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Quasi-proper efficiency: a quantitative enhanced efficiency

Latif POURKARIMI*, **Masoud KARIMI**
Department of Mathematics, Razi University, Kermanshah, Iran

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Abstract: This paper deals with an extension of proper efficiency that considers bounded and unbounded trade-offs between objective functions. While trade-offs between objective functions are unbounded, the rate of growth for these trade-offs is computed by applying a metric. A new concept, namely *quasi-proper efficiency* is introduced that shows that rate of growth of trade-off between objective functions. Two appropriate characterizations for this concept are developed: the first one is based on a scalar function utilizing the Chebyshev norm and the second one is in terms of the concept of stability.

Key words: Multiobjective programming, proper efficiency, quasi-proper efficiency, stable problem

1. Introduction

In many decision-making problems, the decisions are characterized by several criteria. If these criteria cannot be brought to a common scale by some utility functions, we refer to such problems as multicriteria or vector optimization problems. The common practice is to obtain the set of efficient decisions that are not dominated by any others [9]. Because of the conflicting nature of the objectives, Pareto optimal (or efficient) solutions were identified where the value of any objective function without impairing at least one of the others cannot be improved [14]. The concept of efficiency plays a useful role in analyzing the vector optimization problem. In order to exclude certain efficient solutions that display an undesirable anomaly and to provide a more satisfactory characterization, a slightly restricted definition of efficiency, namely properly efficient solution, has been proposed [6].

The notions of proper efficiency are introduced and studied by Kuhn and Tucker [12], Geoffrion [6], Borwein [2, 3], Benson [1], Hartley [8], Henig [9], and Borwein and Zhuang [4, 5]. A comprehensive survey of proper efficiencies can be found in [7].

Trade-off analysis is one of the most important elements in quantitative efficiency analysis. A trade-off denotes the amount of giving up in one of the objective functions that leads to improvement of another objective [14]. There are different concepts of proper efficiency that give different interpretations of trade-offs between the objective functions [11, 15–17].

As seen, the above-mentioned definitions do not lead to any meaningful and quantitative analysis for efficient solutions in which trade-off between some objective functions are unbounded. In order to overcome this deficiency, recently Jiang and Deng in [10] proposed a new concept of enhanced efficiency, namely α-proper efficiency corresponding to a positive parameter α.

*Correspondence: lp_karimi@yahoo.com
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This paper suggests an analysis for evaluating efficient solutions with some unbounded trade-offs. In the following, we analyze unboundedness trade-offs between objective functions. Consider the following multiobjective problem

$$\min_{x \in X} f(x), \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is a feasible set, and $f = (f_1, ..., f_p) : X \to \mathbb{R}^p$ is a vector function. The trade-off between two objective functions f_i and f_j at $\hat{x} \in X$ is defined as follows denoted by $T_{ij}(x, \hat{x})$.

Definition 1.1 *[14] Let $\hat{x} \in X$ and let $i, j \in \{1, ..., p\}$ with $f_i(x) < f_i(\hat{x})$ $j \in \{1, ..., p\}$ with $f_j(\hat{x}) < f_j(x)$. $T_{ij}(x, \hat{x})$ in which*

$$T_{ij}(x, \hat{x}) = \frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \tag{2}$$

is said to be a trade-off between objective functions f_i and f_j at $\hat{x}$.

