Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1166 – 1179

doi:10.3906/mat-1704-107

Some series involving the Euler zeta function

Min-Soo KIM*
Division of Mathematics, Science, and Computers, Kyungnam University, Changwon, Gyeongnam, Republic of Korea

Received: 25.04.2017 • **Accepted/Published Online:** 16.10.2017 • **Final Version:** 08.05.2018

Abstract: In this paper, using the Boole summation formula, we obtain a new integral representation of n-th quasi-periodic Euler functions $\overline{E}_n(x)$ for $n = 1, 2, \ldots$. We also prove several series involving Euler zeta functions $\zeta_E(s)$, which are analogues of the corresponding results by Apostol on some series involving the Riemann zeta function $\zeta(s)$.

Key words: Hurwitz-type Euler zeta functions, Euler zeta functions, Euler polynomials, Boole summation formula, quasi-periodic Euler functions

1. Introduction

The Hurwitz-type Euler zeta function is defined as follows

$$\zeta_E(s,a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s} \tag{1}$$

for complex arguments s with $\mathrm{Re}(s) > 0$ and a with $\mathrm{Re}(a) > 0$, which is a deformation of the well-known Hurwitz zeta function

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{2}$$

for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(a) > 0$. Note that $\zeta(s,1) = \zeta(s)$, the Riemann zeta function. The series (1) converges for $\mathrm{Re}(s) > 0$ and it can be analytically continued to the complex plane without any pole. For further results concerning the Hurwitz-type Euler zeta function, we refer to the recent works in [10] and [14]. Let $a = 1$ in (1); it reduces to the Euler zeta function

$$\zeta_E(s) = \zeta_E(s,1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{3}$$

for $\mathrm{Re}(s) > 0$, which is also a special case of Witten zeta functions in mathematical physics (see [20, p. 248, (3.14)]). In fact, it is shown that the Euler zeta function $\zeta_E(s)$ is summable (in the sense of Abel) to $(1-2^{1-s})\zeta(s)$ for all values of s. Several properties of $\zeta_E(s)$ can be found in [3, 10, 12, 16]. For example, in the form on [1, p. 811], the left-hand side is the special values of the Riemann zeta functions at positive integers,

*Correspondence: mskim@kyungnam.ac.kr
2010 *AMS Mathematics Subject Classification:* 11M35, 33E20, 33B15, 11B68

and the right-hand side is the special values of Euler zeta functions at positive integers. In number theory, the Hurwitz-type Euler zeta function (1) represents the partial zeta function in one version of Stark's conjecture of cyclotomic fields (see [15, p. 4249, (6.13)]). The corresponding L-functions (the alternating L-series) have also appeared in a decomposition of the $(S,\{2\})$-refined Dedekind zeta functions of cyclotomic fields (see [12, p. 81, (3.8)]). Recently, using Log Gamma functions, Can and Dağli proved a derivative formula of these L-functions (see [8, Eq. (4.13)]).

The Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!} \tag{4}$$

for $|t|<\pi$ (see, for details, [11, 21, 27]). They are the special values of (1) at nonpositive integers (see [10, p. 520, Corollary 3], [9, p. 761, (2.3)], [14, p. 2983, (3.1)], [29, p. 41, (3.8)] and (46) below). The integers $E_n=2^nE_n(1/2), n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, are called Euler numbers. For example, $E_0=1, E_2=-1, E_4=5$, and $E_6=-61$. The Euler numbers and polynomials (so called by Scherk in 1825) appear in Euler's famous book, Insitutiones Calculi Differentials (1755, pp. 487-491 and p. 522). Notice that the Euler numbers with odd subscripts vanish, that is, $E_{2m+1}=0$ for all $m\in\mathbb{N}_0$.

For $n\in\mathbb{N}_0$, the n-th quasi-periodic Euler functions are defined by

$$\overline{E}_n(x+1)=-\overline{E}_n(x) \tag{5}$$

for all $x\in\mathbb{R}$, and

$$\overline{E}_n(x)=E_n(x)\ \text{ for } 0\le x<1 \tag{6}$$

(see [7, p. 661]). For arbitrary real numbers x, $[x]$ denotes the greatest integer not exceeding x and $\{x\}$ denotes the fractional part of real number x; thus

$$\{x\}=x-[x]. \tag{7}$$

Then, for $r\in\mathbb{Z}$ and $n\in\mathbb{N}_0$, we have

$$\overline{E}_n(x)=(-1)^{[x]}E_n(\{x\}),\quad \overline{E}_n(x+r)=(-1)^r\overline{E}_n(x) \tag{8}$$

(see [4, (1.2.9)] and [7, (3.3)]). For further properties of the quasi-periodic Euler functions, we refer to [4, 7, 8, 13].

In this paper, we obtain a new integral representation of n-th quasi-periodic Euler functions $\overline{E}_n(x)$ as follows.

