More on sequential order of compact scattered spaces

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Abstract: The proper forcing axiom is shown to imply that a compact scattered sequential space with scattering height at most ω_1 must have sequential order at most ω.

Key words: Sequential order, scattered spaces, PFA

1. Introduction

For all undefined notions we refer the reader to [7, 10]. One of the most interesting questions on the study of sequential spaces is what is the bound of sequential order for a compact sequential space. In 1974, Bashkirov [1] proved that it follows from CH that there are compact sequential spaces of any sequential order up to and including ω_1. Since that time it has been an open problem to determine how large the sequential order of a compact sequential space may be. To date, all the consistency examples have been scattered (e.g., see also [4, 8]) including the prototype example of the one-point compactification of the standard Mrowka space constructed from an infinite maximal almost disjoint family of subsets of ω. To illustrate the Mrowka example, let us recall $M = \omega \cup \mathcal{A}$, where $\mathcal{A}$ is an infinite maximal almost disjoint family of subsets of ω. The topology on the set M is generated by the neighborhood system $\{\mathcal{B}(x)\}_{x \in M}$, where $\mathcal{B}(n) = \{\{n\}\}$ for every natural number n, and $\mathcal{B}(a) = \{\{a\} \cup (a \setminus \{1, 2, 3, \ldots, i\}) : i = 1, 2, 3, \ldots\}$ if $a \in \mathcal{A}$. Since the topology on M is locally compact, take the one-point compactification of M, denoted as $M_0 = M \cup \{\infty\}$, where $\infty \notin M$. M_0 is a sequential space (see, e.g., [12]), and since $\mathcal{A}$ is maximal almost disjoint family, there is no sequence from ω converging to ∞. Thus the sequential order of M_0 is 2 (see [4]). It is a remarkable state of affairs that there is no ZFC result closing the gap between two and ω_1. Moreover, the largest sequential order known to follow from Martin's Axiom is 5 [4]. The author showed in [5] that the proper forcing axiom (PFA) imposes a bound of ω on the sequential order in a restricted class of scattered spaces. See [2] for basic information on proper poset and PFA. This paper improves that result in that it provides information about the limitations of sequential order in arbitrary scattered spaces.

The definitions of sequential space, sequential order, and scattered space are provided below.

Definition 1.1 *For any ordinal θ and θ-sequence $\{x_\alpha : \alpha \in \theta\}$ from a space X, we let $\{x_\alpha : \alpha < \theta\} \to x$ denote the relation that for every neighborhood U of $x \in X$, there is a $\beta < \theta$ such that $\{x_\alpha : \beta < \alpha < \theta\} \subset U$. When $\{x_\alpha : \alpha < \theta\} \to x$ holds, we say that the θ-sequence converges to x.*

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Definition 1.2 *Let X be a space.*

1. *A space X is scattered if every subset of X has a relative isolated point.*

2. *A subset A of a space X is sequentially closed if every ω-sequence from A that converges in X converges to a point in A.*

3. *A space X is sequential if every sequentially closed set is closed.*

4. *The sequential limit operator of a space X is defined on each $A \subset X$ by transfinite recursion with $A^{(0)}$ equaling A. For any $A \subset X$, $A^{(1)}$ is defined to be $A \cup \{x \in X : (\exists \{a_n : n \in \omega\} \subset A) \ \{a_n : n \in \omega\} \to x\}$, and, for an ordinal $\alpha > 1$, we define $A^{(\alpha)}$ to be $\left(\bigcup_{\beta < \alpha} A^{(\beta)} \right)^{(1)}$.*

5. *The sequential order of a sequential space X is the minimum ordinal θ satisfying that, for all $A \subset X$, $A^{(\theta)}$ is sequentially closed, i.e. the sequential order of X is the minimal ordinal θ such that $\bar{A} = A^{(\theta)}$ for every $A \subset X$.*

Our proof of the main result will be a new application of the following result that is a strengthening of a similar result from [5].