As a matter of fact, the concept of trade-off is a gain-to-loss ratio between two objective functions in order to improve one of them while the other one is impaired. In other words, in the definition of $T_{ij}(x, \hat{x})$ the value $f_i(\hat{x}) - f_i(x)$ is the gain and the value $f_j(x) - f_j(\hat{x})$ is the loss while $\hat{x}$ moves to the solution x. Being unboundedness in $T_{ij}(x, \hat{x})$ is a factor for filtering of the efficiency solution of a multiobjective optimization problem. In other words, $\hat{x}$ can move to the solution x and the ratio $T_{ij}(x, \hat{x})$ is infinite, that is, the improvement of f_i is enormous. With these words, Geoffrion defined proper efficiency by eliminating unbounded trade-offs between objective functions [6]. In the following, the definition of efficiency and proper efficiency in Geoffrion's sense are given. Before these definitions we introduce some notations as follows:

$\mathbb{R}^p_+ = \{y : y_i \geqq 0, \quad \forall i \in \{1, 2, ..., p\}\}$, $\mathbb{R}^p_{++} = \{y : y_i > 0, \quad \forall i \in \{1, 2, ..., p\}\}$.

For $y^1,\ y^2 \in \mathbb{R}^p$, we use the following notations:

$$y^1 \leqq y^2 \Leftrightarrow y^2 - y^1 \in \mathbb{R}^p_+,$$
$$y^1 \leq y^2 \Leftrightarrow y^2 - y^1 \in \mathbb{R}^p_+ \text{ and } y_1 \neq y_2,$$
$$y^1 < y^2 \Leftrightarrow y^2 - y^1 \in \mathbb{R}^p_{++}.$$

Definition 1.2 *[14] The element $\hat{x} \in X$ is said to be an efficient solution of the problem (1) if there exists no element $x \in X$ such that $f(x) \leq f(\hat{x})$.*

Definition 1.3 *[6] An efficient solution $\hat{x} \in X$ is called a properly efficient solution in Geoffrion's sense if there exists a positive real number M such that for any $x \in X$, $i \in \{1, ..., p\}$ with $f_i(x) < f_i(\hat{x})$ there is an index $j \in \{1, ..., p\}$ with $f_j(\hat{x}) < f_j(x)$ such that*

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M. \tag{3}$$

The number $M > 0$ satisfying the requirement of Definition 1.3 is called a proper constant of (1) at $\hat{x}$.

Consider the problem $\min_{x\in X} f(x_1,x_2)=(x_1,x_2)$, where $X=\Big\{(x_1,x_2)\in\mathbb{R}^2:\ (x_1-1)^2+(x_2-1)^2\le 1,\ x_1,x_2\in[0,1]\Big\}$. Let $\hat{x}=(1,0)$. Set $x_1^{\varepsilon}:=1-\varepsilon$, where $\varepsilon\in[0,1]$. Then $x_2^{\varepsilon}=1-\sqrt{1-\varepsilon^2}$. In this case, $f_1(x^{\varepsilon})<f_1(\hat{x})$ and $f_2(\hat{x})<f_2(x^{\varepsilon})$. Moreover,

$$\frac{f_1(\hat{x})-f_1(x^{\varepsilon})}{f_2(x^{\varepsilon})-f_2(\hat{x})}=\frac{\varepsilon}{1-\sqrt{1-\varepsilon^2}}\to\infty,\quad \text{whenever } \varepsilon\to 0.$$

Hence $\hat{x}=(1,0)$ is not a properly efficient solution due to the unbounded treatment of $T_{12}(x^{\varepsilon},\hat{x})$ as ε tends to zero. However, $T_{12}(x^{\varepsilon},\hat{x})$ takes large values only when ε is infinitesimal. That is, a enormous improvement is not gained by moving from $\hat{x}$ to x^{ε}. An important and significant point in this analysis is that gained improvement by moving from $\hat{x}$ to x^{ε} is an unrealistic improvement. In other words, unboundedness in this example is obtained for infinitesimal gain. Hence, it is irrational to eliminate such a solution in order to achieve an unrealistic improvement. Now we consider another example in which the ratio $T_{ij}(x,\hat{x})$ at an efficient solution will be infinite whenever the gain improvement is an actual improvement. Consider the problem $\min_{x\in X}(f_1(x),f_2(x))$, where $X=\mathbb{R}\backslash\{0\}$ and $f_1(x)=\frac{1}{f_2(x)}=x$. It is clear that the set of efficient solution is $\{(x,\frac{1}{x})|\ x<0\}$. Consider the efficient solution $\hat{x}=(-1,-1)$. Define $x(n)=(-1-n,\frac{1}{-1-n})$, where n is a natural number. One has

$$T_{ij}(x(n),\hat{x})=\frac{-1-(-1-n)}{\frac{1}{-1-n}-(-1)}=\frac{n}{\frac{n}{n+1}}=n+1.$$

