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On hypersurfaces with parallel Möbius form and constant para-Blaschke eigenvalues

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Abstract: In this paper, we classify umbilic-free hypersurfaces of the unit sphere that have constant para-Blaschke eigenvalues and possess parallel Möbius form. To achieve the classification, we first of all show that, under the condition of having constant para-Blaschke eigenvalues, an umbilic-free hypersurface of the unit sphere is of parallel Möbius form if and only if its Möbius form vanishes identically.

Key words: Blaschke tensor, Möbius form, Möbius second fundamental form, Möbius metric, parallel Möbius form

1. Introduction

In [22], Wang established the Möbius geometry of submanifolds in the unit sphere $\mathbb{S}^{n+p}$ of dimension $(n+p)$ (cf. also [1] for $p=1$). This includes associated with an n-dimensional umbilic-free submanifold $x: M^n \to \mathbb{S}^{n+p}$ the introduction of the Möbius metric g, the Möbius second fundamental form $\mathbf{B}$, the Möbius form $\mathbf{\Phi}$, and the Blaschke tensor $\mathbf{A}$ (for their definitions see Section 2 below).

Since the publication of [22], the study of Möbius geometry of submanifolds in $\mathbb{S}^{n+p}$ has made a lot of progress and many interesting results have been obtained. Among them, we have witnessed the study of both the so-called Möbius isoparametric hypersurfaces and the so-called Blaschke isoparametric hypersurfaces, where a hypersurface in $\mathbb{S}^{n+1}$ is called Möbius isoparametric if it satisfies two conditions that $\mathbf{\Phi}=0$ and all *the eigenvalues of* $\mathbf{B}$ *with respect to* g *(which are called Möbius principal curvatures)* are constant [12]. Similarly, a hypersurface in $\mathbb{S}^{n+1}$ is called Blaschke isoparametric if it satisfies two conditions that $\mathbf{\Phi}=0$ and all the eigenvalues of $\mathbf{A}$ with respect to g are constant [20].

After a series of partial results in [12] and later [3–6, 9, 10, 13], a complete classification of Möbius isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ was finished recently by Li et al. (cf. Theorem 1.1 in [15], together with [12]). Similarly, after many partial results in [7, 17–21], Li and Wang [16] also proved that a Blaschke isoparametric hypersurface in $\mathbb{S}^{n+1}$ with more than two distinct Blaschke eigenvalues is Möbius isoparametric. This, along with applications of the main result reported by Li et al. [15], Li and Zhang [19] and Liu et al. [21], finally completes the classification of Blaschke isoparametric hypersurfaces in $\mathbb{S}^{n+1}$.

Moreover, for the purpose of extending the interesting Möbius geometric characterization of hypersurfaces in space forms with constant mean curvature and constant scalar curvature, due to Li and Wang [13], one also

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considered the so-called *para-Blaschke tensor* (cf. [26]) defined by $\mathbf{D}^{(\lambda)} := \mathbf{A} + \lambda\mathbf{B}$ for a real number λ. After the results in [2] and [26], Li and Wang [16] proved that a hypersurface in $\mathbb{S}^{n+1}$ must be Möbius isoparametric provided that $\mathbf{\Phi} = 0$ and $\mathbf{D}^{(\lambda)}$ (for some $\lambda \in \mathbb{R}$) has more than two distinct constant eigenvalues. Together with results given by Li and Wang [13] and Zhong and Sun [26] and Theorem 1.1 given by Li et al. [15], Li and Wang's [16] above-mentioned result finally completes the classification of umbilic-free hypersurfaces with the conditions $\mathbf{\Phi} = 0$ and that $\mathbf{D}^{(\lambda)}$ has constant eigenvalues for some $\lambda \in \mathbb{R}$.

From the fact that the four Möbius invariants g, $\mathbf{B}$, $\mathbf{A}$, and $\mathbf{\Phi}$ are related by the complicated integrability conditions, in [8] the authors studied Möbius isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ by focusing on the relation between its two conditions. As a result they obtained the following

Theorem 1.1 ([8]) *Let $x : M^n \to \mathbb{S}^{n+1}$ be an umbilic-free hypersurface. Assume that $\mathbf{\Phi}$ is parallel, namely when denoting ∇ the Levi-Civita connection of the Möbius metric g we have $\nabla\mathbf{\Phi} = 0$, and that additionally it satisfies either*

(1) $n = 2$, or

(2) $n \geq 3$ and $\mathbf{B}$ has constant eigenvalues;

then we have $\mathbf{\Phi} = 0$.