Proposition 1.3 ([6]) *PFA implies that if D is a subset of a compact sequential space and if $x \in \overline{D} \setminus D^{(1)}$, then there is an ω_1-sequence $\{x_\alpha : \alpha \in \omega_1\} \subset D^{(1)}$ that converges to x.*

2. Sequential order in scattered spaces

Now we apply Proposition 1.3 to the structure of compact scattered sequential spaces.

Recall that the Cantor–Bendixson derivative of a space is obtained by eliminating all isolated points with the relative topology.

Definition 2.1 *If X is a scattered space, then the Cantor–Bendixson rank of X (also called the height of a scattered space) is the ordinal ρ satisfying that the Cantor–Bendixson derivative process terminates at stage ρ. By transfinite recursion, the scattering levels, $\{X_\alpha : \alpha < \rho\}$, are defined according to the properties that X_0 is the set of isolated points of X, and, for each $\alpha \leq \rho$, X_α is the set of isolated points of $X \setminus \bigcup \{X_\beta : \beta < \alpha\}$. The value of ρ is the minimum ordinal such that $X_{\rho+1}$ is empty.*

Given a scattered space X, we let $\rho_X(x) = \alpha$ (or simply $\rho(x) = \alpha$) for $x \in X$, where $x \in X_\alpha$.

Theorem 2.2 *PFA implies that if X is a compact scattered sequential space with sequential order greater than ω, then the scattering height of X is at least ω_1.*

Proof Assume that the sequential order of X is greater than ω. Choose countable $D \subset X$ so that $D^{(\omega)}$ is not sequentially closed. Since $D^{(\omega+1)} \neq D^{(\omega)}$, fix a point $w \in D^{(\omega+1)} \setminus D^{(\omega)}$ and a sequence $\{w_n : n \in \omega\} \subset D^{(\omega)}$ that converges to w by following Proposition 1.3. Since $D^{(\omega+1)} = \overline{D}$, we may pass to the subspace equaling the closure of D, and thereby assume that D is dense in X. For each $x \in X$, fix a compact open set W_x so that $x \in W_x$ and $\rho(y) < \rho(x)$ for all $y \in W_x \setminus \{x\}$. We may assume that $W_w = X$.

Claim 1 *There is a family of sets $\{D(n,\sigma) : n \in \omega, \sigma \in \omega_1^{<\omega}\}$ together with, for each $n \in \omega$ and $\sigma \in \omega_1^{<\omega}$, points*

$$\{z(n,\sigma,m) : m \in \omega\}, \{x(n,\sigma,m,\alpha) : \alpha \in \omega_1\}, \{y(n,\sigma,\alpha) : \alpha \in \omega_1\}$$

and a subset $J_{n,\sigma} \subset \omega$, satisfying the following, for all n, σ

1. *$D(n,\emptyset)$ is equal to $D \cap W_{w_n}$,*

2. *$D(n,\sigma)$ is a countable subset of $W_{w_n} \cap D^{(k)}$ for some $k \in \omega$,*

3. *$\{z(n,\sigma,m) : m \in \omega\} \subset D(n,\sigma)^{(<\omega_1)}$ and converges to w_n,*

4. *$\{x(n,\sigma,m,\alpha) : \alpha \in \omega_1\} \subset D(n,\sigma)^{(1)}$ converges to $z(n,\sigma,m)$,*

5. *$\{x(n,\sigma,m,\alpha) : m \in J_{n,\sigma}\}$ converges to $y(n,\sigma,\alpha)$,*

6. *$D(n,\sigma) \subset \{y(n,\sigma \restriction j, \alpha) : \sigma(j) \leq \alpha\}$ where $j = \mathrm{dom}(\sigma)$.*

