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Co-maximal signed graphs of commutative rings

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Abstract: Let $\Gamma(R)$ be a graph with element of R (finite commutative ring with unity) as vertices, where two vertices a and b are adjacent if and only if $Ra + Rb = R$. In this paper, we characterize the rings for which a co-maximal meet signed graph $\Gamma_\Sigma(R)$, a co-maximal join signed graph $\Gamma_\Sigma^\vee(R)$, a co-maximal ring sum signed graph $\Gamma_\Sigma^\oplus(R)$, their negation signed graphs $\eta(\Gamma_\Sigma(R))$, $\eta(\Gamma_\Sigma^\vee(R))$, $\eta(\Gamma_\Sigma^\oplus(R))$ respectively and their line signed graphs are balanced, clusterable, and sign-compatible.

Key words: Finite commutative ring, maximal ideal, co-maximal graph, balanced signed graph, co-maximal meet signed graph, co-maximal join signed graph, co-maximal ring sum signed graph

1. Introduction

Istvan Beck [5] introduced the concept of associating a graph with commutative rings. Since then, many researchers have worked in this field. Ashrafi et al.[2] defined the *unit graph* of a commutative ring (R) as the simple graph with vertex set R and two distinct vertices x and y are adjacent if their sum $x + y \in U(R)$, where $U(R)$ is the set of units of R. This graph is denoted by $G(R)$. This kind of work can also be seen in [17]. Let R be a commuatative ring with a nonzero unity and let $Z(R)$ be the set of all zero divisors in R. We recall from [7] that the total graph of R is the simple graph with vertex set R and two distinct vertices x and y are adjacent if their sum $x + y \in Z(R)$. This graph is denoted by $T(\mathit{\Gamma}(R))$. In 1995, Sharma and Bhatwadekar [15] introduced a graph $\Gamma(R)$ on a commutative ring R, whose vertices are elements of R and two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. Further properties of these graphs were established by Maimani et al. [13], and they named this graph the *co-maximal graph* of R, denoted by $\Gamma(R)$. Observe that $G(R)$ is an induced subgraph of the co-maximal graph. Note that if R is a finite ring, then $G(R)$ is the complement graph of $T(\mathit{\Gamma}(R))$ and hence the complement graph of $T(\mathit{\Gamma}(R))$ is an induced subgraph of the co-maximal graph.

Further, in [13], the authors worked on properties of subgraphs $\Gamma_1(R)$, $\Gamma_2(R)$, and $\Gamma_2(R) \setminus J(R)$, where $\Gamma_1(R)$ is the subgraph of $\Gamma(R)$ generated by the units of R, $\Gamma_2(R)$ is the subgraph of $\Gamma(R)$ generated by nonunit elements, and $\Gamma_2(R) \setminus J(R)$ is the subgraph of $\Gamma(R)$ induced on the set of nonunits of R that are not in $J(R)$, where $J(R)$ is the Jacobson radical of R, and also $J(R)$ is the largest 2-sided ideal of R such that $1 - a$ is a unit for all $a \in J(R)$. Let $\Gamma_1(R)$ be the subgraph of $\Gamma(R)$, generated by the units of R, and $\Gamma_2(R)$

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be the subgraph of $\Gamma(R)$, generated by nonunit elements of the ring R. The co-maximal graph $\Gamma(Z_6)$ is shown in Figure 1.

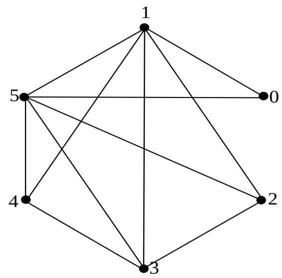

Figure 1. $\Gamma(Z_6)$

For preliminary notations and terminologies in abstract algebra we refer to standard textbooks [8, 9], and for graph theory we refer to [11, 21]. Unless mentioned otherwise, all rings considered in this paper are finite and commutative with unity $1 \neq 0$.

A subring A of a ring R is called a (two-sided) *ideal* of R if for every $r \in R$ and every $a \in A$ both ra and ar are in A. A proper ideal A of R is *maximal ideal* of R if there are no other ideals contained between A and R. An element $a \in R$ is *unit* of the ring R if a^{-1} exists, where $a^{-1} \in R$ is multiplicative inverse of a. A commutative ring is *quasi-local* if it has only finitely many maximal ideals.

In this paper, we denote $\Gamma_2(R) \setminus J(R)$ by $\Gamma_2^{'}(R)$ and $Max(R) = \{M_1, M_2, \ldots, M_n\}$ denotes the set of maximal ideals of R, where M_i is a maximal ideal of R. For a ring R, $U(R)$ denotes the set of units of R.

There are many exciting results proved on subgraphs of co-maximal graphs of rings in [3, 13–15, 19, 22], such as girth, diameter, and some structural properties of $\Gamma_2^{'}(R)$. Some elementary ones are listed below.

Theorem 1 *[13, 19] The following hold for co-maximal graph $\Gamma(R)$ of a commutative ring R:*

(a) Let $\Gamma_1(R)$ be a subgraph of $\Gamma(R)$ whose vertices are the units of R; then $\Gamma_1(R)$ is a complete graph.

(b) Let $\Gamma_2(R)$ be a subgraph of $\Gamma(R)$ whose vertices are the nonunit elements of R; then $a \in J(R)$ if and only if $deg_{\Gamma_2(R)} a = 0$.

(c) $\Gamma_2(R)$ is totally disconnected if and only if R is a local ring.

We extend the theory of the co-maximal graph in the realm of signed graphs. For preliminary notations and terminology for signed graphs, we refer to Zaslavsky [23–25]. A *signed graph* is an ordered pair $\Sigma = (\Sigma^u, \sigma)$, where $\Sigma^u = (V, E)$ is a graph, called the *underlying graph* of Σ and $\sigma : E \to \{+, -\}$ is a function from the edge set E of Σ^u into the set $\{+, -\}$ called the *signature* of Σ. Let $E^+(\Sigma) = \{e \in E(\Sigma^u) : \sigma(e) = +\}$ and $E^-(\Sigma) = \{e \in E(\Sigma^u) : \sigma(e) = -\}$. The elements of $E^+(\Sigma)$ and $E^-(\Sigma)$ are called *positive* and *negative* edges of Σ, respectively. A signed graph is said to be *homogeneous* if all its edges have the same sign and *heterogeneous* otherwise. The *negation* $\eta(\Sigma)$ of a signed graph Σ is a signed graph obtained from Σ by negating the sign of every edge of Σ.

One of the fundamental concepts in the theory of signed graphs is that of balance, clusterability, and $\mathcal{C}$-sign-compatibility. Harary [12] introduced the concept of balanced signed graphs for the analysis of social networks, in which a positive edge stands for a positive relation and a negative edge represents a negative relation. A signed graph is *balanced* if every cycle has an even number of negative edges, and a signed graph that is not balanced is called an unbalanced signed graph.

The following is the well-known result given by Harary in 1956.

Theorem 2 *[12] A signed graph Σ is balanced if and only if its vertex set $V(\Sigma)$ can be partitioned into two subsets V_1 and V_2 (one of them possibly empty) such that every negative edge of Σ joins a vertex of V_1 with one of V_2 while no positive edge does so.*

Now by a *positive section* (*negative section*) [10] in a signed graph Σ, we mean a maximal edge induced weakly connected subsigned graph consisting of only positive (negative) edges of Σ that turn out to be simply a path (semipath) if Σ is a cycle (semicycle). For a signed graph Σ, Behzad and Chartrand [6] defined its *line signed graph* $L(\Sigma)$ as the signed graph in which the edges of Σ are represented as vertices. Two of these vertices are defined to be adjacent whenever the corresponding edge in Σ has a vertex in common; any such edge ef is negative whenever both e and f are negative edges in Σ and positive otherwise.