Theorem 1.1 *implies that the two conditions of Möbius isoparametric hypersurfaces become equivalent to that of $\nabla\mathbf{\Phi} = 0$ and all the Möbius principal curvatures are constant.*

In this paper, instead of $\mathbf{B}$ we consider a natural counterpart of Theorem 1.1 on the Blaschke tensor $\mathbf{A}$, and even more general the para-Blaschke tensor $\mathbf{D}^{(\lambda)}$ for some real number λ. Exactly, we will prove the following result:

Theorem 1.2 *Let $x : M^n \to \mathbb{S}^{n+1}$ $(n \geq 3)$ be an umbilic-free hypersurface such that, for some $\lambda \in \mathbb{R}$, $\mathbf{D}^{(\lambda)}$ has constant eigenvalues. Then the Möbius form satisfies $\nabla\mathbf{\Phi} = 0$ if and only if $\mathbf{\Phi} = 0$.*

Remark 1.3 *Even though it looks similar, when compared with the proof of Theorem 1.1, that of Theorem 1.2 is more involved. In fact, only after finishing the complete classification of submanifolds in the unit sphere with parallel Möbius second fundamental form [11, 24] do we come to realize some key facts in our present proof of Theorem 1.2.*

Remark 1.4 *Related to Theorem 1.2 there have some other similar results. We recall that Zhang [25] showed that if $\nabla\mathbf{\Phi} = 0$ and $\mathbf{A} = \lambda g$ for some smooth function, then $\mathbf{\Phi} = 0$ and λ is a constant. Furthermore, Xia [23] showed that if $\nabla\mathbf{\Phi} = 0$ and $\mathbf{A} + \lambda\mathbf{B} = \mu g$ for some functions λ, μ, then $\mathbf{\Phi} = 0$ and thus the result given by Li and Wang in [13] can also be achieved under some weaker condition.*

Remark 1.5 *The above two theorems and related facts motivate us to raise the following problem: Try to construct an umbilic-free hypersurface $x : M^n \to \mathbb{S}^{n+1}$ for which $\nabla\mathbf{\Phi} = 0$ whereas $\mathbf{\Phi} \neq 0$.*

Finally, a combination of the results in [15, 16, 19, 21] would give the classification of Blaschke isoparametric hypersurface in $\mathbb{S}^{n+1}$. For the convenience of readers, as an immediate consequence of Theorem 1.2, we would state the following results.

Corollary 1.6 *Let $x : M^n \to \mathbb{S}^{n+1}$ be an umbilic-free hypersurface with parallel Möbius form. If for some $\lambda \in \mathbb{R}$ the para-Blaschke tensor $\mathbf{D}^{(\lambda)}$ has constant eigenvalues, then x is Möbius equivalent to an open part of one of the following hypersurfaces:*

(i) a hypersurface with constant mean curvature and constant scalar curvature in $\mathbb{S}^{n+1}$, or the image of σ of a hypersurface with constant mean curvature and constant scalar curvature in $\mathbb{R}^{n+1}$, or the image of τ of a hypersurface with constant mean curvature and constant scalar curvature in the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ of constant sectional curvature -1;

(ii) the image of σ of a cone over a hypersurface with constant mean curvature and constant scalar curvature in $\mathbb{S}^k \subset \mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{n+1}$ for some $k \leq n$;

(iii) the image of τ of a rotational hypersurface over a hypersurface with constant mean curvature and constant scalar curvature in $\mathbb{H}^{k+1} \hookrightarrow \mathbb{H}^{n+1}$ for some $k \leq n$.

Here the notations τ and σ and the construction of "cone" and "rotational hypersurface" are introduced later in Section 4.

We organize the paper as follows. In Section 2, we review the Möbius invariants and integrability conditions for hypersurfaces in $\mathbb{S}^{n+1}$. In Section 3, we prove Theorem 1.2. In Section 4, we complete the proof of Corollary 1.6.

2. Preliminaries

In this section, we recall some fundamental facts and formulas. For proofs and more details, we refer to Wang [22].

For an immersed umbilic-free hypersurface $x : M^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, let $\{e_i\}_{i=1}^n$ be a local orthonormal basis with respect to the induced metric $I = dx \cdot dx$ and $\{\theta_i\}_{i=1}^n$ its dual basis. Let $II = \sum_{i,j} h_{ij}\theta_i \otimes \theta_j$ be the second fundamental form of x, with the squared length $\|II\|^2 = \sum_{i,j}(h_{ij})^2$ and the mean curvature $H = \frac{1}{n}\sum_i h_{ii}$. The Möbius metric g of $x : M^n \to \mathbb{S}^{n+1}$ satisfies $g = \rho^2 dx \cdot dx$, where $\rho^2 = \frac{n}{n-1}(\|II\|^2 - nH^2)$. Let $E_i = \rho^{-1}e_i$, $\omega_i = \rho\theta_i$; then $\{E_1, \ldots, E_n\}$ is an orthonormal basis for (M^n, g) with dual basis $\{\omega_1, \ldots, \omega_n\}$. Let ω_{ij} be the connection 1-form of the Möbius metric g; it is defined by the structure equations $d\omega_i = \sum_j \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$.

