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Vertex covers in graphs with loops

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Abstract: We investigate ideals of vertex covers for the edge ideals associated to considerable classes of connected graphs with loops and exhibit algebraic information about them, such as the existence of linear quotients, the computation of invariant values, and the Cohen–Macaulay property. These algebraic procedures are good instruments for evaluating situations of minimal node coverings in networks.

Key words: Graphs with loops, linear quotients, ideals of vertex covers, Cohen–Macaulay ideals

1. Introduction

This article is devoted to study canonical properties of monomial ideals arising from remarkable classes of graphs with loops using computational and algebraic methods, in order to give analytic models of a network and of its connectivity. More precisely, the ideals of vertex covers for the edge ideals related to certain so-called graphs $\mathcal{K}'$-type are examined and useful algebraic properties of them are considered.

Let $\mathcal{G}$ be any graph on vertex set $[n] = \{v_1, \ldots, v_n\}$, set of edges $E(\mathcal{G})$, set of loops $L(\mathcal{G})$. An algebraic object attached to $\mathcal{G}$ is the edge ideal $I(\mathcal{G})$, a monomial ideal of the polynomial ring in n variables $R = K[X_1, \ldots, X_n]$, K a field.

If $\mathcal{G}$ is a loopless graph, $I(\mathcal{G})$ is generated only by squarefree monomials X_iX_j such that $\{v_i, v_j\} \in E(\mathcal{G})$, $i \neq j$. However, if $\mathcal{G}$ has loops, among the generators of $I(\mathcal{G})$ there are nonsquarefree monomials X_i^2 such that $\{v_i, v_i\} \in L(\mathcal{G})$, for some $i = 1, \ldots, n$.

The *ideal of vertex covers* $I_c(\mathcal{G})$ of $I(\mathcal{G})$ represents the algebraic transposition of the concept of (minimal) vertex cover of a graph $\mathcal{G}$. It is defined to be the ideal of R generated by all monomials $X_{i_1} \cdots X_{i_r}$ such that $(X_{i_1}, \ldots, X_{i_r})$ is an associated minimal prime ideal of $I(\mathcal{G})$.

This terminology is intended to emphasize that a minimal prime "covers" or contains all the monomials in $I(\mathcal{G})$; sometimes, in particular situations, $I(\mathcal{G})$ reflects properties of $I_c(\mathcal{G})$, so that it is sufficient to study $I(\mathcal{G})$ for stating properties of $I_c(\mathcal{G})$.

In [8] a significative class of edge ideals was introduced associated to connected loopless graphs $\mathcal{H}$ on vertex set $[n] = \{v_1, \ldots, v_n\}$ formed by the union of a complete graph K_m, $m < n$, on vertex set $[m] = \{v_{\alpha_1}, \ldots, v_{\alpha_m}\}$, $1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_m = n$ and of star graphs with centers the vertices of K_m.

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Starting from $\mathcal{H}$ and adding loops to some nodes, it was also considered a larger class of connected graphs with loops $\mathcal{K}'$.

The ideals of vertex covers for the class of edge ideals associated to $\mathcal{H}$ and $\mathcal{K}'$ were described by using results about ideals of vertex covers for the edge ideals associated to the complete and the star graphs that make them up.

The explicit description of the generators of the ideals of vertex covers for the class of edge ideals associated to $\mathcal{K}'$ was obtained by generalizing a fundamental property about the structure of such ideals.

In the present paper we investigate the ideals $I_c(\mathcal{K}')$ in order to highlight some algebraic aspects of them.

In particular, we show in what cases the ideal of vertex covers $I_c(\mathcal{K}')$ admits linear quotients, and write them explicitly; we compute standard algebraic invariants of $I_c(\mathcal{K}')$ such as projective dimension, depth, Krull dimension, and Castelnuovo–Mumford regularity, and we establish suitable conditions such that $I_c(\mathcal{K}')$ is a Cohen–Macaulay ideal. For details, see [3, 5–7]. The study of such facts is devoted to find useful tools for improving actual critical situations for the connections in the fields of communications and transport, specifically about minimal node covering.

We treat an optimization problem for the widespread deployment of police patrols on the main crossroads of a big city, in order to control all the streets and residential areas by standard routes.

We will show what should be the minimum number of patrols needed to cover adequately the areas around the intersections of the center and suburbs.

To perform this, a model of an appropriate graph with loops is given.

2. Ideals of vertex covers for graphs $\mathcal{K}'$-type

Let us recall algebraic definitions and notations.

Let $R = K[X_1, \ldots, X_n]$ be a polynomial ring in n variables over an arbitrary field K such that $\deg X_i = 1$, for $i = 1, \ldots, n$.

For any monomial ideal I of R, let $G(I)$ denote the unique minimal set of monomial generators for I.

Definition 2.1 *A vertex cover of $I \subset R$ is a subset C of $\{X_1, \ldots, X_n\}$ such that each $u \in G(I)$ is divided by some $X_i \in C$. C is called minimal if no proper subset of C is a vertex cover of I.*

Let $h(I)$ denote the minimal cardinality of the vertex covers of I.