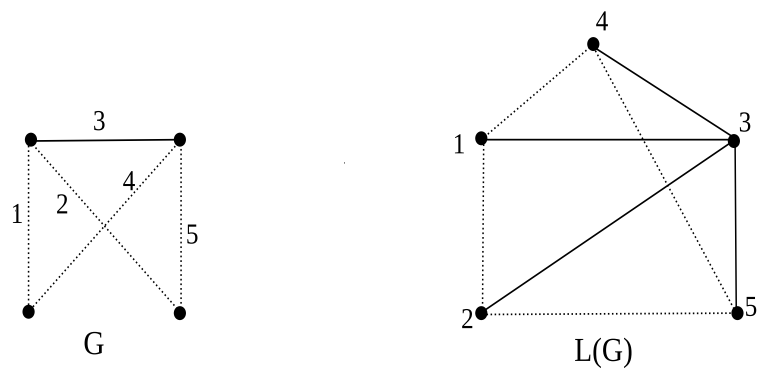

Figure 2. A signed graph G and its line signed graph $L(G)$.

We have the following result that gives the characterization of signed graphs for which their line signed graphs are balanced:

Theorem 3 *[1] For a signed graph Σ, $L(\Sigma)$ is balanced if and only if the following conditions hold:*

(i) for some cycle Z in Σ,

(a) if Z is all-negative, then Z has even length;

(b) if Z is heterogeneous, then Z has an even number of negative sections with even length;

(ii) for $v \in V(\Sigma)$, if $d(v) > 2$, then there is at most one negative edge incident at v in Σ.

A signed graph Σ is said to be *clusterable* if its vertex set can be partitioned into pairwise disjoint subsets called clusters, such that every negative edge joins vertices in different clusters and every positive edge join vertices in the same cluster. Davis in 1967 gave the characterization of clusterable signed graphs as precisely those in which no cycle has exactly one negative edge.

Theorem 4 *[7] A signed graph Σ is clusterable if and only if Σ contains no cycle with exactly one negative edge.*

A *marking* of a given signed graph Σ is a function $\mu : V(\Sigma) \rightarrow \{+,-\}$. A signed graph Σ is said to be *sign-compatible* if there exists a marking μ of its vertices such that the end vertices of every negative edge receive a '$-$' sign in μ, and no positive edge in Σ has both of its ends assigned a '$-$' sign in μ. Further, we establish the characterization of sign-compatible signed graphs.

Theorem 5 *[20] A signed graph Σ is sign-compatible if and only if its vertices can be partitioned into two subsets V_1 and V_2 (one of them possibly empty) such that the all-negative subsigned graph of Σ is precisely the subsigned graph induced by exactly one of the subsets V_1 or V_2.*

Theorem 6 *[20] A signed graph Σ is sign-compatible if and only if Σ does not contain a subsigned graph isomorphic to either of the two signed graphs, Σ_1 formed by taking the path $P_4 = (x,u,v,y)$ with both the edges xu and vy negative and the edge uv positive and Σ_2 formed by taking Σ_1 and identifying the vertices x and y.*

In a signed graph $\Sigma = (\Sigma^u, \sigma), \sigma$ induces a unique marking μ_σ defined by

$$\mu_\sigma(v) = \Pi_{e_j \in E_v} \sigma(e_j), v \in V(\Sigma),$$

which is called the *canonical marking* (or $\mathcal{C}$-marking in short) of Σ, where E_v is a set of edges e_j incident at v in Σ. A canonically marked signed graph Σ is said to be *canonically sign-compatible* (or *$\mathcal{C}$-sign-compatible* in short), if the end vertices of every negative edge receive '$-$' signs and no positive edge has both of its ends assigned '$-$' under μ_σ.

Theorem 7 *[16] A signed graph Σ is $\mathcal{C}$-sign-compatible if and only if the following conditions hold in Σ:*

(a) for every vertex $v \in V(\Sigma)$ either $d^-(v) = 0$ or $d^-(v) \equiv 1 \pmod 2$ and

(b) for every positive edge $e_k = v_i v_j$ in Σ, $d^-(v_i) = 0$ or $d^-(v_j) = 0$

2. Co-maximal meet signed graph

A co-maximal meet signed graph is defined as follows:

Definition 8 *A co-maximal meet signed graph is an ordered pair $\Gamma_\Sigma(R) = (\Gamma(R), \sigma)$, where $\Gamma(R)$ is the co-maximal graph of a commutative ring R and for an edge ab of $\Gamma_\Sigma(R)$, σ is defined as*

$$\sigma(ab) = \begin{cases} + & \textit{if } a \in U(R) \textit{ and } b \in U(R), \\ - & \textit{otherwise.} \end{cases}$$

The co-maximal meet signed graphs $\Gamma_\Sigma(Z_2 * Z_3)$ and $\Gamma_\Sigma(Z_2(x)/\langle x^2 \rangle)$ are shown in Figure 3, in which solid line segments are positive edges and dotted line segments are negative edges.

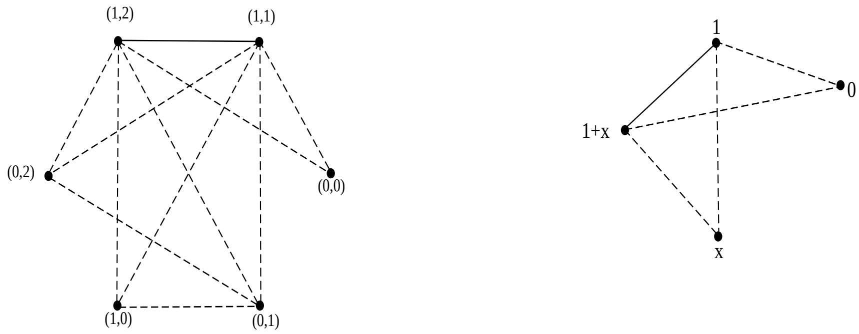

Figure 3. Showing the co-maximal meet signed graphs of $\Gamma_\Sigma(Z_2 * Z_3)$ and $\Gamma_\Sigma(Z_2(x)/\langle x^2\rangle)$.

2.1. Properties of co-maximal meet signed graph

In this section, we describe properties such as balance, clusterability, sign-compatibility, and $\mathcal{C}$-sign-compatibility of co-maximal meet signed graphs. Some of these results were presented at the "International Conference on Current Trends in Graph Theory and Computation(CTGTC-2016)" and were highly appreciated [18].

Theorem 9 *A co-maximal meet signed graph $\Gamma_\Sigma(R)$ is balanced if and only if R is a local ring.*

Proof Let R be a commutative ring with unity(say u). First, we assume that $\Gamma_\Sigma(R)$ is balanced. This implies there does not exist any negative cycle.

If R is not a local ring, then by Theorem 1, $\Gamma_2(R)\setminus J(R)$ is not totally disconnected. There are nonunits $a,b \in R$ such that $Ra+Rb=R$. Since in $\Gamma_\Sigma(R)$ there is a positive edge between two vertices if and only if both the vertices are units of R, then ab is a negative edge and a and b are connected to $u(u\in U(R))$ by a negative edge; therefore $auba$ is a negative cycle, a contradiction.