For $x : M^n \to \mathbb{S}^{n+1}$, we define its Blaschke tensor, its Möbius form, and its Möbius second fundamental form by $\mathbf{A} = \sum_{i,j} A_{ij}\omega_i \otimes \omega_j$, $\mathbf{\Phi} = \sum_i C_i\omega_i$, and $\mathbf{B} = \sum_{i,j} B_{ij}\omega_i \otimes \omega_j$, respectively. The coefficients B_{ij}, A_{ij}, and C_i can be calculated by the associated Euclidean invariants of x as follows (cf. [22]):

$$B_{ij} = \rho^{-1}\big(h_{ij} - H\delta_{ij}\big), \tag{2.1}$$

$$\begin{aligned} A_{ij} = &-\rho^{-2}\Big\{\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}\Big\} \\ &- \tfrac{1}{2}\rho^{-2}\big(\|\nabla(\log\rho)\|^2 - 1 + H^2\big)\delta_{ij}, \end{aligned} \tag{2.2}$$

$$C_i = -\rho^{-2}\Big\{e_i(H) + \sum_j (h_{ij} - H\delta_{ij})e_j(\log\rho)\Big\}, \tag{2.3}$$

where Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to $dx \cdot dx$.

The components of the covariant differentiation of $\mathbf{\Phi}, \mathbf{A}$, and $\mathbf{B}$:

$$\nabla\mathbf{\Phi} = \sum_{i,j} C_{i,j}\omega_i\omega_j,\ \nabla\mathbf{A} = \sum_{i,j,k} A_{ij,k}\omega_i\omega_j\omega_k,\ \nabla\mathbf{B} = \sum_{i,j,k} B_{ij,k}\omega_i\omega_j\omega_k$$

are defined, respectively, by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2.4}$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2.5}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2.6}$$

The integrability conditions of the Möbius invariants are given by

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \tag{2.7}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - A_{ik}B_{kj}), \tag{2.8}$$

$$B_{ij,k} - B_{ik,j} = C_k\delta_{ij} - C_j\delta_{ik}, \tag{2.9}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{jk}\delta_{il} - A_{il}\delta_{jk}, \tag{2.10}$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j}(B_{ij})^2 = \tfrac{n-1}{n}, \tag{2.11}$$

where R_{ijkl} denotes the components of the curvature tensor of g.

The second covariant derivative of C_i is defined by

$$\sum_k C_{i,jk}\omega_k = dC_{i,j} + \sum_k C_{k,j}\omega_{ki} + \sum_k C_{i,k}\omega_{kj}. \tag{2.12}$$

From the exterior differentiation of (2.4), we have the following Ricci identity:

$$C_{i,jk} - C_{i,kj} = \sum_l C_l R_{lijk}. \tag{2.13}$$

3. Proof of Theorem 1.2

Let $x : M^n \to \mathbb{S}^{n+1}$ be an immersed umbilic-free hypersurface; we assume that $\nabla\mathbf{\Phi} = 0$ and, for $\lambda \in \mathbb{R}$, the para-Blaschke tensor $\mathbf{D}^{(\lambda)} := \mathbf{A} + \lambda\mathbf{B}$ has t distinct constant eigenvalues $D_1, D_2, \ldots, D_t$ of multiplicities $m_1, m_2, \ldots, m_t$, respectively. Then around each point we can choose an orthonormal frame field $\{E_i\}$, with $\{\omega_i\}$ its dual, such that $\mathbf{\Phi} = \sum_i C_i\omega_i$ and moreover $\mathbf{D}^{(\lambda)}$ is diagonalized:

$$D^{(\lambda)}_{ij} = A_{ij} + \lambda B_{ij} = d_i\delta_{ij}. \tag{3.1}$$

Without loss of generality, we assume that the matrix (D_{ij}) takes the form

$$(D_{ij}^{(\lambda)}) = \operatorname{diag}\,(\underbrace{D_1,\ldots,D_1}_{m_1},\underbrace{D_2,\ldots,D_2}_{m_2},\ldots,\underbrace{D_t,\ldots,D_t}_{m_t}). \tag{3.2}$$

In the sequel we denote $[i]=\{j\,|\,d_j=d_i\}$ and $\mathcal{I}_s=\{i\,|\,d_i=D_s\}$, $1\le s\le t$.

Lemma 3.1 *The orthonormal frame field $\{E_i\}$ can be chosen such that, in addition to (3.2), the components of $\mathbf{\Phi}$ take the form*

$$(C_1,C_2,\ldots,C_n)=(\underbrace{C_{\bar 1},0,\ldots,0}_{m_1},\underbrace{C_{\bar 2},0,\ldots,0}_{m_2},\cdots,\underbrace{C_{\bar t},0,\ldots,0}_{m_t}) \tag{3.3}$$

Proof According to (3.2) we consider each eigenspace of $\mathbf{D}^{(\lambda)}$ corresponding to its eigenvalue D_s, and aim at finding its new orthonormal basis such that (3.3) holds.