Proof of Claim Assume that $D(n,\sigma)$, like $D(n,\emptyset)$, is a countable subset of $W_{w_n} \cap D^{(k)}$ for some k satisfying that w_n is in the sequential closure of $D(n,\sigma)$, i.e. w_n is in the set of limits of convergent countable sequences from $D(n,\sigma)$. We inductively choose J's as in Claim 1. Choose distinct $\{z(n,\sigma,m) : m \in \omega\}$, also from the sequential closure of $D(n,\sigma)$, converging to w_n. Choose $\{x(n,\sigma,m,\alpha) : \alpha \in \omega_1\} \subset D(n,\sigma)^{(1)}$ converging to $z(n,\sigma,m)$. By PFA, there is an infinite $J_{n,\sigma} \subset \omega$ so that, for each $\alpha \in \omega_1$, the sequence $\{x(n,\sigma,m,\alpha) : m \in J_{n,\sigma}\}$ converges. Indeed, PFA implies $\mathfrak{p} > \omega_1$, where $\mathfrak{p}$ denotes the smallest cardinality of any family of infinite subsets of ω that has a strong finite intersection property but does not have a pseudointersection. Recall that an infinite subset B of ω is a *pseudointersection* of a family $\mathcal{F}$ of infinite subsets of ω if $B \setminus F$ is finite for all $F \in \mathcal{F}$. We can find such a $J_{n,\sigma}$ by induction at each stage α; for more details see, e.g., Theorem 6.9., page 132 in [11].

Let $y(n,\sigma,\alpha)$ be chosen so that $\{x(n,\sigma,m,\alpha) : m \in J_{n,\sigma}\} \to y(n,\sigma,\alpha)$. We check that $\{y(n,\sigma,\alpha) : \alpha \in \omega_1\} \to w_n$. Let A be any compact open subset of W_{w_n} with $w_n \notin A$. Choose m_0 so that $z(n,\sigma,m) \notin A$ for all $m > m_0$. Choose α_0 so that for all $m > m_0$ and all $\alpha > \alpha_0$, $x(n,\sigma,m,\alpha) \notin A$. It then follows that $y(n,\sigma,\alpha) \notin A$ for all $\alpha > \alpha_0$.

For each $\gamma < \omega_1$, w_n is in the sequential closure of the set $\{y(n,\sigma,\alpha) : \gamma \leq \alpha\}$. Since w_n is a limit of the sequence $\{y(n,\sigma,\alpha) : \alpha \in \omega_1\}$, choose a cub, i.e. closed and unbounded set, $C = \{\gamma_\xi : \xi \in \omega_1\} \subset \omega_1$ so that, for each ξ, w_n is in the sequential closure of $\{y(n,\sigma,\alpha) : \gamma_\xi \leq \alpha < \gamma_{\xi+1}\}$. For each $\xi \in \omega_1$, let $D(n,\sigma^\frown\langle\xi\rangle)$ equal $\{y(n,\sigma,\alpha) : \gamma_\xi \leq \alpha < \gamma_{\xi+1}\}$. By construction, $\{x(n,\sigma,m,\alpha) : m \in \omega, \alpha \in \omega_1\}$ is a subset of $D^{(k+1)}$; hence $D(n,\sigma^\frown\langle\xi\rangle)$ is a subset of $D^{(k+2)}$ and w_n is in the sequential closure of $D(n,\sigma^\frown\langle\xi\rangle)$. Then item (6) is satisfied.

By indexing $\omega \times \omega_1^{<\omega}$ in order-type ω_1 (order-preserving with respect to domain on $\omega_1^{<\omega}$), we can recursively choose the sets $J_{n,\sigma}$ as in item (5) so that they form a mod finite chain. This completes the proof of the Claim. □

Fix any sufficiently large κ so that the space X is in $H(\kappa)$; $H(\kappa)$ is the collection of sets that have cardinality less than κ, and such κ always exists; see, e.g., [9]. By the downward Löwenheim–Skolem Theorem, $H(\kappa)$ has an elementary submodel. Recall that a submodel M of H is *elementary* if any statement is true in