Theorem 1.1 *Let $n\in\mathbb{N}_0$ and let $\overline{E}_n(x)$ be the n-th quasi-periodic Euler functions. Then for $x>0$*

$$\overline{E}_n(x)=(-1)^n n!\frac{1}{\pi i}\int_{(c)}\frac{\Gamma(s)}{\Gamma(s+n+1)}\zeta_E(-s-n)x^{-s}\mathrm{d}s,$$

where (c) denotes the vertical straight line from $c-i\infty$ to $c+i\infty$ with $0<c<1$ and $\Gamma(s)$ denotes the Euler gamma function.

Remark 1.2 *We remark that this theorem is an analogue of a result by Li et al. on Riemann zeta functions (see [19, Proposition 1]).*

Furthermore, we also obtain the following two theorems on series involving Euler zeta functions $\zeta_E(s)$. They are the analogues of the corresponding results of Apostol [2] on some series involving the Riemann zeta function.

Theorem 1.3 *Let $\binom{-s}{r}$ denote the binomial symbol defined through the Euler gamma function $\Gamma(s)$ as follows*

$$\binom{-s}{r} = (-1)^r \binom{s+r-1}{r} = (-1)^r \frac{\Gamma(s+r)}{\Gamma(s) r!},$$

where $s \in \mathbb{C}$ and $r \in \mathbb{N}$. Then the following identities hold:

1. *For k odd and $k > 1$, we have*

$$\zeta_E(s)\left(1 - k^{-s}\right) = \frac{1}{2} \sum_{h=1}^{k-1} \frac{(-1)^{h-1}}{h^s} + \sum_{r=1}^{\infty} \binom{-s}{2r} \frac{\zeta_E(s+2r)}{k^{s+2r}} \frac{E_{2r}(k)}{2}.$$

2. *For k odd and $k > 1$, we have*

$$\sum_{h=1}^{k-1} \frac{(-1)^h}{h^s} = \sum_{r=0}^{\infty} \binom{-s}{2r+1} \frac{\zeta_E(s+2r+1)}{k^{s+2r+1}} (E_{2r+1}(k) + E_{2r+1}(0)).$$

Theorem 1.4 *Let μ be the Möbius function. Then for k odd and $k > 1$, we have*

$$\zeta_E(s) \sum_{d|k} \mu(d) d^{-s} = 2 \sum_{r=0}^{\infty} \binom{-s}{2r} \zeta_E(s+2r) k^{-2r-s} H(2r, k) - H(-s, k),$$

where

$$H(\alpha, k) = \sum_{\substack{h=1 \\ (h,k)=1}}^{\left[\frac{k}{2}\right]} (-1)^h h^{\alpha} \quad (\alpha \in \mathbb{C})$$

is the alternating sum of the α-th power of those integers not exceeding $\left[\frac{k}{2}\right]$ that are relatively prime to k.

Remark 1.5 *The evaluations of series involving Riemann zeta function $\zeta(s)$ and related functions have a long history that can be traced back to Christian Goldbach (1690–1764) and Leonhard Euler (1707–1783) (see, for details, [26, Chapter 3]). Ramaswami [24] presented numerous interesting recursion formulas that can be employed to get the analytic continuation of Riemann zeta function $\zeta(s)$ over the whole complex plane. Apostol [2] also gave some formulas involving the Riemann zeta function $\zeta(s)$; some of them are generalizations of Ramaswami's identities. For more results, we refer to, e.g., Apostol [2], Choi and Srivastava [26], Landau [18], Murty and Reece [23], Ramaswami [24], and Srivastava [25].*

2. Proof Theorem 1.1

To derive Theorem 1.1, we need the following lemmas.

In this section, we first present the Boole summation formula as follows:

Lemma 2.1 *([8, Boole summation formula]) Let α, β, and l be integers such that $\alpha < \beta$ and $l > 0$. If $f^{(l)}(t)$ is absolutely integrable over $[\alpha, \beta]$, then*

$$2\sum_{n=\alpha}^{\beta-1}(-1)^n f(n) = \sum_{r=0}^{l-1}\frac{E_r(0)}{r!}\left((-1)^{\beta-1}f^{(r)}(\beta) + (-1)^{\alpha}f^{(r)}(\alpha)\right) + \frac{1}{(l-1)!}\int_{\alpha}^{\beta}\overline{E}_{l-1}(-t)f^{(l)}(t)\mathrm{d}t,$$

where $\overline{E}_n(t)$ is the n-th quasi-periodic Euler functions defended by (6) and (8).

Remark 2.2 *The alternating version of Euler–MacLaurin summation formula is the Boole summation formula (see, for example, [8, Theorem 1.2] and [21, 24.17.1–2]), which is proved by Boole [5], but a similar one may be known by Euler as well (see [22]). Recently, Can and Dağli derived a generalization of the above Boole summation formula involving Dirichlet characters (see [8, Theorem 1.3]).*

A proof of Lemma 2.1 can be found, for example, in [6, Section 5] and [8, Theorem 1.3].

Using the Boole summation formula (see Lemma 2.1 above), we obtain the following formula.