It is clear that $T_{ij}(x(n),\hat{x})\to\infty$ as $n\to\infty$. The gained improvement in this example is n and $o(n)=\|f(x(n))-f(\hat{x})\|$. With these words, eliminating $\hat{x}$ to achieve $x(n)$ is rational and useful, practically, because the improvement is so large. Based on these words, to separate these two cases, unbounded trade-off with infinitesimal gain and unbounded trade-off with enormous improvement, and for an appropriate filtering of efficient solution we define the following concept of efficiency called quasi-proper efficiency.

Definition 1.4 *Let s be a nonnegative real number. An efficient solution $\hat{x}\in X$ is said to be a quasi-properly efficient solution of order s (QPE(s)) for the problem* (1)*, if there exists a positive real number M such that for all $x\in X$ and $i\in\{1,...,p\}$ with $f_i(x)<f_i(\hat{x})$, there is $j\in\{1,...,p\}$ such that $f_j(\hat{x})<f_j(x)$ and*

$$\frac{f_i(\hat{x})-f_i(x)}{f_j(x)-f_j(\hat{x})}\le\frac{M}{\|f(x)-f(\hat{x})\|^s}. \tag{4}$$

It can be seen that for $s=0$, this definition coincides with proper efficiency in Geoffrion's sense.

This paper is organized as follows. Section 2 contains some basic definitions and notations that are used throughout the paper. Section 3 is devoted to introducing a special type of enhanced efficiency, namely quasi-proper efficiency. This concept is illustrated by some examples. In Section 4, we establish two main characterizations of quasi-properly efficient solutions.

2. Quasi-proper efficiency

In order to deal with efficient solutions with some unbounded trade-offs, Jiang and Deng in [10] proposed a new concept of enhanced efficiency, namely α-proper efficiency corresponding to positive parameter α. They showed for some bounded constraint polynomial multiobjective optimization problems in the presence of some condition such as stability and some error bound efficiency implies α-properly efficient with a known estimate on α. The definition of α-proper efficiency due to Jiang and Deng is as follows.

Definition 2.1 *[10] Let α be a positive real number. An efficient solution $\hat{x} \in X$ is said to be an α-proper efficiency solution of problem (1) if there exists a positive number M such that for all $x \in X$*

$$\|[f(\hat{x}) - f(x)]_+\|_\infty \leq M\big(\|[f(x) - f(\hat{x})]_+\|_\infty + \|[f(x) - f(\hat{x})]_+\|_\infty^\alpha\big). \tag{5}$$

Jiang and Deng [10] showed that if $\alpha > 0$ then $\hat{x}$ is an α-proper efficiency solution if and only if there exists an $M > 0$ such that $\hat{x}$ is a minimizer of the following problem:

$$\min_{x \in X} \sum_{i=1}^{p} f_i(x) + M\big(\|[f(x) - f(\hat{x})]_+\|_\infty + \|[f(x) - f(\hat{x})]_+\|_\infty^\alpha\big). \tag{6}$$

In the sequel, we compare the proposed definition, quasi-proper efficiency in Definition 1.4 with α-proper efficiency.

The following example shows that it is possible that there are some efficient solutions that are not considered by Definition 2.1 but are considered by Definition 1.4.

Example 2.2 *Consider problem $\min_{0 \leq x \leq \frac{1}{2}} (f_1(x), f_2(x))$ in which*

$$f_1(x) = \begin{cases} \ln x & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$$

and $f_2(x) = x^2$.