For $s=1$, we make the following orthogonal transformation

$$(\bar E_1,\ldots,\bar E_{m_1})=(E_1,\ldots,E_{m_1})\mathbf{T}_1,\quad \mathbf{T}_1\in \mathrm{SO}(m_1),$$

where if $(C_1,\ldots,C_{m_1})=0$ we take $\mathbf{T}_1=\mathrm{id}$, whereas if $(C_1,\ldots,C_{m_1})\neq 0$ we take $\mathbf{T}_1$ such that

$$\bar E_1=\frac{C_1E_1+C_2E_2+\cdots+C_{m_1}E_{m_1}}{\sqrt{C_1^2+C_2^2+\cdots+C_{m_1}^2}}.$$

Similarly, for $s=2$, we make the following orthogonal transformation

$$(\bar E_{m_1+1},\ldots,\bar E_{m_1+m_2})=(E_{m_1+1},\ldots,E_{m_1+m_2})\mathbf{T}_2,\quad \mathbf{T}_2\in \mathrm{SO}(m_2),$$

where if $(C_{m_1+1},\ldots,C_{m_1+m_2})=0$ we take $\mathbf{T}_2=\mathrm{id}$, whereas if

$$(C_{m_1+1},\ldots,C_{m_1+m_2})\neq 0$$

we take $\mathbf{T}_2$ such that

$$\bar E_{m_1+1}=\frac{C_{m_1+1}E_{m_1+1}+C_{m_1+2}E_{m_1+2}+\cdots+C_{m_1+m_2}E_{m_1+m_2}}{\sqrt{C_{m_1+1}^2+C_{m_1+2}^2+\cdots+C_{m_1+m_2}^2}}.$$

Repeating this procedure up to $s=t$, we will have an orthonormal frame field $\{\bar E_1,\ldots,\bar E_n\}$, defined by

$$(\bar E_1,\ldots,\bar E_n)=(E_1,\ldots,E_n)\begin{pmatrix}\mathbf{T}_1&&&\\&\mathbf{T}_2&&\\&&\ddots&\\&&&\mathbf{T}_t\end{pmatrix},\quad \mathbf{T}_s\in \mathrm{SO}(m_s).$$

Let $\{\bar\omega_i\}$ be the dual frame of $\{\bar E_i\}$ and we write $\mathbf{\Phi}=\sum_i \bar C_i\bar\omega_i$; then it is easily seen that with respect to $\{\bar E_i\}_{i=1}^n$ both (3.2) and (3.3) hold, e.g., if $\sum_{i=1}^{m_1} C_i\omega_i\neq 0$, then, by denoting $\bar E_j=b_{j1}E_1+\cdots+b_{jm_1}E_{m_1}$,

$2 \le j \le m_1$, we have

$$\begin{cases} \bar{C}_1 = \mathbf{\Phi}(\bar{E}_1) = \sum_{i=1}^{n} C_i \omega_i(\bar{E}_1) = \sqrt{C_1^2 + C_2^2 + \cdots + C_{m_1}^2} \neq 0, \\ \bar{C}_j = \mathbf{\Phi}(\bar{E}_j) = \sum_{i=1}^{n} C_i \omega_i(\bar{E}_j) = \sum_{i=1}^{m_1} C_i b_{ji} = 0, \quad 2 \le j \le m_1. \end{cases}$$

Hence we complete the proof of Lemma 3.1. □

From now on, we will take orthonormal frame fields $\{E_i\}_{i=1}^n$ such that both (3.2) and (3.3) hold.

Lemma 3.2 *Assume that* $\nabla \mathbf{\Phi} = 0$ *and that in* (3.1) *all* $\{d_i\}$ *are constants: then we have:*

$$B_{ij} = 0, \quad A_{ij} = 0, \quad \text{if } [i] \neq [j], \tag{3.4}$$

$$D_{ij,k} = 0, \quad \text{if } [i] = [j], \tag{3.5}$$

$$C_{\bar{s}} R_{\bar{s}ijk} = 0, \quad \text{if } [i] = [k] \neq [j],\ j \in \mathcal{I}_s, \tag{3.6}$$

where

$$\sum_k D_{ij,k}\omega_k := dD_{ij}^{(\lambda)} + \sum_k D_{ik}^{(\lambda)}\omega_{kj} + \sum_k D_{kj}^{(\lambda)}\omega_{ki}. \tag{3.7}$$

Proof Since $\nabla \mathbf{\Phi} = 0$, from (2.8) and (3.1), we get

$$0 = (d_j - d_i)B_{ij}, \quad 0 = (d_j - d_i)A_{ij}, \quad \forall\ i, j, \tag{3.8}$$

from which (3.4) immediately follows.

Substitute (3.1) into (3.7), and using the assumption that $d_i = \text{const.}$ we obtain

$$\sum_k D_{ij,k}\omega_k = (d_i - d_j)\omega_{ij}, \quad \forall\ i, j, \tag{3.9}$$

which implies (3.5).

By using (2.10), (3.1), and (3.4), we have

$$R_{lijk} = 0, \quad \text{if } [i] = [k] \neq [j] \text{ and } [l] \neq [j]. \tag{3.10}$$

On the other hand, the condition $\nabla \mathbf{\Phi} = 0$ implies that $C_{i,jk} = 0$. Then, by (2.13) and (3.10), we get

$$0 = \sum_{l \in [j]} C_l R_{lijk}, \quad \text{if } [i] = [k] \neq [j]. \tag{3.11}$$

Combining (3.11) with (3.3), we immediately get (3.6). □

Lemma 3.3 *Assume that* $\nabla \mathbf{\Phi} = 0$ *and* $\mathbf{\Phi} \neq 0$*; then in* (3.3) *at least two elements of* $\{C_{\bar{1}}, C_{\bar{2}}, \ldots, C_{\bar{t}}\}$ *are nonzero.*

Proof Since $\mathbf{\Phi} \neq 0$, without loss of generality, we assume that $C_{\bar{1}} \neq 0$.