Lemma 2.3 *The integral representation*

$$\zeta_E(-u,a) = \frac{1}{2}\sum_{r=0}^{l-1}\binom{u}{r}E_r(0)a^{u-r} + \frac{1}{2(l-1)!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\int_0^{\infty}\overline{E}_{l-1}(-t)(t+a)^{u-l}\mathrm{d}t,$$

holds true for all complex numbers u and $\mathrm{Re}(a) > 0$, where l is any natural number subject only to the condition that $l > \mathrm{Re}(u)$.

Proof The proof from Lemma 2.1 is exactly like the proof given by Can and Dağli [8, Theorem 1.4] when $\chi = \chi_0$, where χ_0 is the principal character modulo 1, and so we omit it. □

Proof of Theorem 1.1 Putting $a = 1$ and $u = s$ in Lemma 2.3, by (3), we find that

$$2\zeta_E(-s) = \sum_{r=0}^{l-1}\binom{s}{r}E_r(0) + \frac{1}{(l-1)!}\frac{\Gamma(s+1)}{\Gamma(s+1-l)}\int_1^{\infty}\overline{E}_{l-1}(1-t)t^{s-l}\mathrm{d}t. \tag{9}$$

By Dirichlet's test in analysis (e.g., [17, p. 333, Theorem 2.6]), the integral on the right-hand side of the above equation converges absolutely for $\mathrm{Re}(s) < l$ and the convergence is uniform in every half-plane $\mathrm{Re}(s) \leq l - \delta$, $\delta > 0$, and so $\zeta_E(-s)$ is an analytic function of s in the half-plane $\mathrm{Re}(s) < l$. Since

$$\overline{E}_{l-1}(1-t) = (-1)^{l-1}\overline{E}_{l-1}(t) \quad (t \in \mathbb{R}) \tag{10}$$

(see [8, (2.7) with $\chi = \chi_0$] and [13, (2.7)]), for $\mathrm{Re}(s) > l-1$, we have

$$\begin{aligned}\int_0^1 \overline{E}_{l-1}(1-t)t^{s-l}\mathrm{d}t &= (-1)^{l-1}\int_0^1 \overline{E}_{l-1}(t)t^{s-l}\mathrm{d}t \\ &= (-1)^{l-1}\int_0^1 E_{l-1}(t)t^{s-l}\mathrm{d}t \\ &= (-1)^{l-1}\sum_{m=0}^{l-1}\binom{l-1}{m}E_m(0)\frac{1}{s-m},\end{aligned} \tag{11}$$

and thus the expression

$$\frac{1}{(l-1)!}\frac{\Gamma(s+1)}{\Gamma(s+1-l)}\int_0^1 \overline{E}_{l-1}(1-t)t^{s-l}\mathrm{d}t = \sum_{k=0}^{l-1}\binom{s}{k}E_k(0), \tag{12}$$

is valid for $\mathrm{Re}(s) > l-1$. Therefore by (9) and (12), for $l-1 < \mathrm{Re}(s) < l$, we have

$$\begin{aligned}2\zeta_E(-s) &= \frac{1}{(l-1)!}\frac{\Gamma(s+1)}{\Gamma(s+1-l)}\int_0^1 \overline{E}_{l-1}(1-t)t^{s-l}\mathrm{d}t \\ &\quad + (-1)^{l-1}\frac{1}{(l-1)!}\frac{\Gamma(s+1)}{\Gamma(s+1-l)}\int_1^\infty \overline{E}_{l-1}(t)t^{s-l}\mathrm{d}t \\ &= \frac{(-1)^{l-1}}{(l-1)!}\frac{\Gamma(s+1)}{\Gamma(s+1-l)}\int_0^\infty \overline{E}_{l-1}(t)t^{s-l}\mathrm{d}t.\end{aligned} \tag{13}$$

Replacing s by $s+l-1$ in (13), for $0 < \mathrm{Re}(s) < 1$, we have

$$\int_0^\infty \overline{E}_{l-1}(t)t^{s-1}\mathrm{d}t = \frac{2(-1)^{l-1}(l-1)!\Gamma(s)}{\Gamma(s+l)}\zeta_E(1-s-l).$$

Finally, by Mellin's inversion formula (see, e.g., [11, p. 49] and [19, p. 1127]), we obtain

$$\overline{E}_{l-1}(t) = 2(-1)^{l-1}(l-1)!\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s)}{\Gamma(s+l)}\zeta_E(1-s-l)t^{-s}\mathrm{d}s,$$

where (c) denotes the vertical straight line from $c-i\infty$ to $c+i\infty$ with $0<c<1$ and $t>0$. Thus the proof of Theorem 1.1 is completed.

3. Proofs of Theorem 1.3 and Theorem 1.4

In this section, we prove Theorem 1.3 and Theorem 1.4 by a method similar to that used by Apostol in [2].

First we need the following lemmas.

