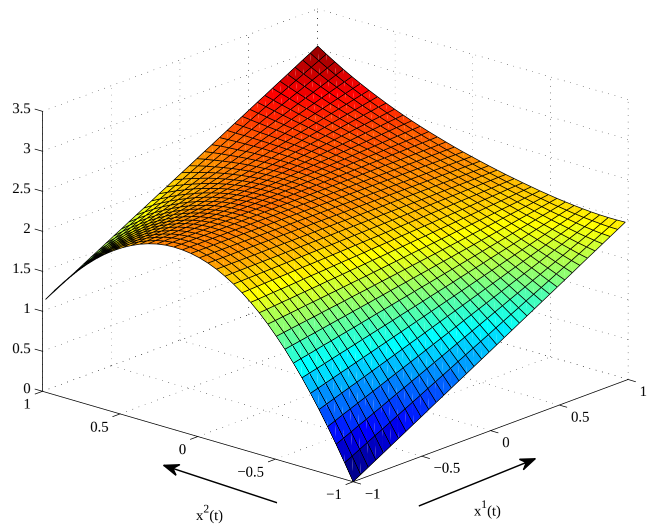

Figure 6. Graph of $\Big[(x^1(t))^2x^2(t)+\arctan(1+x^1(t))+\cos x^1(t)+1-\cos(1)\Big]$.

The Lagrange function L_η in the modified multitime variational problem (MVP2$_\eta(\bar{x})$) is given by

$$L_\eta(x,\lambda^\alpha(t),\mu^\beta(t))=\int_\Omega\Big[((x^1(t))^3+1)(3+\sin(1))+(1+x^2(t))\Big]dv$$
$$+\int_\Omega[\lambda^1((x^1(t))^2-1)+\lambda^2(-1-x^1(t))]dv+\int_\Omega\mu^1(x^1(t)-x^2(t))dv.$$

We observe that $(\bar{x}(t),\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))=((-1,-1),(1,1),0)$ is a saddle-point of this Lagrange function, since the following relations hold:

$$L_\eta(\bar{x},\lambda^\alpha(t),\mu^\beta(t))-L_\eta(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))=0,\ \forall\ \lambda^\alpha(t)\in\mathbb{R}^2_+,\ \mu^\beta(t)\in\mathbb{R}$$

and $L_\eta(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))-L_\eta(x,\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))$

$$=-\int_\Omega[(3+\sin(1))(1+(x^1(t))^3)+(x^1(t))^2-x^1(t)+x^2(t)-1]dv$$
$$\leqq 0,\ \forall\ x(t)\in S.$$

This fact is illustrated in Figure 7.

Since all hypotheses of Theorem 4.1 are fulfilled at $\bar{x}(t)=(-1,-1)$, therefore, $\bar{x}(t)=(-1,-1)$ is an optimal solution to the considered multitime variational problem (MVP2).

Theorem 4.2 *Let $\bar{x}$ be a normal optimal solution to (MVP). Further, assume that $\int_\Omega[\bar{\lambda}^\alpha(t)g^\alpha(\pi_x(t))+\bar{\mu}^\beta(t)h^\beta(\pi_x(t))]dv$ is invex at $\bar{x}$ on S with respect to η. Then $(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))$ is a saddle-point for the Lagrange function in (MVP$_\eta(\bar{x})$).*

Proof Since $\bar{x}$ is a normal optimal solution to (MVP), therefore, the conditions (3)–(5) are satisfied at $\bar{x}$ for