If we set $\hat{x} := 0$, it is seen that $f_1(x) < f_1(\hat{x})$ and $f_2(\hat{x}) < f_2(x)$, for any $0 < x \leq \frac{1}{2}$. Thus $\hat{x}$ is an efficient point and the trade-off between objective functions is unbounded. It can be easily shown that for any number $s > 2$ there exists an $M > 0$ such that

$$\frac{f_1(\hat{x}) - f_1(x)}{f_2(x) - f_2(\hat{x})} = \frac{-\ln x}{x^2} \leq \frac{M}{x^s} < \frac{M}{\|f(x) - f(\hat{x})\|^s}, \quad \text{for all } 0 < x \leq \frac{1}{2}. \tag{7}$$

In other words, the rate of growth of the corresponding unbounded trade-off is not greater than the rate of growth for $\frac{1}{\|f(x)-f(\hat{x})\|^s}$. Thus $\hat{x} = 0$ is a QPE(s) solution.

Now we show that Definition 2.1 does not hold at $\hat{x} = 0$, for any $\alpha \geq 0$.

Clearly, $\|[f(x) - f(0)]_+\|_\infty = x^2$ and $\|[f(0) - f(x)]_+\|_\infty = -\ln x$, for all $0 \leq x \leq \frac{1}{2}$. Consequently, for any $\alpha \geq 0$,

$$\lim_{x \to 0^+} \frac{\|[f(0) - f(x)]_+\|_\infty}{\|[f(x) - f(0)]_+\|_\infty + \|[f(x) - f(0)]_+\|_\infty^\alpha}$$

$$= \lim_{x \to 0^+} \frac{-\ln x}{x^2 + x^{2\alpha}} = \infty.$$

Hence, for any $\alpha \geq 0$ and $M > 0$ there is some $0 < x \leq \frac{1}{2}$ such that

$$\frac{\|[f(\hat{x}) - f(x)]_+\|_\infty}{\|[f(x) - f(\hat{x})]_+\|_\infty + \|[f(x) - f(\hat{x})]_+\|_\infty^\alpha} > M,$$

and therefore

$$\begin{aligned}\|[f(\hat{x})-f(x)]_+\|_\infty >& M\big(\|[f(x)-f(\hat{x})]_+\|_\infty \\ &+\|[f(x)-f(\hat{x})]_+\|_\infty^\alpha\big).\end{aligned}$$

Indeed, Definition 2.1 does not yield any meaningful interpretation for this unbounded trade-off.

3. Characterization of quasi-proper efficiency

This section investigates some characterizations for characterizing quasi-properly efficient solutions based on a scalar function utilizing the Chebyshev norm.

Lemma 3.1 *Let $s \geq 0$. A feasible solution $\hat{x} \in X$ is a QPE(s) solution if and only if there exists a positive real number M such that for all $x \in X$ the following inequality holds:*

$$\|f(x)-f(\hat{x})\|^s\|[f(\hat{x})-f(x)]_+\|_\infty \leq M\|[f(x)-f(\hat{x})]_+\|_\infty. \tag{8}$$

Proof $\Rightarrow$) Let $x \in X$. If $\|[f(\hat{x})-f(x)]_+\|_\infty = 0$, (8) holds. Hence, we can assume that $\|[f(\hat{x})-f(x)]_+\|_\infty > 0$. Let

$$f_{l(x)}(\hat{x})-f_{l(x)}(x) := \|[f(\hat{x})-f(x)]_+\|_\infty,$$

while $l(x) \in \{1,...,p\}$.