Definition 2.2 *$I \subset R$ is said to have linear quotients if there is an ordering $u_1, \ldots, u_t$ of the monomials belonging to $G(I)$ such that the colon ideal $(u_1, \ldots, u_{j-1}) : (u_j)$ is generated by a subset of $\{X_1, \ldots, X_n\}$, for $j = 2, \ldots, t$.*

Remark 2.1 *In [1], Conca and Herzog proved that an ideal generated in the same degree that has linear quotients admits a linear resolution.*

Let $I \subset R$ have linear quotients with respect to the ordering $u_1, \ldots, u_t$ of the monomials of $G(I)$. Let $q_j(I)$ be the number of variables required to generate the ideal $(u_1, \ldots, u_{j-1}) : (u_j)$.

Set $q(I) = \max_{2 \leqslant j \leqslant t} q_j(I)$.

Remark 2.2 *The integer $q(I)$ is independent of the choice of the ordering of the generators that gives linear quotients [4].*

Let us now give information on graphs and their ideals.

Let $\mathcal{G}$ be a connected graph on vertex set $[n] = \{v_1, \ldots, v_n\}$. We indicate $E(\mathcal{G}) = \big\{\{v_i, v_j\} \mid v_i \neq v_j$ vertices of $\mathcal{G}\big\}$ the set of edges of $\mathcal{G}$ and $L(\mathcal{G}) = \big\{\{v_i, v_i\} \mid v_i$ any vertex of $\mathcal{G}\big\}$ the set of loops of $\mathcal{G}$. Hence $\{v_i, v_j\}$ is an edge joining v_i to v_j and $\{v_i, v_i\}$ is a loop of the vertex v_i.

If $L(\mathcal{G}) = \emptyset$, $\mathcal{G}$ is said to be a *simple* or *loopless* graph.

A *complete* graph on vertex set $[m] = \{v_1, \ldots, v_m\}$, denoted by K_m, is a connected graph for which there exists an edge for all pairs $\{v_i, v_j\}$ of vertices of it. K'_m denotes a complete graph with loops on vertex set $[m]$.

A *star* graph on vertex set $[n] = \big\{\{v_i\}, \{v_1, \ldots, v_{i-1}, v_{i+1}, v_n\}\big\}$ with center v_i, denoted by $\text{star}_i(n)$, $i = 1, \ldots, n$, is a complete bipartite graph of the form $K_{1,n-1}$. $\text{star}'_i(n)$ denotes a star graph with loops on vertex set $[n]$ of center v_i.

If $R = K[X_1, \ldots, X_n]$ such that each variable X_i corresponds to the vertex v_i of $\mathcal{G}$, the *edge ideal* $I(\mathcal{G})$ of $\mathcal{G}$ is the monomial ideal $\big(\{X_iX_j \mid \{v_i, v_j\} \in E(\mathcal{G})\} \cup \{X_i^2 \mid \{v_i, v_i\} \in L(\mathcal{G})\}\big) \subset R$.

Definition 2.3 *A subset C of $[n] = \{v_1, \ldots, v_n\}$ is said a* minimal vertex cover *of $\mathcal{G}$ if:*

(i) every edge of $\mathcal{G}$ is incident with one vertex in C, and

(ii) no proper subset of C satisfies (i).

If C verifies condition (i) only, C is called a vertex cover *of $\mathcal{G}$ and it is said to cover all the edges of $\mathcal{G}$.*

The smallest number of vertices in any minimal vertex cover of $\mathcal{G}$ is called the *vertex covering number*.

Remark 2.3 *There exists a one to one correspondence between minimal vertex covers of $\mathcal{G}$ and minimal prime ideals of $I(\mathcal{G})$. In fact, $\wp$ is a minimal prime ideal of $I(\mathcal{G})$ if and only if $\wp = (C)$, for some minimal vertex cover C of $\mathcal{G}$. Thus $I(\mathcal{G})$ has primary decomposition $(C_1) \cap \cdots \cap (C_r)$, where $C_1, \ldots, C_r$ are the minimal vertex covers of $\mathcal{G}$.*

Definition 2.4 *The ideal of vertex covers of the edge ideal $I(\mathcal{G})$, denoted by $I_c(\mathcal{G})$, is the ideal of R generated by all the monomials $X_{i_1} \cdots X_{i_r}$ such that $(X_{i_1}, \ldots, X_{i_r})$ is an associated prime ideal of $I(\mathcal{G})$.*

Note that if $\{v_i, v_i\} \in L(\mathcal{G})$ and v_i belongs to a vertex cover of the graph, $\{v_i, v_i\}$ can be thought to preserve the minimality.

The following generalizes the property for loopless graphs given in [10]

Property 2.1 $I_c(\mathcal{G}) = \Big(\bigcap\limits_{\substack{\{v_i, v_j\} \in E(\mathcal{G}) \\ i \neq j}} (X_i, X_j)\Big) \cap \big(X_k \mid \{v_k, v_k\} \in L(\mathcal{G}), k \neq i, j\big)$.