Lemma 3.1 *Let a be a complex number with a positive real part. The Hurwitz-type Euler zeta function satisfies the following:*

1. *Difference equation: For $k \in \mathbb{N}$,*

$$(-1)^{k-1}\zeta_E(s,a+k) + \zeta_E(s,a) = \sum_{h=0}^{k-1}(-1)^h(a+h)^{-s}.$$

2. *Distribution relation: For an odd positive integer k,*

$$\zeta_E(s,ka)=k^{-s}\sum_{r=0}^{k-1}(-1)^r\zeta_E\left(s,a+\frac{r}{k}\right).$$

Proof From the definition of $\zeta_E(s,a)$, it is easy to show that $\zeta_E(s,a+1)+\zeta_E(s,a)=a^{-s}$. We can rewrite this identity as

$$\zeta_E(s,a+h+1)+\zeta_E(s,a+h)=(a+h)^{-s}, \tag{14}$$

where $h\in\mathbb{N}_0$. Taking the alternating sum on both sides of the above identity as h ranges from 0 to $k-1$, we have

$$(-1)^{k-1}\zeta_E(s,a+k)+\zeta_E(s,a)=\sum_{h=0}^{k-1}(-1)^h(a+h)^{-s},$$

which completes the proof of Part 1.

Part 2 can be derived directly from the definition of $\zeta_E(s,a)$ (see (1) above). $\square$

Lemma 3.2 *The following identities hold:*

1. *Let $a\in\mathbb{R}$ and $a>0$. Then*

$$\zeta_E(s,x+a)=\sum_{r=0}^{\infty}\binom{-s}{r}\zeta_E(s+r,a)x^r,\quad |x|<a,$$

in which we understand $0^0=1$ if $r=0$, and $0^r=0$ otherwise.

2. *Let $|x|<a+1$ with $a\in\mathbb{R}$ and $a>0$. Then*

$$\zeta_E(s,a+1-x)=\sum_{r=0}^{\infty}(-1)^{r-1}\binom{-s}{r}\left\{\zeta_E(s+r,a)-a^{-s-r}\right\}x^r.$$

Remark 3.3 *Part 1 of Lemma 3.2 (and then (4.8) and (4.9) below) is a special case of [23, Theorem 2.4]. Part 2 of Lemma 3.2, when $a=1$, is similar to Eq. (18) in a 2001 book by Srivastava and Choi [26, p. 147].*

Proof of Lemma 3.2 Note that for $|x|<a$

$$\zeta_E(s,x+a)-\zeta_E(s,a)=\sum_{n=0}^{\infty}(-1)^n\left\{\frac{1}{(n+x+a)^s}-\frac{1}{(n+a)^s}\right\}. \tag{15}$$

Writing the summand as

$$\frac{1}{(n+x+a)^s}-\frac{1}{(n+a)^s}=\frac{1}{(n+a)^s}\left(\left(1+\frac{x}{n+a}\right)^{-s}-1\right)$$

and using the binomial theorem,

$$\begin{aligned}\frac{1}{(n+x+a)^s} - \frac{1}{(n+a)^s} &= \frac{1}{(n+a)^s}\left(\sum_{r=0}^{\infty}\binom{-s}{r}\left(\frac{x}{n+a}\right)^r - 1\right) \\ &= \frac{1}{(n+a)^s}\sum_{r=1}^{\infty}\binom{-s}{r}\left(\frac{x}{n+a}\right)^r.\end{aligned} \tag{16}$$

The right side of (15), by (16), is

$$\sum_{r=1}^{\infty}\binom{-s}{r}x^r\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^{s+r}} = \sum_{r=1}^{\infty}\binom{-s}{r}\zeta_E(s+r,a)x^r, \tag{17}$$

where $a>0$. By using (15) and (17), we obtain the first part.

For the second part, note that from the binomial theorem we have

$$(a-x)^{-s} = a^{-s}\left(1-\frac{x}{a}\right)^{-s} = a^{-s}\sum_{r=0}^{\infty}\binom{-s}{r}\left(-\frac{x}{a}\right)^r \tag{18}$$

for $|x|<a$. Setting $h=0$ and replacing a by $a-x$ in (14), we get

$$\zeta_E(s,a-x+1)+\zeta_E(s,a-x) = (a-x)^{-s}. \tag{19}$$

If we replace x by $-x$ in Part 1 and use (18) and (19), we get

$$\begin{aligned}\sum_{r=0}^{\infty}(-1)^r\binom{-s}{r}\left\{\zeta_E(s+r,a)-a^{-s-r}\right\}x^r &= \zeta_E(s,a-x)-(a-x)^{-s} \\ &= -\zeta_E(s,a+1-x).\end{aligned}$$

Thus the result follows.