Now if R is a local ring, then it has only one maximal ideal and $\Gamma_2(R)\setminus J(R)$ is a totally disconnected graph. Therefore, in $\Gamma_\Sigma(R)$ there does not exist any edge between two nonunits. If there is some cycle present in $\Gamma_\Sigma(R)$, then it is of the form $u_l v_m u_n u_l$ (where u_i's are units and v_i's are nonunits), which always contains an even number of negative edges. Therefore, $\Gamma_\Sigma(R)$ is balanced. □

Theorem 10 *A co-maximal meet signed graph $\Gamma_\Sigma(R)$ is always clusterable.*

Proof From the construction of the co-maximal meet signed graph we know that there must be a positive edge between all pairs of units. Simultaneously there must be a negative edge between all pairs of vertices comprising one unit and one nonunit. Additionally there might be a negative edge between some two nonunits also. Now we can partition its vertices into pairwise disjoint subsets $V_1, V_2, V_3 \ldots$ such that all units can be put in one set, say V_1, and all nonunits in different sets $V_2, V_3, V_4 \ldots$. Hence $\Gamma_\Sigma(R)$ is clusterable. □

Theorem 11 *A co-maximal meet signed graph $\Gamma_\Sigma(R)$ is sign-compatible if and only if $|U(R)|=1$.*

Proof Necessity: Let the co-maximal meet signed graph $\Gamma_\Sigma(R)$ be sign-compatible. Let $|U(R)|\geq 2$, $u_1, u_2, 0 \in R$, where u_1, u_2 are units of the ring R and 0 is zero of ring R. From the definition of the co-maximal meet signed graph, we know that $u_1 0$ and $u_2 0$ are negative edges and so we assign negative signs to all three vertices u_1, u_2, and 0. Since $u_1 u_2$ is a positive edge, at least one of the vertices u_1 or u_2 must have a positive sign, which is a contradiction.

Sufficiency is trivial. □

Corollary 12 *If R is a local ring, then $\Gamma_\Sigma(R)$ is sign-compatible if and only if R is isomorphic to Z_2.*

Theorem 13 *If $|U(R)| \geq 2$, then $\Gamma_\Sigma(R)$ is not $\mathcal{C}$-sign-compatible.*

Proof Due to condition (b) of Theorem 7. □

Theorem 14 *If R is a local ring, then $\Gamma_\Sigma(R)$ is $\mathcal{C}$-sign-compatible if and only if R is isomorphic to Z_2.*

2.2. Negation of the co-maximal meet signed graph

In this section, we describe properties such as balance, clusterability, sign-compatibility, and $\mathcal{C}$-sign-compatibility of the negation of the co-maximal meet signed graph.

Theorem 15 *Negation of the co-maximal meet signed graph $\eta(\Gamma_\Sigma(R))$ is balanced if and only if the cardinality of the set of units of ring R is one.*

Proof Necessity: Let us assume that $\eta(\Gamma_\Sigma(R))$ is balanced. In $\eta(\Gamma_\Sigma(R))$ the signs of the edges of $\Gamma_\Sigma(R)$ are changed, that is, some two units are connected with a negative edge, between some two nonunits there is a positive edge, and also there is a positive edge between some two units and nonunits.

If $|U(R)| \geq 2$, then there are two unit elements u_1 and u_2 (say). There exists nonunit $0 \in R$ such that $Ru_1 + R0 = Ru_2 + R0 = Ru_1 + Ru_2 = R$. Thus we have a cycle $u_1 0 u_2 u_1$ with one negative edge between u_1 and u_2, which is a contradiction.

Sufficiency: Now $|U(R)| = 1$ implies that the graph is all-positive. Hence $\eta(\Gamma_\Sigma(R))$ is balanced. □

Theorem 16 *Negation of the co-maximal meet signed graph $\eta(\Gamma_\Sigma(R))$ is clusterable if and only if the cardinality of the set of units of ring R is one.*

Proof Let us assume that $|U(R)| = 1$. Then the negation of the co-maximal meet signed graph $\eta(\Gamma_\Sigma(R))$ is all-positive; therefore trivially is clusterable.

Now suppose $|U(R)| \geq 2$; then there exists at least one negative edge in $\eta(\Gamma_\Sigma(R))$ between the two units u_1, u_2 (say), and also every unit is joined by a positive edge to a nonunit. Let $e_1 \in R$ be some nonunit element. Therefore, there exist positive edges u_1e_1, u_2e_1. Now u_1, u_2 with some nonunit element will form a cycle with exactly one negative edge, contradicting the Davis criterion of clusterability. Hence if $|U(R)| \geq 2$, then $\eta(\Gamma_\Sigma(R))$ is not clusterable. □

Example 17 *If R is isomorphic to either Z_2 or Z_2^r, then negation of the co-maximal meet signed graph $\eta(\Gamma_\Sigma(R))$ is balanced as well as clusterable.*

Theorem 18 *The negation of the co-maximal meet signed graph $\eta(\Gamma_\Sigma(R))$ is always sign-compatible.*

Proof The proof is trivial by giving a negative sign to all units and a positive sign to all nonunits. □

Theorem 19 *The negation of the co-maximal meet signed graph $\eta(\Gamma_\Sigma(R))$ is $\mathcal{C}$-sign-compatible if and only if $|U(R)| = 1$ or even.*

Proof If $|U(R)| = 1$, then the negation of the co-maximal meet signed graph is an all-positive signed graph; therefore $\mathcal{C}$-sign-compatible. If $|U(R)|$ is even, then due to Theorem 7 the signed graph is $\mathcal{C}$-sign-compatible. Next suppose $\eta(\Gamma_\Sigma(R))$ is $\mathcal{C}$-sign-compatible, but on the contrary let $|U(R)|$ be odd. Then $d^-(u_i) \equiv 0 \pmod 2$, where $u_i \in U(R)$, a contradiction due to Theorem 7. □

Example 20 *$\Gamma_\Sigma(R)$ is $\mathcal{C}$-sign-compatible if R is isomorphic to one of the following rings, $Z_n, Z_n * Z_m, Z_2[x]/\langle x^n \rangle (n = 2, 3, \dots)$ or $F = \{0, 1, 2, \dots, p-1\}$ (p is a prime), $F[x]/\langle x^n \rangle (n = 2, 3, \dots)$.*

2.3. Line signed graph of co-maximal meet signed graph

Theorem 21 *The line signed graph of the co-maximal meet signed graph $L(\Gamma_\Sigma(R))$ is balanced if and only if $R \cong Z_2$.*

Proof Necessity: Let the line signed graph $L(\Gamma_\Sigma(R))$ be balanced and the cardinality of ring R be greater than two. Then it has either $|U(R)| \geq 2$ or $|R/U(R)| \geq 2$. Now from the construction of $\Gamma_\Sigma(R)$, if $|U(R)| \geq 2$, then there exists $0 \in R$, where 0 is zero of the ring R, such that $d(0) \geq 2$, precisely $d^-(0) \geq 2$. This is a contradiction due to Theorem 3. On the other hand, if $|R/U(R)| \geq 2$, then there exists $u_i \in R$, where u_i is a unit of ring R such that $d(u_i) \geq 2$, precisely $d^-(u_i) \geq 2$, again a contradiction due to Theorem 3.

Sufficiency: If $|R| \leq 2$, then the proof is trivial since $R \cong Z_2$. □

Theorem 22 *A line signed graph of the negation of the co-maximal meet signed graph $L(\eta(\Gamma_\Sigma(R)))$ is balanced if and only if $|U(R)| \leq 2$.*

Proof Necessity: Let $L(\eta(\Gamma_\Sigma(R)))$ be balanced. On the contrary let $|U(R)| \geq 3$; then from the construction of $\eta(\Gamma_\Sigma(R))$ there exists $u_i \in R$, where u_i is a unit of ring R such that $d^-(u_i) \geq 2$, a contradiction to Theorem 3. Hence $L(\eta(\Gamma_\Sigma(R)))$ is not balanced for $|U(R)| \geq 3$.