In [8] relevant wide classes of squarefree edge ideals associated to connected graphs were examined, in particular those associated to loopless graphs $\mathcal{H}$ on n vertices that consist of a union of a complete graph

K_m, $m < n$, and m star graphs with centers at the vertices of K_m; and, starting from these ones, those associated to graphs $\mathcal{K}$, obtained from $\mathcal{H}$ by adding loops on some vertices of K_m, called graphs $\mathcal{K}$-type.

From now on, we deal with the ideal of vertex covers for the class of connected graphs $\mathcal{K}'$ also introduced in [8], namely the graphs $\mathcal{K}'$-type.

More precisely, $\mathcal{K}'$ is the graph with loops on vertex set $[n] = \{v_1, \ldots, v_n\}$ union of: (i) a complete graph K_m, $m < n$, with vertices $v_{\alpha_1}, \ldots, v_{\alpha_m}$, $1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_m = n$; (ii) star graphs $\text{star}_{\alpha_i}(\alpha_i - \alpha_{i-1})$ with vertices $v_{\alpha_{i-1}+1}, \ldots, v_{\alpha_i}$, $\forall\, i = 1, \ldots, m$, the index α_0 means 0.

Note that the graphs $\mathcal{K}'$-type are larger than the graphs $\mathcal{K}$-type, because $\mathcal{K}'$ may have loops on vertices not belonging to K_m.

The edge ideal of $\mathcal{K}'$ is: $I(\mathcal{K}') = (X_{\alpha_1}X_{\alpha_2}, \ldots, X_{\alpha_1}X_{\alpha_m}, X_{\alpha_2}X_{\alpha_3}, \ldots, X_{\alpha_2}X_{\alpha_m}, \ldots, X_{\alpha_{m-1}}X_{\alpha_m}, X_1X_{\alpha_1}, X_2X_{\alpha_1}, \ldots, X_{\alpha_1-1}X_{\alpha_1}, X_{\alpha_1+1}X_{\alpha_2}, X_{\alpha_1+2}X_{\alpha_2}, \ldots, X_{\alpha_2-1}X_{\alpha_2}, \ldots, X_{\alpha_{m-1}+1}X_{\alpha_m}, \ldots, X_{\alpha_m-1}X_{\alpha_m}, X_{t_1}^2, \ldots, X_{t_l}^2) \subset R = K[X_1, \ldots, X_n],$. with $\{t_1, \ldots, t_l\} \subseteq \{1, \ldots, n\}$.

Remark 2.4 *$I(\mathcal{K}')$ has no linear quotients, because there is not an ordering of the monomials $f_1, \ldots, f_s$ of $G(I(\mathcal{K}'))$ such that $(f_1, \ldots, f_{j-1}) : (f_j)$ is generated by a subset of $\{X_1, \ldots, X_n\}$, for $2 \leqslant j \leqslant s$.*

However, there exists a subgraph $\mathcal{K}$ of $\mathcal{K}'$ whose loops lie only on the vertices belonging to K_m, such that $I(\mathcal{K})$ has linear quotients.

According to Remark 2.1, $I(\mathcal{K})$ has a linear resolution.

In general this is false for $I(\mathcal{K}')$.

The structure of the ideal of vertex covers for graphs $\mathcal{K}'$-type is obtained by using the description of the ideals of vertex covers for the complete graph with loops and the star graph with loops that make it up.

Lemma 2.1 *Let K'_m be a complete graph on vertex set $[m] = \{v_1, \ldots, v_m\}$ having loops on v_1, v_3, v_{m-1}. The ideal of vertex covers of $I(K'_m)$ is generated at most by $m-1$ monomials. In particular,*

(a) if there are loops in all the vertices, $I_c(K'_m) = (X_1X_2 \cdots X_m)$,

(b) if there are loops in $r < m$ vertices, $v_{t_1}, \ldots, v_{t_r}$, $\{t_1, \ldots, t_r\} \subseteq \{1, \ldots, m\}$,

$I_c(K'_m)$ has $m-r$ generators and it is

$\big(\{X_{\sigma_1} \cdots X_{\sigma_{m-1}} \,|\, \sigma_j = t_j,\ \forall\, j = 1, \ldots, r;\ \sigma_i \in \{1, \ldots, m\} \setminus \{t_1, \ldots, t_r\}, \forall\, i \neq j\}\big)$.

Lemma 2.2 *Let $star'_n(n)$ be a star graph on vertex set $[n] = \{v_1, \ldots, v_n\}$ having loops. The ideal of vertex covers of $I(star'_n(n))$ is generated at most by two monomials. In particular,*

(a) if the loops are in the vertices $v_1, \ldots, v_{n-1}$, $I_c(star'_n(n)) = (X_1 \cdots X_{n-1})$,

(b) if the loops are in v_3, v_{n-2}, $I_c(star'_n(n)) = (X_1 \cdots X_{n-1}, X_3X_{n-2}X_n)$,

(c) if there are loops in the center and in the vertices $v_{t_1}, \ldots, v_{t_s}$, $\{t_1, \ldots, t_s\} \subseteq \{1, \ldots, n{-}1\}$, $I_c(star'_n(n)) = (X_{t_1} \cdots X_{t_s}X_n)$.