H is also true in M; for more details see, e.g., [3]. Let M_0 be an elementary submodel of $H(\kappa)$ of cardinality $\aleph_1$ so that M_0 contains ω_1, $\{X\}$ and the sequences constructed in the Claim. By induction on $\gamma \in \omega_1$, choose a mod finite descending sequence $\{J_\gamma : \gamma \in \omega_1\}$ of infinite subsets of J from Claim 1, together with an $\in$-chain of elementary submodels $\{M_\gamma : \gamma \in \omega_1\}$ so that $J_\gamma \in M_{\gamma+1}$ and for each ω-sequence $\langle x_n : n \in \omega \rangle \in X^\omega \cap M_\gamma$, the sequence $\langle x_n : n \in J_\gamma \rangle$ converges to a point in $M_\gamma \cap X$. By PFA, let I be any infinite pseudointersection of the family $\{J_\gamma : \gamma \in \omega_1\}$. Also let M denote the union of the sequence $\{M_\gamma : \gamma \in \omega_1\}$.

The family from Claim 1 is the skeleton of a larger family of points of interest. Given a sequence $\vec{\sigma} = \{\sigma_n : n \in \omega\} \subset \omega_1^{<\omega}$ and an ordinal $\gamma \in \omega_1$, we define the set $V(\vec{\sigma}, \gamma)$ to be a special set of limits of the sequence $\{D(n, \sigma_n^\frown \langle \gamma \rangle) : n \in \omega\}$. That is, we let $V(\vec{\sigma}, \gamma)$ denote the set of $v \in V(\vec{\sigma}, \gamma)$ that satisfy that for each compact open $A \subset W_v \setminus \{v\}$, there is an n_0 such that, for all $n_0 < n \in I$, the set $D(n, \sigma_n^\frown \langle \gamma \rangle) \cap (W_v \setminus A)$ is not empty.

Claim 2 *Let* $\vec{\sigma} = \{\sigma_n : n \in \omega\} \in (\omega_1^{<\omega})^\omega$. *Assume that* $v_\gamma \in V(\vec{\sigma}, \gamma)$ *for each* $\gamma \in \omega_1$. *Then, for each uncountable* $\Gamma \subset \omega_1$, $\{v_\gamma : \gamma \in \Gamma\}$ *converges to* w.

Proof of Claim Let A be any compact open subset of W_w with $A \notin w$. Since w is the limit of the sequence $\{w_n : n \in \omega\}$, choose n_0 so that $w_n \notin A$ for all $n > n_0$. For each $n > n_0$, choose m_n so that $z(n, \sigma_n, m) \in W_w \setminus A$ for all $m > m_n$ by item (3). Choose $\alpha_0 < \omega_1$ so that for each $n > n_0$ and $m > m_n$, $x(n, \sigma_n, m, \alpha) \in W_w \setminus A$ for all $\alpha \geq \alpha_0$ by item (4). It then follows that $\{y(n, \sigma_n, \alpha) : \alpha_0 < \alpha\} \subset W_w \setminus A$ for all $n > n_0$. Note that w_n is in the sequential closure of the set $\{y(n, \sigma, \alpha) : \gamma \leq \alpha\}$ and α_0 is countable. Now it follows that, for each $n > n_0$, $D(n, \sigma_n^\frown \langle \gamma \rangle)$ is contained in $W_w \setminus A$ for all but countably many γ. Since each v_γ is in the closure of $\bigcup\{D(n, \sigma_n^\frown \langle \gamma \rangle) : n_0 < n \in I\}$, we have that $v_\gamma \in W_w \setminus A$ for all but countably many γ. $\square$

Now let Σ denote the set of sequences $\vec{\sigma} = \{\sigma_n : n \in \omega\}$ from $(\omega_1^{<\omega})^\omega$ that are elements of M and that satisfy that the sequence $\{\mathrm{dom}(\sigma_n) : n \in \omega\}$ diverges to infinity. We will be examining the properties of members of $V(\vec{\sigma}, \gamma)$ for elements $\vec{\sigma}$ of Σ and $\gamma \in \omega_1$. For each $\sigma \in \omega_1^{<\omega}$ and each $\gamma \in \omega_1$, let $\vec{\sigma}^\frown \langle \gamma \rangle$ denote the sequence $\{\sigma_n^\frown \langle \gamma \rangle : n \in \omega\}$. Of course $\vec{\sigma}^\frown \langle \gamma \rangle \in \Sigma$ for each $\vec{\sigma} \in \Sigma$.