Sufficiency: Let $|U(R)| \leq 2$ then the negation of the co-maximal meet signed graph contains at most one negative edge. Hence, its line signed graph is all-positive. This implies that $L(\eta(\Gamma_\Sigma(R)))$ is balanced. □

Corollary 23 *If R is a field, then the line signed graph of the negation of the co-maximal meet signed graph $L(\eta(\Gamma_\Sigma(R)))$ is balanced if R is isomorphic to either Z_2 or Z_3.*

Example 24 *If R is isomorphic to Z_2^r or all polynomial rings over Z_2, then the line signed graph of the negation of the co-maximal meet signed graph $L(\eta(\Gamma_\Sigma(R)))$ is balanced.*

Theorem 25 *The line signed graph of co-maximal meet signed graph $L(\Gamma_\Sigma(R))$ is clusterable if and only if $|U(R)| = 1$.*

Proof Necessity: Let $L(\Gamma_\Sigma(R))$ be clusterable. On the contrary let $|U(R)| \geq 2$. Then there exists a cycle in $\Gamma_\Sigma(R)$ of the form $u_1 u_2 0 u_1$, where $u_1, u_2 \in R$ are units of the ring and $0 \in R$ is zero of the ring. Now from the construction of the co-maximal meet signed graph, $u_1 0, u_2 0$ are negative edges whereas $u_1 u_2$ is a positive edge. Clearly, in $L(\Gamma_\Sigma(R))$ there exists at least one cycle with exactly one negative edge, due to the presence of the above-mentioned cycle in $\Gamma_\Sigma(R)$, a contradiction to the Davis criterion stated in Theorem4. Hence, if $|U(R)| \geq 2$, then $L(\Gamma_\Sigma(R))$ is not clusterable.

Sufficiency: If $|U(R)| = 1$, then $\Gamma_\Sigma(R)$ is homogeneous with all-negative edges. Hence, $L(\Gamma_\Sigma(R))$ is also homogeneous with all-negative edges and therefore trivially clusterable. □

Theorem 26 *The line signed graph of the negation of the co-maximal meet signed graph $L(\eta(\Gamma_\Sigma(R)))$ is clusterable if and only if $|U(R)| \leq 2$.*

Proof For the necessity part, suppose that $L(\eta(\Gamma_\Sigma(R)))$ is clusterable. On the contrary let $|U(R)| \geq 3$; then from the construction of $\eta(\Gamma_\Sigma(R))$ we have $u_i \in R$, where u_i is a unit of the ring R such that $d^-(u_i) \geq 2$ and $d^+(u_i) \geq 1$. Let e_1, e_2 be two negative edges incident at u_i and let a_1 be a positive edge incident at u_i. Clearly, in $L(\eta(\Gamma_\Sigma(R)))$ there exists at least one triangle with exactly one negative edge, due to the presence of vertex u_i in $\eta(\Gamma_\Sigma(R))$. This is a contradiction to the Davis criterion of clusterability stated in Theorem 4.

Sufficiency is trivial since for $|U(R)| \leq 2$, $L(\eta(\Gamma_\Sigma(R)))$ is all-positive and hence clusterable. □

3. Co-maximal join signed graph

The definition of a co-maximal join signed graph is as follows:

Definition 27 *A co-maximal join signed graph is an ordered pair $\Gamma^\vee_\Sigma(R) = (\Gamma(R), \sigma)$, where $\Gamma(R)$ is the co-maximal graph of a commutative ring R and for an edge ab of $\Gamma^\vee_\Sigma(R)$, σ is defined as*

$$\sigma(ab) = \begin{cases} + & \text{if } a \in U(R) \text{ or } b \in U(R), \\ - & \text{otherwise.} \end{cases}$$

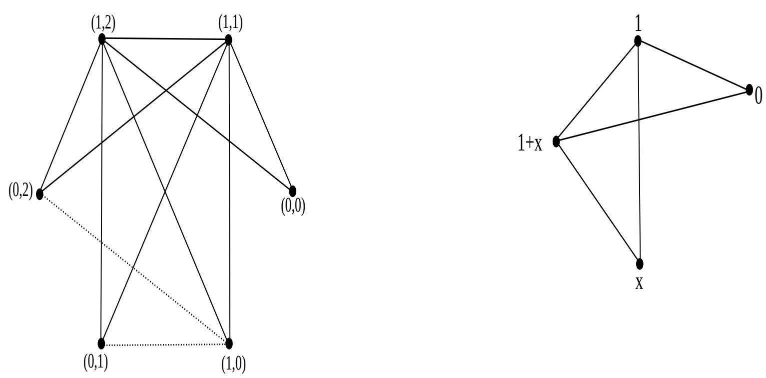

Figure 4. Showing the co-maximal join signed graphs of $\Gamma^\vee_\Sigma(Z_2 * Z_3)$ and $\Gamma^\vee_\Sigma(Z_2(x)/\langle x^2 \rangle)$.

3.1. Properties of co-maximal join signed graph

Theorem 28 *A co-maximal join signed graph $\Gamma^\vee_\Sigma(R)$ is balanced if and only if R is a local ring.*

Proof First, suppose $\Gamma^\vee_\Sigma(R)$ is balanced. On the contrary let R be not a local ring. Then we have nonunits $a_1, a_2 \in R$ (say) such that $Ra_1 + Ra_2 = R$. Let u be some unit in R. Then there exists a cycle ua_1a_2u with exactly one negative edge, which is a contradiction. Now suppose R is a local ring, which implies that there does not exist some edge between nonunits. Thus $\Gamma^\vee_\Sigma(R)$ is an all-positive graph, implying that $\Gamma^\vee_\Sigma(R)$ is balanced. □

Example 29 *If R is isomorphic to either Z_{p^n} or F_q, where F_q is a field of cardinality q, then the co-maximal join signed graph $\Gamma_\Sigma^\vee(R)$ is balanced.*

Theorem 30 *A co-maximal join signed graph $\Gamma_\Sigma^\vee(R)$ is clusterable if and only if R is a local ring.*

Proof First, suppose $\Gamma_\Sigma^\vee(R)$ is clusterable. On the contrary let R be not a local ring. Then we have nonunits $a_1, a_2 \in R$ (say) such that $Ra_1 + Ra_2 = R$. For some unit u in R, there exists a cycle ua_1a_2u with exactly one negative edge, which is a contradiction. If R is a local ring, there does not exist some edge between nonunits. $\Gamma_\Sigma^\vee(R)$ is an all-positive graph, so that we can put all vertices in one cluster. This implies that $\Gamma_\Sigma^\vee(R)$ is clusterable. □

Theorem 31 *If R is a local ring, then the co-maximal join signed graph $\Gamma_\Sigma^\vee(R)$ is $\mathcal{C}$-sign-compatible.*

Proof The proof is trivial. □

Corollary 32 *If R is a field, then the co-maximal join signed graph $\Gamma_\Sigma^\vee(R)$ is $\mathcal{C}$-sign-compatible.*

Example 33 *If $R \cong Z_2 \times Z_2$, then the co-maximal join signed graph $\Gamma_\Sigma^\vee(R)$ is $\mathcal{C}$-sign-compatible.*

Theorem 34 *A co-maximal join signed graph $\Gamma_\Sigma^\vee(R)$ is always sign-compatible.*

Proof The proof is trivial if we give marking to the vertices according to the rule given below: If R is a local ring, then $\Gamma_\Sigma^\vee(R)$ is all-positive, and we can assign positive signs to every vertex. However, if R is not a local ring, then we can assign positive signs to all units and negative signs to all nonunits. □

3.2. Negation of co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$

In this section, we describe properties such as balance, clusterability, sign-compatibility, and $\mathcal{C}$-sign-compatibility of the negation of the co-maximal join signed graph.