The assertions of the preceding two lemmas descend from Property 2.1.

Theorem 2.1 *Let $\mathcal{K}'$ be as above and suppose that it has loops on some of its vertices, say $v_{\alpha_2}, v_{\alpha_4}, v_{\alpha_5}$, $v_{\alpha_{m-3}}$, $v_{\alpha_{m-1}}$, $v_{\alpha_{i-1}+j_1}, \dots, v_{\alpha_{i-1}+j_{p_i}}$, $i = 1, \dots, m$, where $\{j_1, \dots, j_{p_i}\} \subseteq \{1, \dots, \alpha_i - \alpha_{i-1} - 1\}$. The ideal of vertex covers $I_c(\mathcal{K}')$ has at most $m+1$ monomial generators $X_{j_1} \cdots X_{j_{p_1}} X_{\alpha_1} X_{\alpha_1+j_1} \cdots X_{\alpha_1+j_{p_2}} X_{\alpha_2} X_{\alpha_2+j_1} \cdots$ $X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, X_1 \cdots X_{\alpha_1-1} X_{\alpha_1+j_1} \cdots X_{\alpha_1+j_{p_2}} X_{\alpha_2} X_{\alpha_2+j_1} \cdots X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, X_{j_1} \cdots X_{j_{p_1}} X_{\alpha_1} X_{\alpha_1+1} \cdots X_{\alpha_2-1}$ $X_{\alpha_2} X_{\alpha_2+j_1} \cdots X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, \dots\dots\dots,$ $X_{j_1} \cdots X_{\alpha_{m-3}+j_{p_{m-2}}} X_{\alpha_{m-2}} X_{\alpha_{m-2}+1} \cdots X_{\alpha_{m-1}-1} X_{\alpha_{m-1}} X_{\alpha_{m-1}+j_1} \cdots$ $X_{\alpha_{m-1}+j_{p_m}} X_{\alpha_m}, X_{j_1} \cdots X_{\alpha_{m-2}+j_{p_{m-1}}} X_{\alpha_{m-1}} X_{\alpha_{m-1}+1} \cdots X_{\alpha_m-1}$.*

Proof See [8], Theorem 2.2. □

Significant particular cases for determining the structure of the ideal of vertex covers related to a graph $\mathcal{K}'$-type are (see [8]):

(a) some vertices of K_m are not centers of star graphs,

(b) the loops lie only on the vertices of K_m, and

(c) the loops lie only on the vertices not belonging to K_m.

Example 2.1 *Let $\mathcal{K}'$ be the connected graph on vertex set $\{v_1, \dots, v_{11}\}$ given by $K_3 \cup star_4(4) \cup star_8(4)$ $\cup$ $star_{11}(3) \cup \{v_2, v_2\} \cup \{v_5, v_5\} \cup \{v_7, v_7\} \cup \{v_{11}, v_{11}\}$. The ideal of vertex covers for $\mathcal{K}'$ is*

$$I_c(\mathcal{K}') = (X_2X_4X_5X_6X_7X_{11},\ X_2X_4X_5X_7X_8X_{11},\ X_1X_2X_3X_5X_7X_8X_{11}).$$

3. Linear quotients and properties for $I_c(\mathcal{K})'$

In this section we illustrate some algebraic aspects of $I_c(\mathcal{K}')$.

We prove that there exists an ordering on the number of variables of every monomial of $I_c(\mathcal{K}')$ for which this ideal admits linear quotients and write them explicitly; we also compute standard algebraic invariants of $I_c(\mathcal{K}')$ such as projective dimension, depth, Krull dimension, and Castelnuovo–Mumford regularity, and finally, because the ideals of vertex covers $I_c(\mathcal{H})$ are Cohen–Macaulay ([9]), we show that this property is preserved for the ideals of vertex covers $I_c(\mathcal{K}')$ provided that the graphs $\mathcal{K}'$-type have loops on all the vertices corresponding at least to a monomial generator of $I_c(\mathcal{H})$.

Let us examine the existence of linear quotients for the ideal of vertex covers $I_c(\mathcal{K}')$.

Theorem 3.1 *Let $\mathcal{K}'$ be as above, but at most $m-2$ vertices of K_m have loops. Then there exists an order $\leqslant$ on the number of variables in any monomial of $I_c(\mathcal{K}')$, natural order of indices for the same number of variables, such that $I_c(\mathcal{K}')$ has linear quotients.*

Proof Theorem 2.1 and its consequences give us the minimal set of monomial generators for the ideal of vertex covers $I_c(\mathcal{K}')$. Such generators are squarefree and may have different degrees. Put them in increasing order according to the degrees and, for the same degree, the leftmost nonzero difference of the indices is negative.

We have to distinguish two cases and for each of them will explicitly write the linear quotients of $I_c(\mathcal{K}')$.

Case I. – The vertices $v_{\alpha_1}, \dots, v_{\alpha_m}$ of K_m in $\mathcal{K}'$ are centers of star graphs with or without loops; otherwise there is a loop on any vertex of K_m that is not the center of a star graph.