Since $\hat{x}$ is a QPE(s), there exists an index $j \in \{1,...,p\}$ such that

$$\|f(x)-f(\hat{x})\|^s\big(f_{l(x)}(\hat{x})-f_{l(x)}(x)\big) \leq M\big(f_j(x)-f_j(\hat{x})\big),$$

and therefore

$$\begin{aligned}\|f(x)-f(\hat{x})\|^s\|[f(\hat{x})-f(x)]_+\|_\infty =& \|f(x)-f(\hat{x})\|^s\big(f_{l(x)}(\hat{x})-f_{l(x)}(x)\big)\\ \leq& M\big(f_j(x)-f_j(\hat{x})\big)\\ \leq& M\|[f(x)-f(\hat{x})]_+\|_\infty.\end{aligned}$$

$\Leftarrow$) Let $i \in \{1,...,p\}$ and $x \in X$ with $f_i(x) < f_i(\hat{x})$. Thus, $\|[f(\hat{x})-f(x)]_+\|_\infty > 0$. By (8), $\|[f(x)-f(\hat{x})]_+\|_\infty > 0$. Assume that $f_j(x)-f_j(\hat{x}) = \|[f(x)-f(\hat{x})]_+\|_\infty$ ($j \in \{1,...,p\}$). Relation (8) implies that

$$\begin{aligned}\|f(x)-f(\hat{x})\|^s\big(f_i(\hat{x})-f_i(x)\big) \leq& \|f(x)-f(\hat{x})\|^s\|[f(\hat{x})-f(x)]_+\|_\infty\\ \leq& M\|[f(x)-f(\hat{x})]_+\|_\infty\\ =& M\big(f_j(x)-f_j(\hat{x})\big).\end{aligned}$$

Hence, $\hat{x}$ is a QPE(s) of Problem (1). Thus the proof is completed. □

Now we can propose a scalar function to characterize a quasi-properly efficient solution. Corresponding to parameters $M > 0$ and $s \geq 0$ the following scalar function is introduced

$$\bar{f}(x;M,s) = \|f(x)-f(\hat{x})\|^s\Big(\sum_{i=1}^{p}\big(f_i(x)-f_i(\hat{x})\big)\Big) + M\|[f(x)-f(\hat{x})]_+\|_\infty. \tag{9}$$

The following theorem provides a sufficient condition for efficiency of solution $\hat{x}$ using the scalar function given in (9).

Theorem 3.2 *Let $M > 0$. Consider the scalar problem:*

$$\min_{x \in X} \bar{f}(x; M, s). \tag{10}$$

If $\hat{x}$ is a minimizer of Problem (10) then $\hat{x}$ is an efficient solution of Problem (1).

Proof By contrary, assume that $\hat{x}$ is not an efficient solution. Then there exists some $x' \in X$ such that $f(x') \leq f(\hat{x})$. Therefore $\|[f(x') - f(\hat{x})]_+\|_\infty = 0$, $\|f(x') - f(\hat{x})\|^s > 0$, and $\sum_{i=1}^p f_i(x') < \sum_{i=1}^p f_i(\hat{x})$. Hence, $\bar{f}(x'; M, s) < \bar{f}(\hat{x}; M, s) = 0$. This contradiction completes the proof. □

Theorem 3.3 gives a necessary condition for efficiency.

Theorem 3.3 *[10] Let $\hat{x}$ be an efficient solution of Problem (1). Then $\hat{x}$ solves the following problem:*

$$\min_{x \in X} \sum_{i=1}^p f_i(x) + \|[f(\hat{x}) - f(x)]_+\|_1,$$

where $\|.\|_1$ stands for ℓ_1-norm.

Partially similar to the idea of Theorem 3.1 in [10] the following theorem gives a characterization for quasi-properly efficient solutions.

Theorem 3.4 *Let s be a nonnegative real number.*

i) If $\hat{x}$ is a QPE(s) of Problem (1) then there exists a positive real number M such that $\hat{x}$ solves Problem (10).

ii) Let X be a bounded set and $f = (f_1, ..., f_p)$ is a continuous function. If there exists a positive real number M such that $\hat{x}$ solves Problem (10) then $\hat{x}$ is a QPE(s) of Problem (1).