Suppose on the contrary that if $C_{\bar{q}} = 0$ for all $2 \leq q \leq t$, then, by (2.4), we have

$$0 = \sum_j C_{i,j}\omega_j = C_{\bar{1}}\omega_{\bar{1}i}, \quad i \neq \bar{1}.$$

Thus, we obtain $\omega_{\bar{1}i} = 0$ for all i. Then, from (3.9), we obtain that

$$D_{i\bar{1},k} = 0, \quad \forall\, i, k.$$

This combining with (3.1), (3.5), (2.7), and (2.9) gives that

$$0 = D_{i\bar{1},k} - D_{ik,\bar{1}} = B_{ik}C_{\bar{1}} - B_{i\bar{1}}C_k + \lambda(\delta_{i\bar{1}}C_k - \delta_{ik}C_{\bar{1}}), \quad \text{if } [i] = [k]. \tag{3.12}$$

It follows that

$$C_{\bar{1}}(B_{ik} - \lambda\delta_{ik}) = C_k(B_{i\bar{1}} - \lambda\delta_{i\bar{1}}), \quad \text{if } [i] = [k]. \tag{3.13}$$

From (3.13) and (3.4), we obtain

$$B_{ik} = \lambda\delta_{ik}, \quad \text{if } (i,k) \neq (\bar{1},\bar{1}). \tag{3.14}$$

Hence, by (2.11), we have

$$B_{\bar{1}\bar{1}} + (n-1)\lambda = 0, \quad (B_{\bar{1}\bar{1}})^2 + (n-1)\lambda^2 = \tfrac{n-1}{n}.$$

It follows that x has two distinct constant principal curvatures. By Theorem 1.1 we know that $\mathbf{\Phi} = 0$, which is a contradiction.

We complete the proof of Lemma 3.3. □

Using Lemma 3.3, we can further get the following result.

Lemma 3.4 *Assume that* $\nabla\mathbf{\Phi} = 0$ *and* $\mathbf{\Phi} \neq 0$*; then* $t \leq 3$.

Proof According to Lemma 3.3, we can assume that $C_{\bar{1}} \neq 0$ and $C_{\bar{2}} \neq 0$.

If $t \geq 4$, then (3.6) implies that

$$0 = R_{\bar{1}\bar{3}\bar{1}\bar{3}} = B_{\bar{1}\bar{1}}B_{\bar{3}\bar{3}} + D_1 + D_3 - \lambda(B_{\bar{1}\bar{1}} + B_{\bar{3}\bar{3}}), \tag{3.15}$$

$$0 = R_{\bar{1}\bar{4}\bar{1}\bar{4}} = B_{\bar{1}\bar{1}}B_{\bar{4}\bar{4}} + D_1 + D_4 - \lambda(B_{\bar{1}\bar{1}} + B_{\bar{4}\bar{4}}), \tag{3.16}$$

$$0 = R_{\bar{2}\bar{3}\bar{2}\bar{3}} = B_{\bar{2}\bar{2}}B_{\bar{3}\bar{3}} + D_2 + D_3 - \lambda(B_{\bar{2}\bar{2}} + B_{\bar{3}\bar{3}}), \tag{3.17}$$

$$0 = R_{\bar{2}\bar{4}\bar{2}\bar{4}} = B_{\bar{2}\bar{2}}B_{\bar{4}\bar{4}} + D_2 + D_4 - \lambda(B_{\bar{2}\bar{2}} + B_{\bar{4}\bar{4}}). \tag{3.18}$$

From (3.15) and (3.17), we obtain

$$(B_{\bar{3}\bar{3}} - \lambda)(B_{\bar{1}\bar{1}} - B_{\bar{2}\bar{2}}) + D_1 - D_2 = 0. \tag{3.19}$$

Similarly, from (3.15), (3.16), we get

$$(B_{\bar{1}\bar{1}} - \lambda)(B_{\bar{3}\bar{3}} - B_{\bar{4}\bar{4}}) + D_3 - D_4 = 0. \tag{3.20}$$

Since $D_1 \neq D_2$ and $D_3 \neq D_4$, (3.19) and (3.20) show that

$$(B_{\bar{1}\bar{1}} - B_{\bar{2}\bar{2}})(B_{\bar{3}\bar{3}} - B_{\bar{4}\bar{4}}) \neq 0. \tag{3.21}$$

On the other hand, (3.15)+(3.18)-(3.16)-(3.17) immediately gives that

$$0=(B_{\bar{1}\bar{1}}-B_{\bar{2}\bar{2}})(B_{\bar{3}\bar{3}}-B_{\bar{4}\bar{4}}),$$

which is a contradiction to (3.21). This proves Lemma 3.4. □

Proof of Theorem 1.2 It is sufficient to prove the nontrivial part, i.e. if we assume that $\nabla\mathbf{\Phi}=0$ and for some $\lambda\in\mathbb{R}$ the para-Blaschke tensor $\mathbf{D}^{(\lambda)}$ of x has constant eigenvalues, then it must be that $\mathbf{\Phi}=0$.