Theorem 35 *Negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is balanced if and only if $|U(R)| = 1$.*

Proof Necessity: First, suppose the negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is balanced. On the contrary let $|U(R)| \geq 2$. Let $u_1, u_2 \in R$ be units of the ring and $0 \in R$ be zero of the ring. Since $Ru_1 + R0 = Ru_2 + R0 = Ru_1 + Ru_2 = R$, therefore there exist edges between $0, u_1$, and u_2, and all are negative. This implies that there exists at least one negative cycle of length three $0u_1u_20$, which is a contradiction.

For sufficiency, let $|U(R)| = 1$. Now we will show that negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is balanced. If R is a local ring, then there is no cycle by construction, but if R is not a local ring, then there must exist at least one cycle. All cycles are of the form ua_la_mu, where u is a unit of R and a_i's are nonunits of ring R containing an even number of negative edges. Hence $\eta(\Gamma_\Sigma^\vee(R))$ is balanced. □

Corollary 36 *For a local ring, $\eta(\Gamma_\Sigma^\vee(R))$ is balanced if and only if $R \cong Z_2$.*

Theorem 37 *Negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is always clusterable.*

Proof The proof is trivial. □

Theorem 38 *Negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is sign-compatible if and only if R is a local ring.*

Proof Necessity: Let the negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ be sign-compatible. On the contrary let R be not a local ring. Then we have at least two nonunits $a_1, a_2 \in R$(say) such that $Ra_1 + Ra_2 = R = Ru + Ra_1 = Ru + Ra_2$, where $u \in R$ is a unit of R. ua_1, ua_2 being negative edges, we assign negative marks to all three vertices u, a_1, and a_2. However, a_1a_2 is a positive edge and for a graph to be sign-compatible at least one of a_1 or a_2 must be assigned a positive mark, which is a contradiction.

Sufficiency is trivial, if we assign markings to the vertices according to the rule given below: If R is a local ring, then the negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is all-negative and hence sign-compatible. □

Theorem 39 *Negation of co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is $\mathcal{C}$-sign-compatible if and only if R is a local ring such that $|U(R)|$ is odd and $|R|$ is even.*

Proof Necessity: Let $\eta(\Gamma_\Sigma^\vee(R))$ be $\mathcal{C}$-sign-compatible. Then suppose that R is not a local ring. We then have an edge(positive) ab(say) between two nonunit elements $a, b \in R$. Moreover, a and b are connected to $u \in R$, where u is a unit of the ring by negative edges. This is contrary to our assumption, due to Theorem 7. Next, let R be a local ring with $|U(R)|$ being even. Now, since all nonunit elements of the ring are joined to all unit elements of the ring by a negative edge, this implies that $d^-(a_i) = |U(R)|$, where $a_i \in R$ is some nonunit element of R. This is a contradiction to condition (a) of Theorem 7.

Sufficiency: Let R be a local ring where $|U(R)|$ is odd and $|R|$ is even. Therefore, graph $\eta(\Gamma_\Sigma^\vee(R))$ is all-negative. Since the unit elements of the ring are connected to all elements of the ring, $d^-(u_i) = |R| - 1$ (odd) and all nonunit elements of the ring are joined to all the unit elements of the ring by a negative edge. This implies that $d^-(a_i) = |U(R)|$ (odd). Therefore, according to Theorem 7 we can say that $\eta(\Gamma_\Sigma^\vee(R))$ is $\mathcal{C}$-sign-compatible. □

Example 40 *If R is isomorphic to the rings, Z_{p^n}, F, where F is a field, then the negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is sign-compatible.*

Example 41 *If R is isomorphic to Z_2^r or Z_2, then the negation of the co-maximal join signed graph $\eta(\Gamma_\Sigma^\vee(R))$ is $\mathcal{C}$-sign-compatible.*

Example 42 *If R is isomorphic to one of the following rings, $Z_n (n \neq 2)$, Z_{p^n}, where p is prime, $Z_m * Z_n (m, n \neq 2), Z_2[x]/\langle x^2 \rangle, Z_3[x]$, then $\eta(\Gamma_\Sigma^\vee(R))$ is not $\mathcal{C}$-sign-compatible.*

3.3. Line signed graph of co-maximal join signed graph

Theorem 43 *If R is a local ring, then the line signed graph of the co-maximal join signed graph $L(\Gamma_\Sigma^\vee(R))$ is balanced.*

Proof If R is a local ring, then the co-maximal join signed graph $\Gamma_\Sigma^\vee(R)$ is all-positive and hence its line signed graph is also all-positive. This implies that the line signed graph of the co-maximal join signed graph $L(\Gamma_\Sigma^\vee(R))$ is balanced. □

Example 44 *If R is isomorphic to the rings, $Z_2 * Z_2, Z_{p^n}, F$, where F is a field, then the line signed graph of the co-maximal join signed graph $L(\Gamma_\Sigma^\vee(R))$ is balanced.*

Example 45 *If R is isomorphic to $\frac{Z_2[x]}{\langle x^3 \rangle}, Z_n * Z_m (n, m \neq 2)$, $Z_n(n \neq 2, 3, 4$ and prime$)$, then the line signed graph of the co-maximal join signed graph $L(\Gamma_\Sigma^\vee(R))$ is not balanced.*

Theorem 46 *The line signed graph of the negation of the co-maximal join signed graph $L(\eta(\Gamma_\Sigma^\vee(R)))$ is balanced if and only if $R \cong Z_2$.*

Proof Necessity: If $L(\eta(\Gamma_\Sigma^\vee(R)))$ is balanced, then our claim is that $R \cong Z_2$.

To achieve our claim, we examine the structure of $\eta(\Gamma_\Sigma^\vee(R))$, which depends upon the following two cases:

(1) when R is isomorphic to a nonlocal ring.

(2) when R is isomorphic to a local ring.

Case 1: If R is isomorphic to a nonlocal ring, then there will be at least two maximal ideals, say M_1, M_2 of R. Let $a_1 \in M_1$ and $a_2 \in M_2$, where a_1, a_2 are nonunit elements of the ring R and $u \in R$ is a unit element of the ring. Then, from the construction of $\eta(\Gamma_\Sigma^\vee(R))$ $a_1 a_2$ is a positive edge whereas ua_1 and ua_2 are negative edges. This implies that $d^-(u) \geq 2$, which is contrary to our assumption due to Theorem 3.

Case 2: If R is isomorphic to a local ring. Claim: $R \cong Z_2$; we shall prove the claim by the contradiction. If $|R| \geq 3$, then there are two possibilities:

(a) $|U(R)| \geq 2$.

If $|U(R)| \geq 2$, then $d^-(0) \geq 2$, where $0 \in R$ is zero of the ring. This is a contradiction due to Theorem 3

(b) $|U(R)| = 1$. If $|U(R)| = 1$, then from the construction of $\eta(\Gamma_\Sigma^\vee(R))$ $d^-(u) \geq 2$, where $u \in R$ is a unit of the ring. This is again a contradiction due to Theorem 3

Sufficiency is trivial. □

Theorem 47 *If R is a local ring, then the line signed graph of the co-maximal join signed graph $L(\Gamma_\Sigma^\vee(R))$ is clusterable.*

Proof The proof is trivial. □

Example 48 *If R is isomorphic to $Z_2 * Z_2$, then $L(\Gamma_\Sigma^\vee(R))$ is clusterable.*

Example 49 *If R is isomorphic to $Z_m * Z_n (n, m \neq 2)$, then $L(\Gamma_\Sigma^\vee(R))$ is not clusterable.*

Theorem 50 *The line signed graph of the negation of the co-maximal join signed graph* $L(\eta(\Gamma_\Sigma^\vee(R)))$ *is clusterable if and only if* R *is a local ring.*

Proof Necessity: Let $L(\eta(\Gamma_\Sigma^\vee(R)))$ be clusterable. On the contrary let R be not a local ring. Then in $\eta(\Gamma_\Sigma^\vee(R))$, one can easily determine a cycle of length three, viz., ua_1a_2u with one positive and two negative edges. Therefore, in $L(\eta(\Gamma_\Sigma^\vee(R)))$ there will be a cycle with exactly one negative edge, which is contrary to the Davis criterion of clusterability [Theorem 4].