For each $\vec{\sigma} \in \Sigma$, let $\mu(\vec{\sigma})$ denote the minimum ordinal satisfying that there is a sequence $\{v_\gamma : \gamma \in \omega_1\}$ satisfying that $v_\gamma \in V(\vec{\sigma}, \gamma)$ and $\rho(v_\gamma) \leq \mu(\vec{\sigma})$ for all $\gamma \in \omega_1$. We may assume there is a bound $\bar{\mu} \in \omega_1$ such that $\mu(\vec{\sigma}) \leq \bar{\mu}$ for all $\vec{\sigma} \in \Sigma$ because it follows from Claim 2 that $\rho(w)$ is at least as large as $\mu(\vec{\sigma})$ for any $\vec{\sigma} \in \Sigma$.

Define the natural ordering $<^*$ on Σ to mean, for $\vec{\sigma}, \vec{\tau} \in \Sigma$, $\vec{\sigma} <^* \vec{\tau}$ providing $\sigma_n \subset \tau_n$ for all but finitely many n. Now let us define $\mu^*(\vec{\sigma})$ to be the minimum of the set $\{\mu(\vec{\tau}) : \vec{\sigma} <^* \vec{\tau} \in \Sigma\}$.

Claim 3 *For each* $\vec{\sigma} \in \Sigma$, *there is a* γ_0 *so that* $\mu(\vec{\sigma}) \leq \mu^*(\vec{\sigma}^\frown \langle \gamma \rangle)$ *for all* $\gamma_0 < \gamma \in \omega_1$.

Proof of Claim It follows from the definitions of $\mu(\vec{\sigma}^\frown \langle \gamma \rangle)$ and $\mu^*(\vec{\sigma}^\frown \langle \gamma \rangle)$ that we can choose some $v_\gamma \in V(\vec{\sigma}, \gamma)$ with $\rho(v_\gamma) = \mu^*(\vec{\sigma}^\frown \langle \gamma \rangle)$. Since each is an element of ω_1, there is some $\xi < \omega_1$ such that $\rho(v_\gamma) = \xi$ for cofinally many $\gamma \in \omega_1$. Of course, by definition of $\mu(\vec{\sigma})$, we have that $\mu(\vec{\sigma}) \leq \xi$. $\square$

Claim 4 *There is a* $\vec{\sigma} \in \Sigma$ *such that* $\mu^*(\vec{\sigma}) = \mu(\vec{\sigma})$.

Proof of Claim If $\vec{\sigma}, \vec{\tau}$ are in Σ, and $\vec{\sigma} <^* \vec{\tau}$, then $\mu^*(\vec{\sigma}) \leq \mu^*(\vec{\tau})$. Now define $\beta(\vec{\sigma})$ to be the minimum ordinal such that $\mu^*(\vec{\tau}) \leq \beta(\vec{\sigma})$ for all $\vec{\tau} \in \Sigma$ such that $\vec{\sigma} <^* \vec{\tau}$. It follows that if $\vec{\sigma} <^* \vec{\tau}$, then $\beta(\vec{\tau}) \leq \beta(\vec{\sigma})$. For this reason we can choose $\vec{\sigma}$ so that $\beta(\vec{\sigma})$ is minimal.

If there is a $\vec{\tau}$ such that $\vec{\sigma} <^* \vec{\tau}$ and $\mu^*(\vec{\tau}) = \beta(\vec{\sigma})$, then for almost all γ, $\beta(\vec{\sigma}) = \mu^*(\vec{\tau}) \leq \mu(\vec{\tau}) \leq \mu^*(\vec{\tau}^\frown\langle\gamma\rangle) = \beta(\vec{\sigma})$.