First of all, according to Lemma 3.3, the assertion holds if $t=1$. Alternatively, another proof of this case can be found in [23].

Next, suppose on the contrary that $\mathbf{\Phi}\neq 0$; making use of Lemma 3.4, we will derive a contradiction by dividing the remaining discussions into two independent cases:

(i) $t=2,\quad n\geq 3$;

(ii) $t=3,\quad n\geq 3$.

Case (i). $t=2,\quad n\geq 3$. We assume that $C_{\bar{1}}\neq 0$ and $C_{\bar{2}}\neq 0$.

From (3.5), (3.1) and (2.7), (2.9), we have

$$\begin{cases} 0=D_{a\bar{1},b}-D_{ab,\bar{1}}=B_{ab}C_{\bar{1}}-B_{a\bar{1}}C_b+\lambda(\delta_{a\bar{1}}C_b-\delta_{ab}C_{\bar{1}}), & \text{if } a,b\in\mathcal{I}_1,\\ 0=D_{p\bar{2},q}-D_{pq,\bar{2}}=B_{pq}C_{\bar{2}}-B_{p\bar{2}}C_q+\lambda(\delta_{p\bar{2}}C_q-\delta_{pq}C_{\bar{2}}), & \text{if } p,q\in\mathcal{I}_2.\end{cases} \tag{3.22}$$

The above equations and (3.3) imply that

$$B_{ab}=\lambda\delta_{ab},\quad \text{if } a,b\in\mathcal{I}_1 \text{ and } (a,b)\neq(\bar{1},\bar{1}), \tag{3.23}$$

$$B_{pq}=\lambda\delta_{pq},\quad \text{if } p,q\in\mathcal{I}_2 \text{ and } (p,q)\neq(\bar{2},\bar{2}). \tag{3.24}$$

Hence, by (3.23), (3.24), and (2.11), we get

$$\begin{cases}\sum_i B_{ii}=B_{\bar{1}\bar{1}}+B_{\bar{2}\bar{2}}+(n-2)\lambda=0,\\ \sum_{i,j}(B_{ij})^2=(B_{\bar{1}\bar{1}})^2+(B_{\bar{2}\bar{2}})^2+(n-2)\lambda^2=\frac{n-1}{n}.\end{cases} \tag{3.25}$$

Since $\lambda=\text{const.}$, by (3.25) we know that

$$B_{\bar{1}\bar{1}}=\text{const.},\quad B_{\bar{2}\bar{2}}=\text{const.} \tag{3.26}$$

Hence all Möbius principal curvatures are constant. By Theorem 1.1 we obtain $\mathbf{\Phi}=0$, which is a contradiction.

This completes the proof of **Case (i)**.

Case (ii). $t=3,\ n\geq 3$.

In this case, we assume that the para-Blaschke tensor $\mathbf{D}^{(\lambda)}$ has three distinct constant eigenvalues D_1,D_2,D_3 of multiplicities m_1,m_2,m_3, respectively.

If $n = 3$, according to Lemma 3.3, without loss of generality, we can assume that $C_1 \neq 0$ and $C_2 \neq 0$; then, by (3.6), (2.10), and (3.1), we get

$$0 = R_{1212} = B_{11}B_{22} + D_1 + D_2 - \lambda(B_{11} + B_{22}), \tag{3.27}$$

$$0 = R_{1313} = B_{11}B_{33} + D_1 + D_3 - \lambda(B_{11} + B_{33}), \tag{3.28}$$

$$0 = R_{2323} = B_{22}B_{33} + D_2 + D_3 - \lambda(B_{22} + B_{33}). \tag{3.29}$$

Then the summation (3.27)+(3.28) gives that

$$B_{11}(B_{22} + B_{33}) + 2D_1 + D_2 + D_3 - \lambda(2B_{11} + B_{22} + B_{22}) = 0. \tag{3.30}$$

This, together with the fact $B_{11} + B_{22} + B_{33} = 0$, implies that

$$(B_{11})^2 + \lambda B_{11} - (2D_1 + D_2 + D_3) = 0.$$

Hence $B_{11} = \text{const.}$ and thus by (2.11) we see that all the Möbius principal curvatures are constant. By Theorem 1.1 we obtain the desired contradiction.