Proof

i) Since $\hat{x}$ is an efficient solution, by Theorem 3.3

$$\sum_{i=1}^p f_i(\hat{x}) \leq \sum_{i=1}^p f_i(x) + \|[f(\hat{x}) - f(x)]_+\|_1,$$

for all $x \in X$. Therefore, for all $x \in X$,

$$\begin{aligned} 0 \leq & \|f(x) - f(\hat{x})\|^s \sum_{i=1}^p \big(f_i(x) - f_i(\hat{x})\big) + \|f(x) - f(\hat{x})\|^s \|[f(\hat{x}) - f(x)]_+\|_1 \\ \leq & \|f(x) - f(\hat{x})\|^s \sum_{i=1}^p \big(f_i(x) - f_i(\hat{x})\big) + p\|f(x) - f(\hat{x})\|^s \|[f(\hat{x}) - f(x)]_+\|_\infty \\ \leq & \|f(x) - f(\hat{x})\|^s \sum_{i=1}^p \big(f_i(x) - f_i(\hat{x})\big) + p\bar{M}\|[f(x) - f(\hat{x})]_+\|_\infty, \end{aligned}$$

for some $\bar{M} > 0$ satisfying Lemma 3.1.

Setting $M := p\bar{M}$. Hence, $\bar{f}(\hat{x}; M, s) = 0 \leq \bar{f}(x; M, s)$, for all $x \in X$.

ii) Consider $x \in X$ and $i \in \{1,...,p\}$ with $f_i(x) < f_i(\hat{x})$. Then $\|[f(x)-f(\hat{x})]_+\|_\infty > 0$. Because, in the case $\|[f(x)-f(\hat{x})]_+\|_\infty = 0$ then $f(x) \leq f(\hat{x})$. Hence $\bar{f}(x;M,s) < 0$, which is a contradiction because $\hat{x}$ is a minimizer of $\bar{f}(\cdot;M,s)$. Since $\hat{x}$ solves Problem (10), for any $i \in \{1,...,p\}$

$$\|f(x)-f(\hat{x})\|^s\big(f_j(\hat{x})-f_j(x)\big) \leq \|f(x)-f(\hat{x})\|^s \sum_{\substack{i=1\\ i\neq j}}^{p} \big(f_i(x)-f_i(\hat{x})\big).$$

Without loss of generality, assume that

$$\|[f(\hat{x})-f(x)]_+\|_\infty = f_j(\hat{x})-f_j(x), \text{ for some } j \in \{1,...,p\}.$$

Therefore,

$$\begin{aligned}\|f(x)-f(\hat{x})\|^s\|[f(\hat{x})-f(x)]_+\|_\infty &= \|f(x)-f(\hat{x})\|^s\big(f_j(\hat{x})-f_j(x)\big)\\ &\leq \|f(x)-f(\hat{x})\|^s \sum_{\substack{i=1\\ i\neq j}}^{p}\big(f_i(x)-f_i(\hat{x})\big) + M\|[f(x)-f(\hat{x})]_+\|_\infty\\ &\leq L(p-1)\|[f(x)-f(\hat{x})]_+\|_\infty + M\|[f(x)-f(\hat{x})]_+\|_\infty\\ &\leq (L(p-1)+M)\|[f(x)-f(\hat{x})]_+\|_\infty,\end{aligned}$$

where L is an upper bound of $\|f(x)-f(\hat{x})\|^s$ because of boundedness of X and continuity of f. Setting $\hat{M} := L(p-1)+M$, conclude that

$$\|f(x)-f(\hat{x})\|^s\|[f(\hat{x})-f(x)]_+\|_\infty \leq \hat{M}\|[f(x)-f(\hat{x})]_+\|_\infty, \text{ for all } x \in X;$$

thus, $\hat{x}$ is a QPE(s). □

The boundedness assumption in Theorem 3.4 (ii) is redundant for locally efficient solutions. Hence, it can be stated as an "if and only if" proposition for local solutions as stated in Corollary 3.5.