▷ There are no loops on the vertices of K_m. The linear quotients are:

∘ $(X_{\alpha_1}),\dots,(X_{\alpha_m})$
if there are ends not covered by loops for each star graph;

∘ $(X_{\alpha_1}),\dots,(X_{\alpha_{m-1}})$
if a star graph with its center in v_{α_m} has loops in all its remaining vertices.

▷ There is a loop on the vertex v_{α_m} of K_m. The linear quotients are:

∘ $(X_{\alpha_1}),\dots,(X_{\alpha_{m-1}})$
if there are ends not covered by loops for each star graph whose center has no loop, or v_{α_m} is not the center of a star graph;

∘ $(X_{\alpha_1}),\dots,(X_{\alpha_{m-2}})$
if a star graph with center in $v_{\alpha_{m-1}}$ has loops in all its remaining vertices.

▷ There are loops on vertices $v_{\alpha_{m-1}}, v_{\alpha_m}$ of K_m. The linear quotients are:

∘ $(X_{\alpha_1}),\dots,(X_{\alpha_{m-2}})$
if there are ends not covered by loops for each star graph whose center has no loop, or $v_{\alpha_{m-1}}, v_{\alpha_m}$ are not centers of star graphs;

∘ $(X_{\alpha_1}),\dots,(X_{\alpha_{m-3}})$
if a star graph with its center in $v_{\alpha_{m-2}}$ has loops in all its remaining vertices.

....

▷ There are loops on vertices $v_{\alpha_3},\dots,v_{\alpha_m}$ of K_m. The linear quotients are:

∘ $(X_{\alpha_1}),(X_{\alpha_2})$
if there are ends not covered by loops for each star graph whose center has no loop, or $v_{\alpha_3},\dots,v_{\alpha_m}$ are not centers of star graphs;

∘ (X_{α_1})
if a star graph with its center in v_{α_2} has loops in all its remaining vertices.

Observe that if there are loops on $m-1$ or m vertices of K_m, $I_c(\mathcal{K}')$ has no linear quotients because it is generated by a unique monomial whose variables correspond to the vertices with loops of $\mathcal{K}'$.

Case II. $-$ The vertices without loops of K_m in $\mathcal{K}'$ are not all centers of star graphs with or without loops.

▷ There are no loops on the vertices of K_m. The linear quotients are:

∘ $(X_{\alpha_{i_1}}),\dots,(X_{\alpha_{i_{m-1}}})$
$\forall\, i_1 \neq \dots \neq i_{m-1} \in \{\alpha_1,\alpha_2,\dots,\alpha_m\}$.

▷ There is a loop on the vertex v_{α_m} that can be the center of a star graph with or without loops. The linear quotients are:

- $(X_{\alpha_{i_1}}),\dots,(X_{\alpha_{i_{m-2}}})$
  when none, some, or all the vertices $v_{\alpha_{i_1}},\dots,v_{\alpha_{i_{m-2}}}$ of K_m are centers of star graphs with or without loops, $\forall\, i_1 \neq \dots \neq i_{m-2} \in \{\alpha_1,\dots,\alpha_{m-1}\}$.

▷ There are loops on vertices $v_{\alpha_{m-1}}, v_{\alpha_m}$ and each one can be a center of a star graph with or without loops. The linear quotients are:

- $(X_{\alpha_{i_1}}),\dots,(X_{\alpha_{i_{m-3}}})$
  when none, some, or all the vertices $v_{\alpha_{i_1}},\dots,v_{\alpha_{i_{m-3}}}$ of K_m are centers of star graphs with or without loops, $\forall\, i_1 \neq \dots \neq i_{m-3} \in \{\alpha_1,\dots,\alpha_{m-2}\}$.

  ....

▷ There are loops on vertices $v_{\alpha_4},\dots,v_{\alpha_m}$ and each one can be a center of a star graph with or without loops. The linear quotients are:

- $(X_{\alpha_{i_1}}),(X_{\alpha_{i_2}})$
  when none, one, or both the vertices $v_{\alpha_{i_1}}, v_{\alpha_{i_2}}$ of K_m are centers of star graphs with or without loops, $\forall\, i_1 \neq i_2 \in \{\alpha_1,\alpha_2,\alpha_3\}$.

▷ There are loops on vertices $v_{\alpha_3},\dots,v_{\alpha_m}$ and each one can be a center of a star graph with or without loops. The linear quotients are:

- (X_{α_i}) when $v_{\alpha_i} \in K_m$ is or not the center of a star graph with or without loops, $\forall\, i \in \{\alpha_1,\alpha_2\}$.