If $n \geq 4$, again we assume that $C_{\bar{1}} \neq 0$ and $C_{\bar{2}} \neq 0$. Then, by (3.6), we obtain

$$R_{\bar{1}i\bar{1}i} = R_{\bar{2}k\bar{2}k} = 0, \quad \forall\, i \in \mathcal{I}_2 \cup \mathcal{I}_3,\ \forall\, k \in \mathcal{I}_1 \cup \mathcal{I}_3.$$

It follows from (3.1) and (2.10) that

$$B_{\bar{2}\bar{2}}B_{kk} - \lambda(B_{\bar{2}\bar{2}} + B_{kk}) = -(D_1 + D_2), \quad \forall\, k \in \mathcal{I}_1, \tag{3.31}$$

$$B_{\bar{1}\bar{1}}B_{ii} - \lambda(B_{\bar{1}\bar{1}} + B_{ii}) = -(D_1 + D_2), \quad \forall\, i \in \mathcal{I}_2, \tag{3.32}$$

$$B_{\bar{1}\bar{1}}B_{jj} - \lambda(B_{\bar{1}\bar{1}} + B_{jj}) = -(D_1 + D_3), \quad \forall\, j \in \mathcal{I}_3, \tag{3.33}$$

$$B_{\bar{2}\bar{2}}B_{kk} - \lambda(B_{\bar{2}\bar{2}} + B_{kk}) = -(D_2 + D_3), \quad \forall\, k \in \mathcal{I}_3. \tag{3.34}$$

Now the subtraction (3.32)–(3.33) gives that

$$(B_{\bar{1}\bar{1}} - \lambda)(B_{ii} - B_{jj}) = D_3 - D_2 \neq 0, \quad i \in \mathcal{I}_2,\ j \in \mathcal{I}_3.$$

Analogously, from (3.32), (3.33), and (3.34), we can get

$$(B_{\bar{2}\bar{2}} - \lambda)(B_{kk} - B_{jj}) = D_3 - D_1 \neq 0, \quad k \in \mathcal{I}_1,\ j \in \mathcal{I}_3,$$

$$(B_{\bar{3}\bar{3}} - \lambda)(B_{\bar{2}\bar{2}} - B_{\bar{1}\bar{1}}) = D_1 - D_2 \neq 0.$$

The above equations imply that

$$B_{\bar{1}\bar{1}} \neq \lambda, \quad B_{\bar{2}\bar{2}} \neq \lambda, \quad B_{\bar{3}\bar{3}} \neq \lambda. \tag{3.35}$$

Again, using the equations (3.31)–(3.34), we get

$$(B_{\bar{1}\bar{1}} - \lambda)(B_{ii} - B_{jj}) = 0, \quad \text{if } [i] = [j] \neq [1], \tag{3.36}$$

$$(B_{\bar{2}\bar{2}} - \lambda)(B_{ii} - B_{jj}) = 0, \quad \text{if } [i] = [j] = [1]. \tag{3.37}$$

Hence, we have

$$B_{ii} = B_{jj} \neq \lambda, \quad \text{if } [i] = [j]. \tag{3.38}$$

Claim 1. $m_1 = m_2 = 1$ and $C_{\bar{3}} = 0$.

In fact, if $m_1 \geq 2$, similar to (3.22), from (3.5), (3.1) and (2.7), (2.9), we get

$$0 = D_{a\bar{1},a} - D_{aa,\bar{1}} = B_{aa}C_{\bar{1}} - B_{a\bar{1}}C_a + \lambda(\delta_{a\bar{1}}C_a - \delta_{aa}C_{\bar{1}}), \quad \text{if } a \in \mathcal{I}_1. \tag{3.39}$$

It follows that $B_{aa} = \lambda$ for all $a \in \mathcal{I}_1, a \neq \bar{1}$, a contradiction to (3.38).

Hence we have $m_1 = 1$. Similarly, we also have $m_2 = 1$.

Finally, if $C_{\bar{3}} \neq 0$, then we have $1 = m_3 = n - 2 \geq 2$, still a contradiction.

This verifies Claim 1.

From (3.6), (2.10) and (3.1), (3.4), we get

$$0 = R_{\bar{1}i\bar{1}k} = B_{\bar{1}\bar{1}}B_{ik} + A_{ik} = (B_{\bar{1}\bar{1}} - \lambda)B_{ik}, \quad \text{if } i, k \in \mathcal{I}_3, \ i \neq k.$$

From the fact $B_{\bar{1}\bar{1}} \neq \lambda$, we have

$$B_{ik} = 0, \quad \text{if } i, k \in \mathcal{I}_3, \ i \neq k. \tag{3.40}$$

From the above discussion, we get

$$(B_{ij}) = \text{diag}(B_{11}, B_{22}, B_{\bar{3}\bar{3}}, \ldots, B_{\bar{3}\bar{3}}). \tag{3.41}$$

Using the fact $B_{11} + B_{22} + m_3 B_{\bar{3}\bar{3}} = 0$ we get

$$(B_{11})^2 + B_{22}B_{11} + m_3 B_{\bar{3}\bar{3}} B_{11} = 0.$$

This combining with (3.32) and (3.33) gives that

$$(B_{11})^2 + \lambda m_3 B_{11} - [(1 + m_3)D_1 + D_2 + m_3 D_3] = 0.$$

Thus $B_{11} = \text{const.}$ and therefore, by (2.11), all B_{ii} $(1 \leq i \leq n)$ are constant. By Theorem 1.1 we know that $\mathbf{\Phi} = 0$, again a contradiction.