Lemma 3.4 *Suppose k is an odd positive integer. Then we have*

$$\zeta_E(s)\left(1-k^{-s}\right) = \sum_{r=1}^{\infty}(-1)^r\binom{-s}{r}\frac{\zeta_E(s+r)}{k^{s+r}}\frac{E_r(k)+E_r(0)}{2}.$$

Proof Suppose k is an odd positive integer. If we take $a=1$ and $x=-h/k, 0\le h\le k-1$ in Part 1 of Lemma 3.2, multiply by $(-1)^h$, and sum over h, then we have

$$\sum_{h=0}^{k-1}(-1)^h\zeta_E\left(s,1-\frac{h}{k}\right) = \sum_{r=0}^{\infty}(-1)^r\binom{-s}{r}\frac{\zeta_E(s+r)}{k^r}\sum_{h=0}^{k-1}(-1)^h h^r, \tag{20}$$

in which we understand $0^r=1$ if $r=0$, and $0^r=0$ otherwise. Note that for an odd positive integer k we have

$$\left\{1,1-\frac{1}{k},\dots,1-\frac{k-1}{k}\right\} = \left\{\frac{1}{k},\frac{2}{k},\dots,\frac{1}{k}+\frac{k-1}{k}\right\}. \tag{21}$$

If we put $a = 1/k$ in Part 2 of Lemma 3.1 and use (21), we get

$$\begin{aligned}\sum_{h=0}^{k-1}(-1)^h\zeta_E\left(s,1-\frac{h}{k}\right) &= \sum_{h=0}^{k-1}(-1)^h\zeta_E\left(s,\frac{1}{k}+\frac{h}{k}\right)\\ &= k^s\zeta_E(s,1)\\ &= k^s\zeta_E(s).\end{aligned}\tag{22}$$

Hence, by (20) and (22), we have

$$\begin{aligned}\zeta_E(s) &= \sum_{r=0}^{\infty}(-1)^r\binom{-s}{r}\frac{\zeta_E(s+r)}{k^{s+r}}\sum_{h=0}^{k-1}(-1)^h h^r\\ &= \zeta_E(s)k^{-s} + \sum_{r=1}^{\infty}(-1)^r\binom{-s}{r}\frac{\zeta_E(s+r)}{k^{s+r}}\sum_{h=0}^{k-1}(-1)^h h^r\end{aligned}\tag{23}$$

for odd k. Moreover, it is easily seen that

$$\sum_{h=0}^{k-1}(-1)^h h^r = \frac{E_r(k)+E_r(0)}{2} \quad \text{for odd } k \tag{24}$$

(see [21, Equation 24.4.8] and [27, Theorem 2.1]). Thus, the proof is completed by (23) and (24). □

Lemma 3.5 *Suppose k is an odd positive integer with $k > 1$. Then we have*

$$\zeta_E(s)\left(1-k^{-s}\right) = \sum_{h=1}^{k-1}\frac{(-1)^{h-1}}{h^s} + \sum_{r=1}^{\infty}\binom{-s}{r}\frac{\zeta_E(s+r)}{k^{s+r}}\frac{E_r(k)+E_r(0)}{2}.$$

Proof Suppose $k \in \mathbb{N}$. If we take $a = 1$ and $x = h/k, 0 \le h \le k-1$ in Part 1 of Lemma 3.2, multiply by $(-1)^h$, and sum over h, then we have

$$\begin{aligned}\sum_{h=0}^{k-1}(-1)^h\zeta_E\left(s,1+\frac{h}{k}\right) &= \sum_{r=0}^{\infty}\binom{-s}{r}\frac{\zeta_E(s+r)}{k^r}\sum_{h=0}^{k-1}(-1)^h h^r\\ &= \sum_{r=1}^{\infty}\binom{-s}{r}\frac{\zeta_E(s+r)}{k^r}\frac{E_r(k)+E_r(0)}{2} + \zeta_E(s).\end{aligned}\tag{25}$$

Now, setting $a = 1$ in Part 1 of Lemma 3.1, we obtain

$$(-1)^{k-1}\zeta_E(s,k+1) + \zeta_E(s,1) = \sum_{h=0}^{k-1}(-1)^h(h+1)^{-s} \quad (k \in \mathbb{N}),$$

which is equivalent to

$$(-1)^k\zeta_E(s,k) + \zeta_E(s) = \sum_{h=1}^{k-1}(-1)^{h-1}h^{-s} \tag{26}$$

for $k \geq 2$. We set $a = 1$ in Part 2 of Lemma 3.1 and use (26); then the first term of (25) equals

$$\begin{aligned}\sum_{h=0}^{k-1}(-1)^h \zeta_E\left(s, 1+\frac{h}{k}\right) &= k^s \zeta_E(s,k) \\ &= k^s\left(\zeta_E(s) - \sum_{h=1}^{k-1}(-1)^{h-1}h^{-s}\right)\end{aligned} \tag{27}$$

for odd $k > 1$, and so by combining (25) and (27) we obtain the result. □

Now we give proofs of Theorem 1.3 and Theorem 1.4, respectively.

Proof of Theorem 1.3 It needs to be noted that

$$E_k(0) = 0$$

if k is even ([27, p. 5, Corollary 1.1(ii)]). Using the above identity, adding Lemma 3.4 and Lemma 3.5, we obtain Part 1 of Theorem 1.3. Subtracting Lemma 3.5 from Lemma 3.4, we have Part 2 of Theorem 1.3.