As previously observed, if there are loops on at least $m-1$ vertices of K_m, $I_c(\mathcal{K}')$ has no linear quotients. □

Remark 3.1 *If $\mathcal{K}'$ satisfies the hypotheses of Theorem 3.1, $I_c(\mathcal{K}')$ has linear quotients. By Theorem 2.1, $I_c(\mathcal{K}')$ is not necessarily generated in the same degree. According to Remark 2.1, $I_c(\mathcal{K}')$ has not in general a linear resolution.*

The existence of linear quotients for a monomial ideal $I \subset R$ makes the computation of some algebraic invariants of R/I easier. In fact:

◇ the *projective dimension*, $\mathrm{pd}_R(R/I) = q(I) + 1$,

◇ the *depth*, $\mathrm{depth}_R(R/I) = \dim R - \mathrm{pd}_R(R/I)$,

where $\dim R = \dim_R(R/I) + h(I)$ is the *Krull dimension*,

◇ the *Castelnuovo–Mumford regularity*,

$\mathrm{reg}_R(R/I) = \max\{\deg f \mid f \text{ minimal generator of } I\} - 1$.

Lemma 3.1 *Let $R = K[X_1,\dots,X_n]$ and $I_c(\mathcal{K}') \subset R$. Then $h(I_c(\mathcal{K}')) = 1$.*

Proof Being $I_c(\mathcal{K}')$ an ideal of vertex covers associated to a graph with at least a loop v_i, the variable X_i is common in all its generators. Thus there exists a vertex cover of $I_c(\mathcal{K}')$ having the unique element X_i, namely a set of minimal cardinality among the vertex covers of $I(\mathcal{K}')$. □

Lemma 3.2 *Let $R = K[X_1,\dots,X_n]$ and $I_c(\mathcal{K}') \subset R$. Suppose that $I_c(\mathcal{K}')$ has linear quotients. Then $q(I_c(\mathcal{K}')) = 1$.*

Proof As a consequence of Theorem 3.1, $I_c(\mathcal{K}')$ has linear quotients if it is generated at least by two monomials. In this case, the number of variables for generating the ideal $(f_1, \ldots, f_{h-1}) : (f_h)$ is 1, for all $h = 1, \ldots, t$, $t \leqslant m+1$, m the number of vertices in K_m. □

Let us study algebraic invariants values for the ideal of vertex covers $I_c(\mathcal{K}')$.

Theorem 3.2 *Let $\mathcal{K}'$ be as above. Let σ denote the maximal number of vertices among the star graphs of $\mathcal{K}'$. Let $R = K[X_1, \ldots, X_n]$ and $I_c(\mathcal{K}') \subset R$ be the ideal of vertex covers related to $\mathcal{K}'$. If $I_c(\mathcal{K}')$ has linear quotients, then:*

1) $pd_R(R/I_c(\mathcal{K}')) = 2$.

2) $depth_R(R/I_c(\mathcal{K}')) = n-2$.

3) $dim_R(R/I_c(\mathcal{K}')) = n-1$.

4) $(m-1)+(\sigma-2) \leqslant reg_R(R/I_c(\mathcal{K}')) \leqslant n-2$.

Proof 1) The length of a minimal free resolution of $R/I_c(\mathcal{K}')$ over R is equal to $q(I_c(\mathcal{K}'))+1$ ([4], Corollary 1.6). Hence, by Lemma 3.2, $\mathrm{pd}_R(R/I_c(\mathcal{K}')) = 2$.
2) As a consequence of 1), by the Auslander–Buchsbaum formula, one has that
$\mathrm{depth}_R(R/I_c(\mathcal{K}')) = n - \mathrm{pd}_R(R/I_c(\mathcal{K}')) = n-2$.
3) It results in $\dim_R(R/I_c(\mathcal{K}')) = \dim_R R - h(I_c(\mathcal{K}'))$ [2]. Hence, by Lemma 3.1, $\dim_R(R/I_c(\mathcal{K}')) = n-1$.
4) By [1], Lemma 4.1, $\mathrm{reg}(I_c(\mathcal{K}'))$ is the maximum of the degrees for any minimal generator of $I_c(\mathcal{K}')$. By hypothesis, $I_c(\mathcal{K}')$ cannot be generated by a unique monomial and so any element of it has at most $n-1$ variables. Indeed, the lower bound is given by the maximal degree of the generators of the correspondent $I_c(\mathcal{H})$ ([8], Prop. 2.1, Thm. 2.1). □

If $I_c(\mathcal{K}')$ is generated by a unique monomial, it has no linear quotients. However, the following result holds:

Corollary 3.1 *Let $R = K[X_1, \ldots, X_n]$ and $I_c(\mathcal{K}') \subset R$. If $I_c(\mathcal{K}')$ is generated by a unique monomial, then:*

1) $pd_R(R/I_c(\mathcal{K}')) = 1$.

2) $depth_R(R/I_c(\mathcal{K}')) = n-1$.

3) $dim_R(R/I_c(\mathcal{K}')) = n-1$.

4) $reg_R(R/I_c(\mathcal{K}')) \leqslant n-1$.

Example 3.1 *Let $\mathcal{K}' = K_5 \cup star_3(3) \cup star_6(3) \cup star_8(2) \cup star_{12}(3) \cup \{v_5, v_5\} \cup \{v_8, v_8\} \cup \{v_{12}, v_{12}\}$ on vertex set $\{v_1, \ldots, v_{12}\}$.*

$I_c(\mathcal{K}') = (X_3X_5X_6X_8X_{12}, X_3X_4X_5X_8X_9X_{12}, X_1X_2X_5X_6X_8X_9X_{12})$.