This completes the proof of **Case (ii)**.

We have completed the proof of Theorem 1.2.

4. Proof of Corollary 1.6

Let $\mathbb{H}^{n+p}$ denote the $(n+p)$-dimensional hyperbolic space of constant sectional curvature -1, which can be defined by

$$\mathbb{H}^{n+p} = \{(y_0, y_1) \in \mathbb{R}^+ \times \mathbb{R}^{n+p} | -y_0^2 + y_1 \cdot y_1 = -1\},$$

where $\cdot$ denotes the canonical Euclidean inner product. Let $\mathbb{S}^{n+p}_+$ be the open hemisphere in $\mathbb{S}^{n+p}$ whose first coordinate is positive. Then we have two conformal diffeomorphisms $\sigma : \mathbb{R}^{n+p} \hookrightarrow \mathbb{S}^{n+p} \setminus \{(-1, 0)\}$ and $\tau : \mathbb{H}^{n+p} \hookrightarrow \mathbb{S}^{n+p}_+$ as follows:

$$\sigma(u) = \left(\tfrac{1-|u|^2}{1+|u|^2}, \tfrac{2u}{1+|u|^2}\right), \quad u \in \mathbb{R}^{n+p}, \tag{4.1}$$

$$\tau(y_0, y_1) = \left(\tfrac{1}{y_0}, \tfrac{y_1}{y_0}\right), \quad (y_0, y_1) \in \mathbb{H}^{n+p}. \tag{4.2}$$

By use of σ and τ, we can regard submanifolds in $\mathbb{R}^{n+p}$ and $\mathbb{H}^{n+p}$ as submanifolds in $\mathbb{S}^{n+p}$, respectively.

Definition 4.1 *([14]) Given an immersed r-dimensional submanifold $u : M^r \to \mathbb{S}^{r+p}$, for $n \geq r+1$, the cone $\mathfrak{C}$ over u in $\mathbb{R}^{n+p}$ is defined by*

$$\mathfrak{C} : M^r \times \mathbb{R}^+ \times \mathbb{R}^{n-r-1} \to \mathbb{R}^{r+p+1} \times \mathbb{R}^{n-r-1} := \mathbb{R}^{n+p}$$

with $\mathfrak{C}(q, t, v) = (tu(q), v)$, where $q \in M^r$, $t \in \mathbb{R}^+$ and $v \in \mathbb{R}^{n-r-1}$.

Definition 4.2 *([14]) Let $\mathbb{R}^{k+1}_+ = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} | x_{k+1} > 0\}$ be the upper half-space, and $u = (u_1, \ldots, u_{k+1}) : M^k \to \mathbb{R}^{k+1}_+$ be an immersed hypersurface. The rotational hypersurface over u in $\mathbb{R}^{n+1}$ is defined as*

$$f : M^k \times \mathbb{S}^{n-k} \to \mathbb{R}^{n+1},$$
$$f(q, v) = (u(q), v) = (u_1, \ldots, u_k, u_{k+1}v),$$

where $q \in M^k$ and $v \in \mathbb{S}^{n-k}$.

Proof of Corollary 1.6.

Assume that the umbilic-free hypersurface $x : M^n \to \mathbb{S}^{n+1}$ satisfies $\nabla\mathbf{\Phi} = 0$ and, for $\lambda \in \mathbb{R}$, the para-Blaschke tensor $\mathbf{D}^{(\lambda)}$ has constant eigenvalues.

First, from Theorem 1.2 we have $\mathbf{\Phi} = 0$.

Next, if $\mathbf{D}^{(\lambda)}$ has exactly one eigenvalue, then by [13], x is Möbius equivalent to one of the hypersurfaces as stated in (i).

If $\mathbf{D}^{(\lambda)}$ has two distinct eigenvalues, then, according to Zhong and Sun [26], x is Möbius equivalent to an isoparametric hypersurface with two principal curvatures in $\mathbb{S}^{n+1}$, or a hypersurface as indicated in Example 3.2 or Example 3.3 of [26]. It can be verified that the hypersurface in Example 3.2 there is Möbius equivalent to the cone in Definition 4.1, and the hypersurface in Example 3.2 there is Möbius equivalent to the rotational hypersurface in Definition 4.2. Thus, in this case, x is Möbius equivalent to one of the hypersurfaces as stated in (i), or (ii), or (iii). Here we would mention that the above Examples 3.2 and 3.3 in [26] were restated as hypersurfaces $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$ in Theorem 5.9 of [7], respectively.

Finally, if $\mathbf{D}^{(\lambda)}$ has more than two distinct eigenvalues, then, according to [16], x is Möbius isoparametric and, by the main theorem of [15], x is locally Möbius equivalent to either the image of σ of an isoparametric hypersurface in $\mathbb{S}^{n+1}$, or the cone over an isoparametric hypersurface in $\mathbb{S}^k \subset \mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{n+1}$ $(k \leq n)$. Hence, x is locally Möbius equivalent to one of the hypersurfaces as stated in (i) or (ii).

This completes the proof of Corollary 1.6.

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