Proof of the Theorem 1.4 For $\alpha \in \mathbb{C}$, we introduce the alternating sum

$$H(\alpha, k) = \sum_{\substack{h=1 \\ (h,k)=1}}^{\left[\frac{k}{2}\right]} (-1)^h h^\alpha.$$

From now on, let k denote an odd integer and $k > 1$. By taking $a = 1$ and $x = h/k, (h,k) = 1$ in Part 1 of Lemma 3.2, $1 \leq h \leq \left[\frac{k}{2}\right]$, multiplying by $(-1)^h$, and summing over h, we obtain

$$\sum_{\substack{h=1 \\ (h,k)=1}}^{\left[\frac{k}{2}\right]} (-1)^h \zeta_E\left(s, 1+\frac{h}{k}\right) = \sum_{r=0}^{\infty}\binom{-s}{r}\zeta_E(s+r)k^{-r}H(r,k). \tag{28}$$

Similarly, we have

$$\sum_{\substack{h=1 \\ (h,k)=1}}^{\left[\frac{k}{2}\right]} (-1)^{h-1} \zeta_E\left(s, 1-\frac{h}{k}\right) = -\sum_{r=0}^{\infty}(-1)^r\binom{-s}{r}\zeta_E(s+r)k^{-r}H(r,k). \tag{29}$$

Setting $h = 0$ in (14), the left-hand side of (28) equals

$$-\sum_{\substack{h=1 \\ (h,k)=1}}^{\left[\frac{k}{2}\right]} (-1)^h \zeta_E\left(s, \frac{h}{k}\right) + k^s H(-s,k). \tag{30}$$

If k is odd, $(k-1)/2$ is an integer and so we get

$$\begin{aligned}\frac{k-1}{2}=\left[\frac{k}{2}\right] &\Leftrightarrow \frac{k}{2}=\left[\frac{k}{2}\right]+\frac{1}{2}\\ &\Leftrightarrow k=2\left[\frac{k}{2}\right]+1 \\ &\Leftrightarrow k-\left[\frac{k}{2}\right]=\left[\frac{k}{2}\right]+1.\end{aligned} \tag{31}$$

Hence

$$1-\frac{\left[\frac{k}{2}\right]}{k}=\frac{\left[\frac{k}{2}\right]+1}{k}, 1-\frac{\left[\frac{k}{2}\right]-1}{k}=\frac{\left[\frac{k}{2}\right]+2}{k}, \dots, 1-\frac{1}{k}=\frac{k-1}{k},$$

which leads easily to the required

$$-\sum_{\substack{h=1\\(h,k)=1}}^{\left[\frac{k}{2}\right]}(-1)^h\zeta_E\left(s,1-\frac{h}{k}\right)=\sum_{\substack{h=\left[\frac{k}{2}\right]+1\\(h,k)=1}}^{k}(-1)^h\zeta_E\left(s,\frac{h}{k}\right),$$

that is,

$$\sum_{\substack{h=1\\(h,k)=1}}^{\left[\frac{k}{2}\right]}(-1)^h\left\{\zeta_E\left(s,\frac{h}{k}\right)-\zeta_E\left(s,1-\frac{h}{k}\right)\right\}=\sum_{\substack{h=1\\(h,k)=1}}^{k}(-1)^h\zeta_E\left(s,\frac{h}{k}\right). \tag{32}$$

Now subtracting (28) from (29), from (30) and (32), we have

$$\sum_{\substack{h=1\\(h,k)=1}}^{k}(-1)^h\zeta_E\left(s,\frac{h}{k}\right)=k^sH(-s,k)-2\sum_{r=0}^{\infty}\binom{-s}{2r}\zeta_E(s+2r)k^{-2r}H(2r,k). \tag{33}$$

By the definition of the Möbius functions, for $n\in\mathbb{N}$, we have

$$\sum_{d|n}\mu(d)=\left[\frac{1}{n}\right]=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n>1\end{cases}$$

(see [2, p. 25, Theorem 2.1]). Recalling from Part 2 of Lemma 3.1 that

$$\zeta_E(s,ka)=k^{-s}\sum_{r=0}^{k-1}(-1)^r\zeta_E\left(s,a+\frac{r}{k}\right), \tag{34}$$

and letting $a=1/k$ in (34), we obtain

$$\begin{aligned}\zeta_E(s)&=k^{-s}\sum_{r=0}^{k-1}(-1)^r\zeta_E\left(s,\frac{r+1}{k}\right)\\ &=k^{-s}\sum_{r=1}^{k}(-1)^{r-1}\zeta_E\left(s,\frac{r}{k}\right),\end{aligned} \tag{35}$$

where k is odd. Hence the left-hand side of (33) may be rewritten as

$$\begin{aligned}
\sum_{\substack{h=1\\(h,k)=1}}^{k}(-1)^h\zeta_E\left(s,\frac{h}{k}\right) &= \sum_{h=1}^{k}(-1)^h\sum_{d|(h,k)}\mu(d)\zeta_E\left(s,\frac{h}{k}\right)\\
&= \sum_{h=1}^{k}(-1)^h\sum_{d|h,d|k}\mu(d)\zeta_E\left(s,\frac{h}{k}\right)\\
&= \sum_{d|k}\mu(d)\sum_{m=1}^{k/d}(-1)^{md}\zeta_E\left(s,\frac{md}{k}\right)\\
&= \sum_{d|k}\mu(d)\sum_{m=1}^{k/d}(-1)^{m}\zeta_E\left(s,\frac{m}{k/d}\right)\\
&\qquad \text{(use replace } k/d \text{ by } k \text{ in (35))}\\
&= -k^s\zeta_E(s)\sum_{d|k}\mu(d)d^{-s},
\end{aligned} \tag{36}$$

since d is odd in the case k is odd. Thus, by combining (33) and (36), the proof of Theorem 1.4 is completed.