$I_c(\mathcal{K}')$ has linear quotients $(X_3), (X_6)$, but it is not generated in the same degree and so it has no linear resolution. In fact:

$$0 \to R(-7) \oplus R(-8) \to R(-5) \oplus R(-6) \oplus R(-7) \to R/I_c(\mathcal{K}') \to R \to 0.$$

A computation for getting algebraic invariants gives:

1) $pd_R(R/I_c(\mathcal{K}')) = 2$.

2) $depth_R(R/I_c(\mathcal{K}')) = 10$.

3) $dim_R(R/I_c(\mathcal{K}')) = 11$.

4) $reg_R(R/I_c(\mathcal{H})) = 6$.

Let us examine the Cohen–Macaulay property for $I_c(\mathcal{K}')$.

Recall that an ideal $I \subset R$ is *Cohen–Macaulay* if and only if R/I is Cohen–Macaulay, i.e. $\text{depth}(R/I) = \dim(R/I)$.

Starting from some characteristics of the loopless graph $\mathcal{H}$, formed by the union of a complete graph K_m, $m < n$, together with star graphs with centers the vertices of K_m, and from the structure of the edge ideal $I(\mathcal{H})$ of $\mathcal{H}$, for the ideal of vertex covers $I_c(\mathcal{H})$ the following holds.

Proposition 3.1 *$I_c(\mathcal{H})$ is a Cohen–Macaulay ideal.*

Proof See [9], Proposition 2.4. □

The next result shows that such a property is preserved for the ideals $I_c(\mathcal{K}')$ related to the graphs $\mathcal{K}'$-type that have loops on all the vertices corresponding at least to a monomial generator of $I_c(\mathcal{H})$.

Theorem 3.3 *Let $I_c(\mathcal{H})$ be the ideal of vertex covers for the edge ideal associated to the graph $\mathcal{H}$ and $f_1, \dots, f_t$, $t \leqslant m-1$ be its monomial generators. Let us consider the graph $\mathcal{K}'$ obtained from $\mathcal{H}$ by inserting loops at least in the vertices that correspond to all the variables on any generator f_i of $I_c(\mathcal{H})$. Then the ideal of vertex covers $I_c(\mathcal{K}')$ is Cohen–Macaulay.*

Proof In [8] the structure of the generators of the ideal of vertex covers $I_c(\mathcal{H})$ was studied. When a loop $\{v_i, v_i\}$ is added to the graph $\mathcal{H}$, all the generators of the ideal of vertex covers for the edge ideal associated to the new graph $\mathcal{K}'$ will have the common variable X_i. If the loops added are relative at least to all the variables in any generator of $I_c(\mathcal{H})$, the ideal of vertex covers $I_c(\mathcal{K}')$ will be generated by a unique monomial. The assertion follows from Corollary 3.1, because $\text{depth}_R(R/I_c(\mathcal{K}')) = \dim_R(R/I_c(\mathcal{K}'))$. □

Example 3.2 *Let $\mathcal{H} = K_5 \cup star_3(3) \cup star_6(3) \cup star_8(2) \cup star_{12}(3)$ on vertex set $\{v_1, \dots, v_{12}\}$. $I_c(\mathcal{H}) = (X_1X_2X_6X_8X_9X_{12}, X_3X_4X_5X_8X_9X_{12},\ X_3X_6X_7X_9X_{12}, X_3X_6X_8X_9X_{10}X_{11}, X_3X_6X_8X_{12})$.*

Let $\mathcal{K}' = K_5 \cup star_3(3) \cup star_6(3) \cup star_8(2) \cup star_{12}(3) \cup \{v_2, v_2\} \cup \{v_3, v_3\} \cup \{v_4, v_4\} \cup \{v_5, v_5\} \cup \{v_8, v_8\} \cup \{v_9, v_9\} \cup \{v_{10}, v_{10}\} \cup \{v_{12}, v_{12}\}$ be the graph obtained from $\mathcal{H}$ by inserting loops on the vertices correspondent to the monomial generator $X_3X_4X_5X_8X_9X_{12}$ of $I_c(\mathcal{H})$ and in other ones.

It is $I_c(\mathcal{K}') = (X_2X_3X_4X_5X_8X_9X_{10}X_{12})$.

One has: $depth_R(R/I_c(\mathcal{K}')) = 11 = dim_R(R/I_c(\mathcal{K}'))$.

Then $I_c(\mathcal{K}')$ is Cohen–Macaulay.

The last theorem extends to any connected graph with loops, as the following formulates

Theorem 3.4 *Let $\mathcal{G}$ be a connected loopless graph on vertex set $[n] = \{v_1, \dots, v_n\}$ and edge ideal $I(\mathcal{G}) \subset R = K[X_1, \dots, X_n]$. Let $I_c(\mathcal{G})$ be the ideal of vertex covers for the edge ideal associated to $\mathcal{G}$ and suppose* *that $I_c(\mathcal{G})$ is Cohen–Macaulay. Let us consider the graph $\mathcal{G}'$ built from $\mathcal{G}$ by adding loops at least on the vertices that correspond to all the variables in any generator of $I_c(\mathcal{G})$. Then the ideal of vertex covers $I_c(\mathcal{G}')$ is Cohen–Macaulay.*

Proof A generalization of the proof of Theorem 3.3, bearing in mind that the hypotheses make $I_c(\mathcal{G}')$ a principal ideal. □

4. An optimization problem of police control

Let us evaluate an application for the widespread deployment of a minimum number of police patrols in crucial crossroads or squares of a big city, in order to control all the streets and residential areas by standard routes.