Corollary 3.5 *Let $s \in \mathbb{R}$ be a nonnegative number. Then $\hat{x}$ is a locally QPE(s) of Problem* (1) *if and only if there exists a positive real number M such that $\hat{x}$ is a local minimizer of Problem* (10).

In the sequel, quasi-properly efficient solutions are characterized based on the notion of stability. To this aim, consider the following problem

$$\begin{aligned}\bar{P}(0):\quad \min\quad & \varphi(x)\\ \text{s.t.}\quad & f(x) \leq 0,\\ & x \in X,\end{aligned}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary scalar function. Now, related to any $y \in \mathbb{R}^p$, the perturbation Problem $\bar{P}(y)$ is defined as follows:

$$\begin{aligned}\bar{P}(y):\quad \min\quad & \varphi(x)\\ \text{s.t.}\quad & f(x) \leq y,\\ & x \in X,\end{aligned}$$

Denote by $A(y)$ the feasible set of Problem $\bar{P}(y)$ and set

$$v(y) = \begin{cases} \inf\{\varphi(x) : x \in A(y)\}, & \text{if } A(y) \neq \emptyset \\ \infty, & \text{if } A(y) = \emptyset. \end{cases}$$

Assume that $\hat{x}$ is a minimizer of Problem $\bar{P}(0)$, that is

$$v(0) = \inf\{\varphi(x) : x \in A(0)\} = \varphi(\hat{x}).$$

In this case, Problem $\bar{P}(0)$ is said to be stable [13] at $\hat{x}$ if there exists an $M > 0$ such that

$$\frac{v(y) - v(0)}{\|y\|_1} \geq -M, \text{ for all } y \neq 0. \tag{11}$$

Theorem 3.6 *Assume that $\hat{x} \in X$ is an efficient solution of the problem (1) and s be a nonnegative real number. Then*

i) If $\hat{x}$ is a QPE(s) solution of (1) then $P(0)$ is stable at $\hat{x}$, where

$$\begin{aligned} P(y): \quad v(y) := \min \quad & \|f(x) - f(\hat{x})\|^s \textstyle\sum_{i=1}^p \big(f_i(x) - f_i(\hat{x})\big) \\ s.t. \quad & f(x) - f(\hat{x}) \leq y, \\ & x \in X. \end{aligned}$$

ii) Let X be a bounded set. If $P(0)$ is stable at $\hat{x}$ then $\hat{x}$ is a QPE(s) solution of (1).

Proof i) The first part is proved by contradiction. Assume that $P(0)$ is not stable at $\hat{x}$. Thus there exists $\{y_k\} \subset \mathbb{R}^p_+$ with $\|y_k\|_1 > 0$ for all $k = 1, 2, \ldots$, such that

$$\frac{v(y_k) - v(0)}{\|y_k\|_1} \to -\infty,$$

and $v(y_k) < v(0)$, for all $k \in \mathbb{N}$. Hence, for any $k \in \mathbb{N}$ there is $x_k \in X$ such that $f(x_k) - f(\hat{x}) \leq y_k$ and

$$\frac{\|f(x_k) - f(\hat{x})\|^s \sum_{i=1}^p \big(f_i(x_k) - f_i(\hat{x})\big)}{\|y_k\|_1} \to -\infty. \tag{12}$$

Since $\hat{x}$ is a QPE(s), by Theorem 3.4 there is an $M > 0$ such that for any $x \in X$

$$0 \leq \|f(x) - f(\hat{x})\|^s \sum_{i=1}^p \big(f_i(x) - f_i(\hat{x})\big) + M\|[f(x) - f(\hat{x})]_+\|_\infty.$$