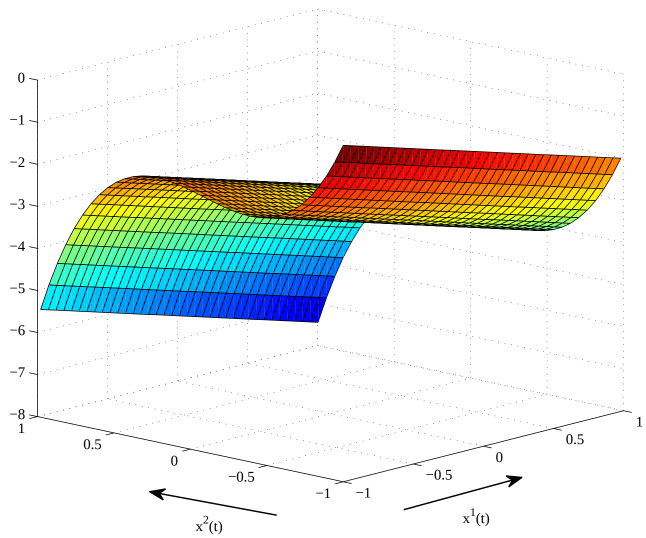

Figure 7. Graph of $-\Big[(3+\sin(1))(1+(x^1(t))^3)+(x^1(t))^2-x^1(t)+x^2(t)-1\Big]$.

$\bar{\lambda}^\alpha(t),\ \bar{\mu}^\beta(t)$. From (3), we have

$$\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t))+\bar{\lambda}^\alpha(t)\frac{\partial g^\alpha}{\partial x}(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)\frac{\partial h^\beta}{\partial x}(\pi_{\bar{x}}(t))=D_\gamma\Big(\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))+\bar{\lambda}^\alpha(t)\frac{\partial g^\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)\frac{\partial h^\beta}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big).$$

Let x be any feasible solution to (MVP). Multiplying the above equation by $\eta(\pi_x(t),\pi_{\bar{x}}(t))$ and then integrating over Ω, we get

$$\int_\Omega \eta(\pi_x(t),\pi_{\bar{x}}(t))\Big[\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t))+\bar{\lambda}^\alpha(t)\frac{\partial g^\alpha}{\partial x}(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)\frac{\partial h^\beta}{\partial x}(\pi_{\bar{x}}(t))\Big]dv$$
$$=\int_\Omega \eta(\pi_x(t),\pi_{\bar{x}}(t))D_\gamma\Big[\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))+\bar{\lambda}^\alpha(t)\frac{\partial g^\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)\frac{\partial h^\beta}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv.$$

Using Remark 2.1, the above equation can be rewritten as

$$\int_\Omega \eta(\pi_x(t),\pi_{\bar{x}}(t))\Big[\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t))+\bar{\lambda}^\alpha(t)\frac{\partial g^\alpha}{\partial x}(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)\frac{\partial h^\beta}{\partial x}(\pi_{\bar{x}}(t))\Big]dv$$
$$=-\int_\Omega D_\gamma(\eta(\pi_x(t),\pi_{\bar{x}}(t))\Big[\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))+\bar{\lambda}^\alpha(t)\frac{\partial g^\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)\frac{\partial h^\beta}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv. \tag{16}$$

Since $\int_\Omega[\bar{\lambda}^\alpha(t)g^\alpha(\pi_x(t))+\bar{\mu}^\beta(t)h^\beta(\pi_x(t))]dv$ is invex at $\bar{x}$ on S with respect to η, therefore, by Definition 2.2, we have

$$\int_\Omega[\bar{\lambda}^\alpha(t)g^\alpha(\pi_x(t))+\bar{\mu}^\beta(t)h^\beta(\pi_x(t))]dv-\int_\Omega[\bar{\lambda}^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)h^\beta(\pi_{\bar{x}}(t))]dv$$
$$\geqq\int_\Omega\Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial}{\partial x}[\bar{\lambda}^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)h^\beta(\pi_{\bar{x}}(t))]$$
$$+D_\gamma\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial}{\partial x_\gamma}[\bar{\lambda}^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))+\bar{\mu}^\beta(t)h^\beta(\pi_{\bar{x}}(t))]\Big]dv.$$

By (16), the above inequality yields

$$\int_\Omega [\bar{\lambda}^\alpha(t)g^\alpha(\pi_x(t)) + \bar{\mu}^\beta(t)h^\beta(\pi_x(t))]dv - \int_\Omega [\bar{\lambda}^\alpha(t)g^\alpha(\pi_{\bar{x}}(t)) + \bar{\mu}^\beta(t)h^\beta(\pi_{\bar{x}}(t))]dv$$
$$\geqq - \int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv.$$