Sufficiency is trivial, if R is a local ring, then $\eta(\Gamma_\Sigma^\vee(R))$ is an all-negative graph. Hence, $L(\eta(\Gamma_\Sigma^\vee(R)))$ will also be an all-negative graph and therefore clusterable. □

4. Co-maximal ring sum signed graph

Definition 51 *A co-maximal ring sum signed graph is an ordered pair* $\Gamma_\Sigma^\oplus(R) = (\Gamma(R), \sigma)$, *where* $\Gamma(R)$ *is the co-maximal graph of a commutative ring* R *and for an edge* (ab) *of* $\Gamma_\Sigma(R)^\oplus$, σ *is defined as*

$$\sigma(ab) = \begin{cases} + & \text{either } a \in U(R) \text{ or } b \in U(R), \\ - & \text{otherwise.} \end{cases}$$

The co-maximal ring sum signed graphs $\Gamma_\Sigma^\oplus(Z_2 * Z_3)$ and $\Gamma_\Sigma^\oplus(Z_2(x)/\langle x^2\rangle)$ are shown in Figure 3, in which solid line segments are positive edges and dotted line segments are negative edges.

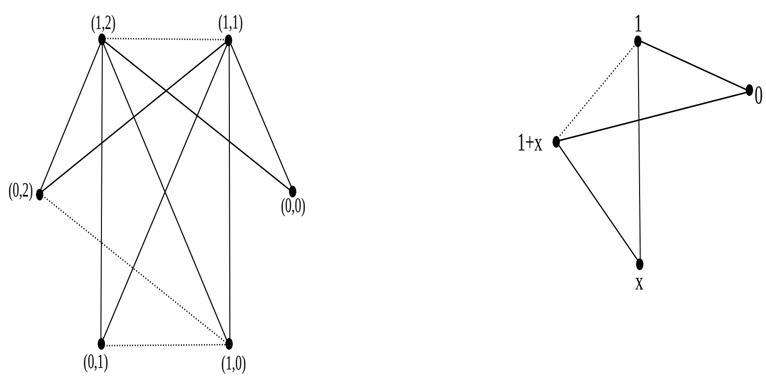

Figure 5. Showing the co-maximal ring sum signed graphs of $\Gamma(Z_2 * Z_3)$ and $\Gamma(Z_2(x)/\langle x^2\rangle)$.

4.1. Properties of co-maximal ring sum signed graphs

Theorem 52 *The co-maximal ring sum signed graph* $\Gamma_\Sigma^\oplus(R)$ *is balanced if and only if* R *is isomorphic to* Z_2.

Proof Necessity: Let the co-maximal ring sum signed graph $\Gamma_\Sigma^\oplus(R)$ be balanced. If we suppose that R is not a local ring, then there will be at least two maximal ideals, say m_1, m_2. Let $a_1 \in m_1, a_2 \in m_2$ be nonunit elements of the ring R and $u \in R$ be a unit of the ring. From the definition of a co-maximal ring sum signed graph, one can easily determine the presence of a cycle ua_1a_2u with exactly one negative edge a_1a_2, which is a contradiction to our assumption that $\Gamma_\Sigma^\oplus(R)$ is balanced. Next, if $|U(R)| \geq 2$ and R is a local ring, then there exists a cycle $u_10u_2u_1$, where $u_1, u_2 \in R$ are units of R and $0 \in R$ is zero of ring R. From the definition of a co-maximal ring sum signed graph, $u_1 0$ and $u_2 0$ are positive edges and $u_1 u_2$ is a negative edge. Therefore, cycle $u_1 0 u_2 u_1$ has exactly one negative edge, which is a contradiction.

Sufficiency: If R is isomorphic to Z_2, then the co-maximal ring sum signed graph is all-positive and hence $\Gamma_{\Sigma}^{\oplus}(R)$ is balanced. □

Theorem 53 *Co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ is clusterable if and only if R is isomorphic to Z_2.*

Proof Necessity: Let the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ be clusterable. From the proof of Theorem 52, one can easily determine that if R is not a local ring or R is a local ring with $|U(R)| \geq 2$, then there exists a cycle with exactly one negative edge, which is a contradiction to the Davis criterion of clusterability [Theorem 4].

Sufficiency: If R is isomorphic to Z_2, then the co-maximal ring sum signed graph is all-positive and hence $\Gamma_{\Sigma}^{\oplus}(R)$ is clusterable. □

Theorem 54 *Co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ is sign-compatible if and only if $|U(R)| = 1$ or R is a local ring.*

Proof Necessity: Let the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ be sign-compatible. Let $|U(R)| \geq 2$, R be not a local ring, and $u_1, u_2, a_1, a_2 \in R$, where u_1, u_2 are units of R and a_1, a_2 are nonunits of R. Also $Ru_1 + Ru_2 = Ru_1 + Ra_1 = Ru_2 + Ra_2 = Ra_1 + Ra_2 = R$ implies that $a_1 a_2$, $u_1 u_2$ are negative edges. For $\Gamma_{\Sigma}^{\oplus}(R)$ to be sign-compatible end vertices $a_1, a_2; u_1, u_2$ must be assigned negative marking. However, $u_1 a_1$, $u_2 a_2$ are positive edges and therefore at least one of u_1 or a_1 and u_2 or a_2 must have positive marking, which is a contradiction.

Sufficiency is trivial. If R is a local ring, then on marking all units '$-$' and all nonunits '$+$', $\Gamma_{\Sigma}^{\oplus}(R)$ becomes sign-compatible. However, if R is not a local ring and $|U(R)| = 1$, then marking all nonunits '$-$' and all units '$+$', we get sign-compatible graph $\Gamma_{\Sigma}^{\oplus}(R)$. □

Example 55 *If R is isomorphic to either Z_2^r, then the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ is sign-compatible.*

Theorem 56 *If R is a local ring, then the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ is $\mathcal{C}$-sign-compatible if and only if $|U(R)| = 1$ or $|U(R)|$ is even.*

Proof Necessity: Let $\Gamma_{\Sigma}^{\oplus}(R)$ be $\mathcal{C}$-sign-compatible. If $|U(R)| \neq 1$ is odd, then $d^-(v) \neq 0$ and $d^-(v) \equiv 0 \pmod 2$, and, according to Theorem 7, $\Gamma_{\Sigma}^{\oplus}(R)$ is not $\mathcal{C}$-sign-compatible.