Hence, for any $k \in \mathbb{N}$

$$\begin{aligned} 0 \leq & \|f(x_k) - f(\hat{x})\|^s \sum_{i=1}^p \big(f_i(x_k) - f_i(\hat{x})\big) + M\|[f(x_k) - f(\hat{x})]_+\|_\infty \\ \leq & \|f(x_k) - f(\hat{x})\|^s \sum_{i=1}^p \big(f_i(x_k) - f_i(\hat{x})\big) + M\|y_k\|_\infty \\ \leq & \|f(x_k) - f(\hat{x})\|^s \sum_{i=1}^p \big(f_i(x_k) - f_i(\hat{x})\big) + M\|y_k\|_1. \end{aligned}$$

Thus,

$$\frac{\|f(x_k)-f(\hat{x})\|^s\sum_{i=1}^{p}\big(f_i(x_k)-f_i(\hat{x})\big)}{\|y_k\|_1}\geq -M,$$

which contradicts (12). Hence, the proof of the first part is completed.

ii) To prove the second part, assume that $P(0)$ is stable at $\hat{x}$. By contradiction, assume that $\hat{x}$ is not QPE(s). Thus, for any sequence $\{M_k\}$ with $M_k\to+\infty$, from Theorem 3.4 (ii), there exists a sequence $\{x_k\}$ such that

$$\|f(x_k)-f(\hat{x})\|^s\sum_{i=1}^{p}\big(f_i(x_k)-f_i(\hat{x})\big)+M_k\|[f(x_k)-f(\hat{x})]_+\|_\infty<0. \tag{13}$$

From (13) and $\sum_{i=1}^{p}[f_i(x_k)-f_i(\hat{x})]_+\leq p\|[f(x_k)-f(\hat{x})]_+\|_\infty$, for any $k\in\mathbb{N}$, the following inequality holds:

$$\|f(x_k)-f(\hat{x})\|^s\sum_{i=1}^{p}\big(f_i(x_k)-f_i(\hat{x})\big)+M_k\frac{1}{p}\sum_{i=1}^{p}[f_i(x_k)-f_i(\hat{x})]_+<0. \tag{14}$$

Since $\hat{x}$ is an efficient solution, (14) implies that $\sum_{i=1}^{p}[f_i(x_k)-f_i(\hat{x}]_+>0$. Now set $y_k:=[f(x_k)-f(\hat{x})]_+$. Therefore $\|y_k\|_1>0$ and $f(x_k)-f(\hat{x})\leq y_k$. Hence, x_k is a feasible point for $P(y_k)$, for all k. On the other hand, $v(y_k)\leq\|f(x_k)-f(\hat{x})\|^s\sum_{i=1}^{p}\big(f_i(x_k)-f_i(\hat{x})\big)$ and $v(0)=0$. Thus,

$$\begin{aligned}\frac{v(y_k)-v(0)}{\|y_k\|_1}\leq&\frac{\|f(x_k)-f(\hat{x})\|^s\sum_{i=1}^{p}\big(f_i(x_k)-f_i(\hat{x})\big)}{\|y_k\|_1}\\ \leq&-M_k\frac{\frac{1}{p}\|y_k\|_1}{\|y_k\|_1}\\ =&\frac{-M_k}{p}\to-\infty,\ \text{as }k\to+\infty,\end{aligned}$$

which contradicts (11). Thus $\hat{x}$ is a QPE(s). □

It should be noted that Lee et al. obtained similar necessary and sufficient conditions for properly efficient solutions in Hartley's sense using stability [13].

4. Conclusion

In this paper, we introduce a new concept of efficiency, namely quasi-proper efficiency, and characterize it. In the interactive optimization literature proper efficiency and quasi-proper efficiency play important roles and can be used as an efficient guideline in applications. It should be noted that scalar problems given in this paper just propose some characterizations for quasi-proper efficiency. In order to have a computational procedure, by comparing available approximation of the efficient frontier, we can consider some certain values of " M " and determine so-called " M -proper efficiency " and " M -quasi-proper efficiency ".

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