To prove there is a $\vec{\sigma} <^* \vec{\tau}$, so that $\mu^*(\vec{\tau}) = \beta(\vec{\sigma})$ we use the idea that ω-chains in $<^*$ are bounded. The only complication to this is that we are restricted to chains in M. Choose an increasing sequence $\{\beta_n : n \in \omega\} \in M_0$ cofinal in $\beta(\vec{\sigma})$. Using cofinality, clearly $H(\kappa)$ models that there is a function $f : \omega \to \Sigma$ such that, for each $n \in \omega$, $\vec{\sigma} <^* f(n) <^* f(n+1)$ and satisfying that $\beta_n < \mu^*(f(n))$. Since all $f(n) \in \Sigma$ and Σ consists of elements from $M = \bigcup_\gamma M_\gamma$, one can choose an $\alpha \in \omega_1$ so that $f(n) \in M_\alpha$ for all n. Now we work in $M_{\alpha+1}$. Set $\Sigma_\alpha = \Sigma \cap M_\alpha$, which is an element of $M_{\alpha+1}$. Using $J_\alpha \in M_\alpha$ as a parameter, we define, for $\vec{\tau} \in \Sigma_\alpha$ and $\gamma \in \omega_1$, $V(\vec{\tau}, \gamma, J_\alpha)$ to be all those $v \in X \cap M_\alpha$ satisfying that for each compact open $A \subset W_v \setminus \{v\}$ there is an n_0 such that for all $n_0 < n \in J_\alpha$, the set $D(n, \tau_n^\frown\langle\gamma\rangle) \cap (W_v \setminus A)$ is not empty. It is easily checked, by elementarity, that if $v \in M_\alpha$ then $v \in V(\vec{\tau}, \gamma)$ if and only if $v \in V(\vec{\tau}, \gamma, J_\alpha)$. We can make use of the fact that $V(\vec{\tau}, \gamma, J_\alpha)$ is in $M_{\alpha+1}$. Similarly, we can define $\Sigma_\alpha, V(\vec{\tau}, \gamma, J_\alpha)$-versions of $\mu(\vec{\tau}; J_\alpha)$ and $\mu^*(\vec{\tau}; J_\alpha)$. It also follows from elementarity of M_α that these will equal the original values. Now $H(\kappa)$ recognizes that there is a $<^*$-increasing sequence from $\Sigma \cap M_\alpha$, namely f, such that $\vec{\sigma} < *f(0)$ and $\beta_n < \mu^*(f(n); J_\alpha)$ for each n. Thus if we follow inductive method, by f elementarity, in the next step again there is a function $g \in M_{\alpha+1}$ with this same property. We define $\vec{\tau} \in \Sigma$ so that $g(n) <^* \vec{\tau}$ for all n. It follows now that $\beta_n < \mu^*(g(n)) \leq \mu^*(\vec{\tau})$ for all n. This implies that $\mu^*(\vec{\tau}) = \beta(\vec{\sigma})$ as required. □

For each $\gamma < \delta \in \omega_1$ we may choose $v_{\gamma,\delta} \in V(\vec{\sigma}^\frown\langle\gamma, \delta\rangle)$ such that $\rho(v_{\gamma,\delta}) = \mu(\vec{\sigma})$. Recall that, by Claim 2, $\{v_{\gamma,\delta} : \delta \in \omega_1\}$ converges to w. For each $\gamma < \delta$ and n, choose $d(n, \gamma, \delta) \in D(n, \sigma_n^\frown\langle\gamma, \delta\rangle)$ so that for all but finitely many $n \in I$, $d(n, \gamma, \delta) \in W_{v_{\gamma,\delta}}$. Recall that there is a $\alpha = \alpha(n, \gamma, \delta)$ so that $d(n, \gamma, \delta) = y(n, \sigma_n, \alpha)$. Fix such a value $\alpha(n, \gamma, \delta)$ for each $n \in \omega$ and $\gamma < \delta \in \omega_1$. Recall that when γ_δ was in the cub defining $D(n, \sigma_n^\frown\langle\gamma, \delta\rangle)$ we have that $\gamma_\delta \leq \alpha(n, \gamma, \delta)$. Therefore the sequence $\{x(n, \sigma_n^\frown\langle\gamma\rangle, m, \alpha(n, \gamma, \delta)) : \delta \in \omega_1\}$ converges to $z(n, \sigma^\frown\langle\gamma\rangle, m)$.