Sufficiency: If $|U(R)| = 1$ or $|U(R)|$ is even and R is a local ring, then either $d^-(v) = 0$ or $d^-(v) \equiv 1 \pmod 2$ and $d^-(v_i) = 0$, where $v_i \in R$ is a nonunit of ring R. Hence, as per Theorem 7, $\Gamma_{\Sigma}^{\oplus}(R)$ is $\mathcal{C}$-sign-compatible. □

Example 57 *$\Gamma_{\Sigma}^{\oplus}(R)$ is $\mathcal{C}$-sign-compatible if R is isomorphic to one of the following rings, $F = \{0, 1\}$ and $F[x]/\langle x^n \rangle (n = 2, 3, \dots)$, $F_1 = \{0, 1, \dots, p-1\}$ (p is prime) and $F_1[x]/\langle x^n \rangle (n = 2, 3, \dots)$.*

Corollary 58 *If $R \cong F_{p^n}$, where F_{p^n} is a field of cardinality p^n, then the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ is $\mathcal{C}$-sign-compatible if and only if $p \neq 2$ and $n \geq 1$.*

Example 59 *If R is isomorphic to Z_{p^n}, where p is prime, then the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ is $\mathcal{C}$-sign-compatible.*

Theorem 60 *In the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ if $|U(R)| \geq 2$ and R is not a local ring, then the graph is not $\mathcal{C}$-sign-compatible.*

Proof Proof is trivial, from construction of $\Gamma_{\Sigma}^{\oplus}(R)$ and Theorem 7. □

Example 61 *If R is isomorphic to Z_{p^r}, then the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ graph is $\mathcal{C}$-sign-compatible.*

4.2. Negation of co-maximal ring sum signed graph

Theorem 62 *Negation of the co-maximal ring sum signed graph $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ is always balanced.*

Proof Proof is trivial from the construction of $\eta(\Gamma_{\Sigma}^{\oplus}(R))$. We can see that if there are some cycles, then they must contain an even number of negative edges. Hence $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ is balanced. □

Theorem 63 *Negation of the co-maximal ring sum signed graph $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ is always clusterable.*

Proof Negation of the co-maximal ring sum signed graph $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ is clusterable, if all the units are put in set V_1 and all the nonunits in V_2. □

Theorem 64 *Negation of the co-maximal ring sum signed graph $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ is sign-compatible if and only if $R \cong Z_2$.*

Proof Necessity: Let the negation of the co-maximal ring sum signed graph $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ be sign-compatible. Suppose that $|U(R)| \geq 2$. If $u_1, u_2 \in R$ are units of the ring R and $0 \in R$ is zero of ring R such that $u_1 0, u_2 0$ edges are negative, the end vertices u_1, u_2 and 0 will be marked '$-$'. However, $u_1 u_2$ edge is positive and for a signed graph to be sign-compatible at least one of the vertices u_1 and u_2 must be assigned marking '$+$', which is a contradiction. Again let R be not a local ring and $|U(R)| = 1$. Assume that $a_1, a_2 \in R$ are nonunits of the ring R and $u \in R$ is a unit of ring R such that $a_1 u, a_2 u$ edges are negative. Assigning the mark '$-$' to the vertices a_1, a_2 and u, $a_1 a_2$ being positive edges at least one of the vertices a_1 and a_2 should be marked by '$+$', which is a contradiction.

Sufficiency is trivial; all the vertices are marked '$-$'. □

Theorem 65 *Negation of the co-maximal ring sum signed graph $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ is $\mathcal{C}$-sign-compatible if and only if $R \cong Z_2$.*

Proof Necessity: Let the negation of the co-maximal ring sum signed graph $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ be $\mathcal{C}$-sign-compatible. Suppose $|U(R)| \geq 2$. Then $u_1 u_2$ being a positive edge, the condition for $\eta(\Gamma_{\Sigma}^{\oplus}(R))$ to be $\mathcal{C}$-sign-compatible is that at least one of the units u_1 or u_2 must have a marking of '+' but $u_1 0$ and $u_2 0$ both being negative edges, mark of u_1, u_2 and 0 must be '$-$', a contradiction.

If $|U(R)| = 1$ and R is not a local ring, then there will be nonunits $a_1, a_2 \in R$ and unit $u \in R$ such that $Ra_1 + Ra_2 = Ru + Ra_1 = Ru + Ra_2 = R$, where $a_1 a_2$ is a positive edge and ua_1, ua_2 are negative edges. Due to Theorem 7, $d^-(a_1) = 0$ or $d^-(a_2) = 0$, this is again a contradiction.

Sufficiency is trivial. □

4.3. Line signed graph of co-maximal ring sum signed graph

Theorem 66 *If $|U(R)| \geq 3$, then the line signed graph of the co-maximal ring sum signed graph is not balanced.*

Proof Suppose that $|U(R)| \geq 3$; then in the co-maximal ring sum signed graph $d^-(u_i) \geq 2$ and $d(u_i) > 2$, where $u_i \in R$ is a unit of the ring. Then, according to Theorem 3, the line signed graph of the co-maximal ring sum signed graph is not balanced. □

Theorem 67 *If R is a local ring, then $L(\Gamma_{\Sigma}^{\oplus}(R))$ is balanced if and only if $|U(R)| \leq 2$.*

Proof Suppose that $L(\Gamma_{\Sigma}^{\oplus}(R))$ is balanced. On the contrary let $|U(R)| \geq 3$, which will be a contradiction to our assumption as per Theorem 66.

Next, let $|U(R)| \leq 2$ then in $\Gamma_{\Sigma}^{\oplus}(R)$ there will exist at most one negative edge. Hence, its line signed graph is all-positive and therefore $L(\Gamma_{\Sigma}^{\oplus}(R))$ is balanced. □

Example 68 *If R is isomorphic to one of the following rings:$Z_2, Z_3, Z_4, Z_2 * Z_2, \frac{Z_2[x]}{\langle x^2 \rangle}$, then the line signed graph of the co-maximal ring sum signed graph is balanced.*

Theorem 69 *The line signed graph of the negation of the co-maximal ring sum signed graph is balanced if and only if $R \cong Z_2$.*

Proof Necessity: Let the line signed graph of the negation of the co-maximal ring sum signed graph be balanced. First, suppose that R is not a local ring. Then there exist at least two maximal ideals M_1, M_2 (say). Now, from the definition of $\eta(\Gamma_{\Sigma}^{\oplus}(R))$, one can easily conclude that $d^-(u_i) \geq 2$, where $u_i \in R$ is unit of the ring, which is a contradiction to our assumption as per Theorem 3. Next suppose R is a local ring and $|U(R)| \geq 2$. Let $0 \in R$ be zero of the ring. Then from the construction of $\eta(\Gamma_{\Sigma}^{\oplus}(R))$, $d^-(0) \geq 2$, which is again a contradiction as per Theorem 3. Hence, the line signed graph of the negation of the co-maximal ring sum signed graph is not balanced.

Sufficiency is trivial. □

Theorem 70 *If $|U(R)| \geq 3$, then the line signed graph of the co-maximal ring sum signed graph is not clusterable.*

Proof Let $|U(R)| \geq 3$. Now from the construction of the co-maximal ring sum signed graph there will exist a unit, say $u \in R$, such that $d^-(u) \geq 2$ and $d^+(u) \geq 1$. Therefore, in its line signed graph there must exist a cycle of length three with exactly one negative edge, which is a contradiction as per the Davis criterion. □

Theorem 71 *If R is a local ring, then the line signed graph of the co-maximal ring sum signed graph is clusterable if and only if $|U(R)| \leq 2$.*

Proof First suppose that the line signed graph of the co-maximal ring sum signed graph is clusterable. If $|U(R)| \geq 3$, then as per Theorem 70 we get a contradiction to our assumption.

Next suppose that $|U(R)| \leq 2$ and R is a local ring. From the definition of a co-maximal ring sum signed graph one can easily conclude that there exists at most one negative edge in $\Gamma_{\Sigma}^{\oplus}(R)$. Therefore, its line signed graph $L(\Gamma_{\Sigma}^{\oplus}(R))$ is all-positive and hence clusterable. □

Theorem 72 *The line signed graph of the negation of the co-maximal ring sum signed graph $L(\eta(\Gamma_{\Sigma}^{\oplus}(R)))$ is clusterable if and only if $R \cong Z_2$.*

Proof Necessity: Let the line signed graph $L(\eta(\Gamma_{\Sigma}^{\oplus}(R)))$ be clusterable. First suppose that R is not a local ring. Now, from the definition of the negation of the co-maximal ring sum signed graph, one can easily conclude that there exists a cycle of length three with exactly one positive edge and two negative edges. Therefore, in its line signed graph this cycle maps onto a cycle of length three with exactly one negative edge, which is contrary to the Davis criterion, as per Theorem 4. Next suppose that R is a local ring with $|U(R)| \geq 2$. Then there will exist a cycle $u_1u_2au_1$ with two negative and one positive edges(where $u_1, u_2 \in R$ are units of the ring and $a \in R$ is a nonunit of the ring). Therefore, in its line signed graph $L(\eta(\Gamma_{\Sigma}^{\oplus}(R)))$ this cycle maps onto a cycle with exactly one negative edge, again in contradiction to the Davis criterion, Theorem 4.