To perform this, we can think of modeling the city through a graph $\mathcal{K}'$-type. To be more precise:

- vertices represent crucial nodes, such as crossroads or squares;
- every edge can be seen as the route assigned to the patrol stationed in a vertex to reach an adjacent one;
- every loop can be intended as the route assigned to the patrol to return to the starting vertex without going through anyone else.

The least number of patrols needed to adequately cover the areas around the intersections of the center and suburbs is given by the monomials with the minimum number of variables that generate the ideal of vertex covers of the edge ideal related to $\mathcal{K}'$.

They just identify the crossroads or squares that have to be monitored. If there are different generators of the ideal of vertex covers with the smallest number of variables, there will be several solutions to the minimum problem.

Remark 4.1 *If the ideal of vertex covers for $\mathcal{K}'$ has only one generator, then it is Cohen–Macaulay. This can be interpreted thinking of the graphs $\mathcal{K}'$-type have a unique covering, namely one solution to the problem of minimal vertex covers.*

Example 4.1 *Let us suppose the map of any city can be modeled through the graph in the Figure.*

The minimum number of police patrols necessary for covering the main crossroads or squares, represented by the vertices of the graph, in order to control all the streets and residential areas by standard routes, outlined by edges and loops of it, is linked to the generators with the minimum number of variables of the ideal of vertex covers $I_c(\mathcal{K}')$.

According to Theorem 2.1, a computation yields the displayed monomial generators for it:
$X_1X_2X_3X_7X_9X_{10}X_{12}X_{14}X_{15}X_{16}X_{17}X_{19}X_{20}X_{23}X_{25}X_{26}X_{27}X_{28}X_{29}X_{30}X_{32}X_{33}$,
$X_2X_3X_4X_7X_9X_{10}X_{11}X_{12}X_{13}X_{15}X_{16}X_{17}X_{19}X_{20}X_{23}X_{25}X_{26}X_{27}X_{28}X_{29}X_{30}X_{32}X_{33}$,
$X_2X_3X_4X_7X_9X_{10}X_{12}X_{14}X_{15}X_{16}X_{17}X_{18}X_{20}X_{21}X_{22}X_{25}X_{26}X_{27}X_{28}X_{29}X_{30}X_{32}X_{33}$,
$X_2X_3X_4X_7X_9X_{10}X_{12}X_{14}X_{15}X_{16}X_{17}X_{18}X_{20}X_{23}X_{25}X_{26}X_{27}X_{28}X_{29}X_{30}X_{32}X_{33}$,
$X_2X_3X_4X_7X_9X_{10}X_{12}X_{14}X_{15}X_{16}X_{17}X_{19}X_{20}X_{21}X_{22}X_{25}X_{26}X_{27}X_{28}X_{29}X_{30}X_{32}X_{33}$,
$X_2X_3X_4X_7X_9X_{10}X_{12}X_{14}X_{15}X_{16}X_{17}X_{19}X_{20}X_{23}X_{25}X_{26}X_{27}X_{28}X_{29}X_{32}X_{33}$.
They correspond to minimal vertex covers of $\mathcal{K}'$. The only one with the smallest number of vertices is:
$C = \{2, 3, 4, 7, 9, 10, 12, 14, 15, 16, 17, 19, 20, 23, 25, 26, 27, 28, 29, 32, 33\}$.
Thus, the requested minimum number of patrols is the vertex covering number of $\mathcal{K}'$, namely twenty-one; they have to be set out on the nodes of C.

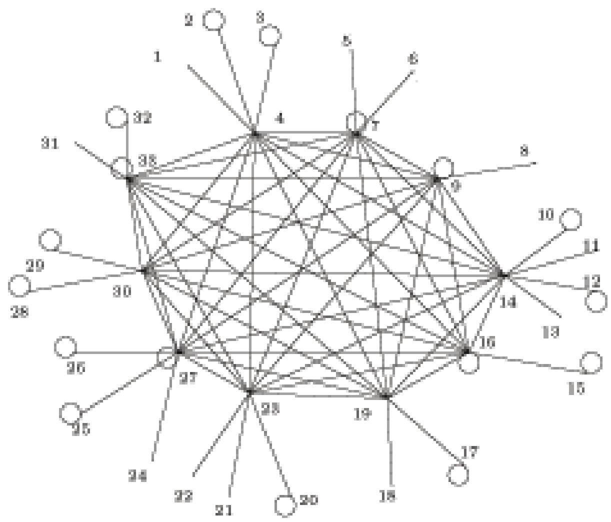

Figure 1. $\mathcal{K}'$-type graph

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