4. Some further identities

In the spirit of Euler, by working with the formal power series, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\zeta_E(-n)t^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k k^n\right)\frac{(-1)^n t^n}{n!}\\
&= \sum_{k=1}^{\infty}(-1)^k\left(\sum_{n=0}^{\infty}\frac{(-kt)^n}{n!}\right).
\end{aligned} \tag{37}$$

The last term of (37) converges to

$$-\frac{1}{e^t+1}. \tag{38}$$

Thus, directly from definition (4), (38) may be written

$$-\frac{1}{e^t+1} = -\frac{1}{2}\sum_{n=0}^{\infty}E_n(0)\frac{t^n}{n!} \tag{39}$$

Applying the reflection formula of Euler polynomials (see [21, 24.4.4]):

$$E_n(1-x) = (-1)^n E_n(x), \tag{40}$$

with $x=0$, by (37), (38), and (39), we obtain

$$\zeta_E(-n) = \frac{(-1)^n}{2}E_n(0) = \frac{1}{2}E_n(1) \tag{41}$$

for $n \in \mathbb{N}_0$, which imply $\zeta_E(-1) = 1/4, \zeta_E(-2) = 0, \zeta_E(-3) = -1/8, \ldots$ (see [10, p. 520, Corollary 3]). The following identity involving Euler polynomials

$$E_n(x) = 2x^n - \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} E_{n-r}(0) x^r \quad (n \in \mathbb{N}_0) \tag{42}$$

follows from the known formula (see [11, p. 41, (6)] and [21, 24.4.2])

$$E_n(x+1) + E_n(x) = 2x^n \quad (n \in \mathbb{N}_0), \tag{43}$$

in the case we replace $E_n(x+1)$ by $\sum_{r=0}^{n} \binom{n}{r} E_{n-r}(1) x^r$ in (43), then set $x = 0$, and replace n by $n - r$ in (40).

Putting $a = 1$ and $s = -n$ in Part 1 of Lemma 3.2, we obtain the result

$$\zeta_E(-n, x+1) = \sum_{r=0}^{n} \binom{n}{r} \zeta_E(r-n) x^r, \quad |x| < 1. \tag{44}$$

Next setting $a = x, s = -n$, and $h = 0$ in (14), we have

$$\zeta_E(-n, x+1) + \zeta_E(-n, x) = x^n. \tag{45}$$

Combining (44) and (45), we have

$$\zeta_E(-n, x) = x^n - \sum_{r=0}^{n} \binom{n}{r} \zeta_E(r-n) x^r,$$

and by (41) and (42), we have

$$\begin{aligned} \zeta_E(-n, x) &= \frac{1}{2} \left(2x^n - \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} E_{n-r}(0) x^r \right) \\ &= \frac{1}{2} E_n(x) \end{aligned} \tag{46}$$

for $n \in \mathbb{N}_0$ (see [10, p. 520, (3.20)], [16, p. 4, (1.22)], and [29, p. 41, (3.8)]).

For $a = 1$, Part 2 of Lemma 3.2 yields

$$\zeta_E(s, 2-x) = \sum_{r=0}^{\infty} (-1)^{r-1} \binom{-s}{r} \{\zeta_E(s+r) - 1\} x^r, \tag{47}$$

where $|x| < 2$ (cf. [26, p. 146, (18)]). Replacing the summation index r in (47) by $r+1$, and setting $x = 1$, we arrive immediately at an analogue form of (2.3) in [25]:

$$\sum_{r=1}^{\infty} (-1)^r \binom{-s}{r} \{\zeta_E(s+r) - 1\} + 2\zeta_E(s) = 1. \tag{48}$$

Letting $x=-1$ in (47) and using (14) with $a=1,2$ and $h=0$, that is, $\zeta_E(s,3)=\zeta_E(s)+1/2^s-1$, we find that

$$\zeta_E(s)=1-\frac{1}{2^{s+1}}-\frac{1}{2}\sum_{r=1}^{\infty}\binom{-s}{r}\left\{\zeta_E(s+r)-1\right\}, \tag{49}$$

which provides a companion of Landau's formula (see [18, p. 274, (3)] and [28, p. 33, (2.14.1)]). Setting $x=1/2$ in (47), and using (14) with $a=1/2$ and $h=0$, that is, $\zeta_E(s,3/2)+\zeta_E(s,1/2)=2^s$, we obtain a series representation for $\beta(s)$:

$$\begin{aligned}\beta(s)&=1+\sum_{r=0}^{\infty}\frac{(-1)^r}{2^{r+s}}\binom{-s}{r}\left\{\zeta_E(s+r)-1\right\}\\&=1+\sum_{r=0}^{\infty}\frac{1}{2^{r+s}}\binom{s+r-1}{r}\left\{\zeta_E(s+r)-1\right\},\end{aligned} \tag{50}$$