Sufficiency is trivial. □

5. Isomorphism of co-maximal signed graphs

Theorem 73 *For a co-maximal graph $\Gamma(R)$, the co-maximal join signed graph and the co-maximal ring sum signed graph are isomorphic if and only if $|U(R)| = 1$.*

Proof Necessity: Suppose $\Gamma_{\Sigma}^{\vee}(R) \cong \Gamma_{\Sigma}^{\oplus}(R)$. If possible, let $|U(R)| \geq 2$. Now, for some two vertices i and j in the co-maximal graph, there are the following three possibilities:

(1) $i \in U(R)$ and $j \in U(R)$ or

(2) $i \notin U(R)$ and $j \notin U(R)$ or

(3) $i \in U(R)$ and $j \notin U(R)$ or vice versa.

In condition (1) ij is a positive edge in $\Gamma_{\Sigma}^{\vee}(R)$, while in $\Gamma_{\Sigma}^{\oplus}(R)$ ij is a negative edge. In condition (2) ij is a negative edge in both $\Gamma_{\Sigma}^{\vee}(R)$ and $\Gamma_{\Sigma}^{\oplus}(R)$. Similarly, in condition (3) ij is a positive edge in both $\Gamma_{\Sigma}^{\vee}(R)$ and $\Gamma_{\Sigma}^{\oplus}(R)$. Since i and j are arbitrary vertices in a co-maximal graph, this is true for all the vertices. Thus, the number of negative edges in $\Gamma_{\Sigma}^{\oplus}(R) > \Gamma_{\Sigma}^{\vee}(R)$, which is a contradiction to the hypothesis.

Sufficiency: Suppose $|U(R)| = 1$. It is clear that the underlying structures of $\Gamma_{\Sigma}^{\oplus}(R)$ and $\Gamma_{\Sigma}^{\vee}(R)$ are the same. Suppose ij is a negative edge in $\Gamma_{\Sigma}^{\vee}(R)$; then according to the definition of $\Gamma_{\Sigma}^{\vee}(R)$, $i \notin U(R)$ and $j \notin U(R)$. Now, as per the definition of $\Gamma_{\Sigma}^{\oplus}(R)$, ij is a negative edge in $\Gamma_{\Sigma}^{\oplus}(R)$ also. Now suppose ij is a positive edge in $\Gamma_{\Sigma}^{\vee}(R)$. Then we have the following possibilities:

(i) $i \in U(R)$ and $j \in U(R)$ or

(ii) $i \in U(R)$ and $j \notin U(R)$ or vice versa.

Since $|U(R)| = 1$, condition (i) is not possible. Then the only possibilities are $i \in U(R)$ and $j \notin U(R)$ or vice versa. As per the definition of $\Gamma_{\Sigma}^{\vee}(R)$, ij is a positive edge in $\Gamma_{\Sigma}^{\oplus}(R)$ also. Therefore, $\Gamma_{\Sigma}^{\oplus}(R) \cong \Gamma_{\Sigma}^{\vee}(R)$. Hence the Theorem holds. □

Theorem 74 *For a co-maximal graph $\Gamma(R)$, the co-maximal meet signed graph and the co-maximal join signed graph are never isomorphic.*

Proof Given the co-maximal graph, there are two possibilities, i.e. either $|U(R)| = 1$ or $|U(R)| \geq 2$.

Case I: Suppose $|U(R)| = 1$. From the construction, the co-maximal meet signed graph $\Gamma_{\Sigma}(R)$ is all-negative, while the co-maximal join signed graph $\Gamma_{\Sigma}^{\vee}(R)$ is not an all-negative signed graph. Thus the number of negative edges in $\Gamma_{\Sigma}(R) > \Gamma_{\Sigma}^{\vee}(R)$. Thus, $\Gamma_{\Sigma}(R)$ is not isomorphic to $\Gamma_{\Sigma}^{\vee}(R)$, when $|U(R)| = 1$.

Case II: Suppose $|U(R)| \geq 2$. Now, for some two vertices i and j in a co-maximal graph, we have the following three possibilities:

(i) $i \in U(R)$ and $j \in U(R)$ or

(ii) $i \notin U(R)$ and $j \notin U(R)$ or

(iii) $i \in U(R)$ and $j \notin U(R)$ or vice versa.

In condition (i) ij is a positive edge in both $\Gamma_{\Sigma}(R)$ and $\Gamma_{\Sigma}^{\vee}(R)$. In condition (ii) ij is a negative edge in both $\Gamma_{\Sigma}(R)$ and $\Gamma_{\Sigma}^{\vee}(R)$. In condition (iii) ij is a positive edge in $\Gamma_{\Sigma}^{\vee}(R)$ but in $\Gamma_{\Sigma}(R)$ ij is a negative edge. Since i and j are arbitrary vertices in a co-maximal graph, this is true for all the vertices of a co-maximal graph. Thus, the number of negative edges in $\Gamma_{\Sigma}(R) > \Gamma_{\Sigma}^{\vee}(R)$. Hence, $\Gamma_{\Sigma}(R)$ is not isomorphic to $\Gamma_{\Sigma}^{\vee}(R)$, when $|U(R)| \geq 2$. Hence the Theorem holds. □

Theorem 75 *For a co-maximal graph $\Gamma(R)$, the co-maximal meet signed graph and the co-maximal ring sum signed graph are never isomorphic.*

Proof Given the co-maximal graph, there are two possibilities i.e. either $|U(R)| = 1$ or $|U(R)| \geq 2$.

Case I: Suppose $|U(R)| = 1$. Now, from the construction, the co-maximal meet signed graph $\Gamma_{\Sigma}(R)$ is all-negative, while the co-maximal ring sum signed graph $\Gamma_{\Sigma}^{\oplus}(R)$ is not an all-negative signed graph. Therefore, the number of negative edges in $\Gamma_{\Sigma}(R) > \Gamma_{\Sigma}^{\oplus}(R)$. Thus, $\Gamma_{\Sigma}(R)$ is not isomorphic to $\Gamma_{\Sigma}^{\oplus}(R)$, when $|U(R)| = 1$.

Case II: Suppose $|U(R)| \geq 2$. Now, for some two vertices i and j in a co-maximal graph, there are the following three possibilities:

(i) $i \in U(R)$ and $j \in U(R)$ or

(ii) $i \notin U(R)$ and $j \notin U(R)$ or

(iii) $i \in U(R)$ and $j \notin U(R)$ or vice versa.

In condition (i) ij is a positive edge in $\Gamma_\Sigma(R)$ but in $\Gamma_\Sigma^\oplus(R)$ ij is a negative edge. In condition (ii) ij is a negative edge in both $\Gamma_\Sigma(R)$ and $\Gamma_\Sigma^\vee(R)$. In condition (iii) ij is a positive edge in $\Gamma_\Sigma^\oplus(R)$ but in $\Gamma_\Sigma(R)$ ij is a negative edge. Since i and j are arbitrary vertices in a co-maximal graph, this is true for all the vertices of a co-maximal graph. Thus, the number of negative edges in $\Gamma_\Sigma(R) > \Gamma_\Sigma^\oplus(R)$. Hence, $\Gamma_\Sigma(R)$ is not isomorphic to $\Gamma_\Sigma^\oplus(R)$, when $|U(R)| \geq 2$. Hence the Theorem holds. □

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