Choose a function $f_{\gamma,\delta} \in \omega^\omega$ so that, for all $n \in I$ such that $d(n, \gamma, \delta) \in W_{v_\gamma}$, the sequence $\{x(n, \sigma_n, m, \alpha) : m \in I \setminus f_{\gamma,\delta}(n)\}$ (here $f_{\gamma,\delta}(n)$ denotes the nth-component of the function $f_{\gamma,\delta}$) is contained in W_{v_γ}. Since PFA implies that $\omega_1 < \mathfrak{b}$, $\mathfrak{b}$ is the minimal cardinality of an unbounded family in ω^ω, there is a function $h \in \omega^\omega$ such that $f_{\gamma,\delta} <^* h$ for all $\gamma < \delta \in \omega_1$.

For each $\gamma \in \omega_1$, choose a point $w_{\gamma,h} \in X$ that is the limit of a converging subsequence of $\{z(n, \sigma_n^\frown\langle\gamma\rangle, h(n)) : n \in I\}$. Note that $w_{\gamma,h} \in D^{(\omega)}$; hence $w_{\gamma,h} \neq w$. Since $\{v_{\gamma,\delta} : \gamma < \delta \in \omega_1\}$ converges to w, there is a δ_0 such that $v_{\gamma,\delta} \notin W_{w_{\gamma,h}}$ for $\gamma > \delta_0$. Using that, for each n, the sequence $\{x(n, \sigma^\frown\langle\gamma\rangle, h(n), \alpha) : \alpha \in \omega_1\}$ converges to $z(n, \sigma_n^\frown\langle\gamma\rangle, h(n))$, choose a δ_γ such that $v_{\gamma,\delta_\gamma} \notin W_{w_{\gamma,h}}$, and for infinitely many $n \in I$, $x(n, \sigma^\frown\langle\gamma\rangle, h(n), \alpha(n, \gamma, \delta_\gamma))$ is in $W_{w_{\gamma,h}}$. Let v'_γ denote a point in $W_{w_{\gamma,h}} \cap W_{v_{\gamma,\delta_\gamma}}$ that is a limit of the sequence $\{x(n, \sigma^\frown\langle\gamma\rangle, h(n), \alpha(n, \gamma, \delta_\gamma)) : n \in I\}$. Although h need not have been chosen from M, and so there is no guarantee that v'_γ is from $V(\vec{\sigma}, \gamma)$, we can use the fact that $v_{\gamma,\delta_\gamma} = v_\gamma$ is in $V(\vec{\sigma}, \gamma)$ and apply elementarity of M to assert that, for each $\gamma \in \omega_1$, there is a pair $v_\gamma, v'_\gamma \in V(\vec{\sigma}, \gamma)$ such that $v_{\gamma'} \in W_{v_\gamma} \setminus \{v_\gamma\}$ and $\rho(v_\gamma) = \mu(\vec{\sigma})$. However, we now have a contradiction. By Claim 2, the sequence $\{v'_\gamma : \gamma \in \omega_1\}$ fulfills the defining conditions

of $\mu(\vec{\sigma})$, and so the value of $\rho(v'_\gamma)$ is at least $\mu(\vec{\sigma})$ for uncountably many γ. On the other hand, for such a γ, $\rho(v'_\gamma) < \rho(v_\gamma) = \mu(\vec{\sigma})$. This shows there is no countable bound on $\mu(\vec{\tau})$ for $\vec{\tau} \in \Sigma$ and completes the proof. $\square$

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