where $\beta(s)$ denotes the Dirichlet beta function defined by (see [1, p. 807, 23.2.21])

$$\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}.$$

The above series converges for all $\mathrm{Re}(s)>0$. Setting $s=2$ in (50), we deduce

$$\text{Catalan's constant } G=\beta(2)=1+\sum_{r=1}^{\infty}\frac{r}{2^{r+1}}\left\{\zeta_E(r+1)-1\right\}, \tag{51}$$

which is one of the basic constants whose irrationality and transcendence (though strongly suspected) remain unproven (cf. [26, p. 29, (16)]).

Acknowledgment

The author thanks the referees for their very careful reading of this manuscript and for their very insightful comments and suggestions.

References

[1] Abramowitz M, Stegun I. Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. New York, NY, USA: Dover, 1972.

[2] Apostol TM. Some series involving the Riemann zeta function. Proc Amer Math Soc 1954; 5: 239-243.

[3] Ayoub R. Euler and the zeta function. Amer Math Monthly 1974; 81: 1067-1086.

[4] Bayad A. Arithmetical properties of elliptic Bernoulli and Euler numbers. Int J Algebra 2010; 4: 353-372.

[5] Boole G. A Treatise on the Calculus of Finite Differences. New York, NY, USA: Dover, 1960.

[6] Borwein JM, Calkin NJ, Manna D. Euler-Boole summation revisited. Amer Math Monthly 2009; 116: 387-412.

[7] Carlitz L. Multiplication formulas for products of Bernoulli and Euler polynomials. Pacific J Math 1959; 9: 661-666.

[8] Can M, Dağli MC. Character analogue of the Boole summation formula with applications. Turk J Math 2017; 41: 1204-1223.

[9] Chang CH, Ha CW. A multiplication theorem for the Lerch zeta function and explicit representations of the Bernoulli and Euler polynomials. J Math Anal Appl 2006; 315: 758-767.

[10] Choi J, Srivastava HM. The multiple Hurwitz zeta function and the multiple Hurwitz-Euler eta function. Taiwanese J Math 2011; 15: 501-522.

[11] Erdélyi A, Magnus W, Oberhettinger F, Francesco GT. Higher Transcendental Functions, Vol. 1. New York, NY, USA: McGraw-Hill, 1953.

[12] Hu S, Kim MS. The $(S,\{2\})$-Iwasawa theory. J Number Theory 2016; 158: 73-89.

[13] Hu S, Kim D, Kim MS. On reciprocity formula of Apostol-Dedekind sum with quasi-periodic Euler functions. J Number Theory 2016; 162: 54-67.

[14] Kim MS, Hu S. On p-adic Hurwitz-type Euler zeta functions. J Number Theory 2012; 132: 2977-3015.

[15] Kim MS, Hu S. On p-adic Diamond-Euler Log Gamma functions. J Number Theory 2013; 133: 4233-4250.

[16] Kim T. Euler numbers and polynomials associated with zeta functions. Abstr Appl Anal 2008; 2008: Article ID 581582, 11 pages.

[17] Lang S. Undergraduate Analysis. New York, NY, USA: Springer-Verlag, 1997.

[18] Landau E. Handbuch der Lehre von der Verteilung der Primzahlen. New York, NY, USA: Chelsea Publishing Co., 1953.

[19] Li HL, Ma J, Zhang WP. On some Diophantine Fourier series. Acta Math Sinica (Engl. Ser.) 2010; 26: 1125-1132.

[20] Min J. Zeros and special values of Witten zeta functions and Witten L-functions. J Number Theory 2014; 134: 240-257.

[21] NIST. Digital Library of Mathematical Functions. Available at http://dlmf.nist.gov/.

[22] Nörlund NE. Vorlesungen uber Differenzenrechnung. Berlin, Germany: Springer-Verlag, 1924.

[23] Murty MR, Reece M. A simple derivation of $\zeta(1-K)=-B_K/K$. Funct Approx Comment Math 2000; 28: 141-154.

[24] Ramaswami V. Notes on Riemann's ζ-Function. J London Math Soc 1934; 9: 165-169.

[25] Srivastava HM. Sums of certain series of the Riemann zeta function. J Math Anal Appl 1988; 134: 129-140.

[26] Srivastava HM, Choi J. Series associated with the zeta and related functions. Dordrecht, Netherlands: Kluwer Academic Publishers, 2001.

[27] Sun ZW. Introduction to Bernoulli and Euler polynomials. A lecture given in Taiwan on June 6, 2002. http://math.nju.edu.cn/ zwsun/BerE.pdf.

[28] Titchmarsh EC. The theory of the Riemann zeta-function. New York, NY, USA: The Clarendon Press, Oxford University Press, 1986.

[29] Williams KS, Zhang NY. Special values of the Lerch zeta function and the evaluation of certain integrals. Proc Amer Math Soc 1993; 119: 35-49.