Since $\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t)) = 0$, the above inequality can be rewritten as

$$\int_\Omega \Big[\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$+ \int_\Omega \bar{\lambda}^\alpha(t)g^\alpha(\pi_x(t))dv + \int_\Omega \bar{\mu}^\beta(t)h^\beta(\pi_x(t))dv$$
$$\geqq \int_\Omega \Big[\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$+ \int_\Omega \bar{\lambda}^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))dv + \int_\Omega \bar{\mu}^\beta(t)h^\beta(\pi_{\bar{x}}(t))dv.$$

Hence, by the definition of the Lagrange function L_η, we have

$$L_\eta(x,\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)) \geqq L_\eta(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)),\ \forall\ x \in S. \tag{17}$$

On the other hand, since $\bar{x} \in S$, therefore, the following inequality,

$$\int_\Omega \lambda^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))dv + \int_\Omega \mu^\beta(t)h^\beta(\pi_{\bar{x}}(t))dv \leqq 0,$$

holds for all $\lambda^\alpha(t) \in \mathbb{R}^m_+, \mu^\beta(t) \in \mathbb{R}^q$.

Using (4) together with the feasibility of $\bar{x}$, the above inequality can be rewritten as

$$\int_\Omega \lambda^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))dv + \int_\Omega \mu^\beta(t)h^\beta(\pi_{\bar{x}}(t))dv \leqq \int_\Omega \bar{\lambda}^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))dv + \int_\Omega \bar{\mu}^\beta(t)h^\beta(\pi_{\bar{x}}(t))dv.$$

Again, since $\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t)) = 0$, it follows that

$$\int_\Omega \Big[\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_x(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$+ \int_\Omega \lambda^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))dv + \int_\Omega \mu^\beta(t)h^\beta(\pi_{\bar{x}}(t))dv$$
$$\leqq \int_\Omega \Big[\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x}(\pi_{\bar{x}}(t)) + D_\gamma\eta(\pi_{\bar{x}}(t),\pi_{\bar{x}}(t))\frac{\partial f}{\partial x_\gamma}(\pi_{\bar{x}}(t))\Big]dv$$
$$+ \int_\Omega \bar{\lambda}^\alpha(t)g^\alpha(\pi_{\bar{x}}(t))dv + \int_\Omega \bar{\mu}^\beta(t)h^\beta(\pi_{\bar{x}}(t))dv,$$

which, by the definition of Lagrange function L_η, yields

$$L_\eta(\bar{x},\lambda^\alpha(t),\mu^\beta(t)) \leqq L_\eta(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t)),\ \forall\ \lambda^\alpha(t) \in \mathbb{R}^m_+, \mu^\beta(t) \in \mathbb{R}^q. \tag{18}$$

Thus, from (17) and (18), we conclude that $(\bar{x},\bar{\lambda}^\alpha(t),\bar{\mu}^\beta(t))$ is a saddle-point of the Lagrange function in $(\text{MVP}_\eta(\bar{x}))$. This completes the proof. □

5. Conclusion

In this paper, we have used the modified objective function approach to obtain a new characterization of optimality in the considered multitime variational problem. By using this approach, optimal solutions of the original multitime variational problem are characterized by minimizers of an approximated multitime variational problem with the modified objective function. Then, we can characterize solvability of the original multitime variational problem, in general, by the help of a less complex approximated multitime variational problem constructed in the used approach. In some cases, an approximated multitime variational problem with the modified objective function is linear and/or convex (and such a case was illustrated in the paper). This is an important property of the analyzed method since optimal solutions of nonconvex multitime variational problems with complex objective functions can be characterized by the help of minimizers of linear and/or convex multitime variational problems. Further, we have presented the characterization of a saddle-point of the Lagrange function defined for a modified multitime variational problem constructed in the modified objective function method. This property of the modified objective function method can be a basis for introducing some numerical algorithms, which can be an aim of our future research.

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