Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2018) 42: 1242 – 1254

doi:10.3906/mat-1704-4

Just non-Artinian modules over some group rings

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Received: 03.04.2017 • **Accepted/Published Online:** 24.10.2017 • **Final Version:** 08.05.2018

Abstract: Let D be a Dedekind domain and G be a periodic Abelian-by-finite group. In this paper we study DG-modules in which every factor-module, apart from the trivial one, is DG-Artinian. In particular we prove that such modules cannot be D-periodic and that G must be subject to some restrictions. Finally, we give a detailed description of such modules when G is periodic Abelian and the spectrum of D is infinite.

Key words: Almost Artinian module, Dedekind domain

1. Introduction

Artinian modules are one of the classical objects of study in algebra. Their applications have been decisive in studying finiteness conditions in groups, modules, Lie algebras, and other algebraic structures. In particular, the knowledge of some details about the structure of Artinian modules over group rings is often necessary to study generalized soluble groups. Nowadays, the theory of Artinian modules over group rings is very well developed; it is rich in many important results and it has its own goals and themes of different nature. Many famous algebraists contributed to this theory (see, for instance, [8], where some results on this subject are collected). Certainly, the structure of an Artinian module A over some group ring RG depends essentially on the structure of the group G. Artinian DG-modules, where D is a Dedekind domain, were fully described only when G is a periodic Abelian-by-finite group of finite section rank (see [8, Chapter 12]). For periodic groups, which are not Abelian-by-finite, the situation is very complicated. In fact, if G is a countable 2-group of exponent 4 such that $[G, G] = Z(G)$ is a group of order 2, and p is an odd prime, then there exists an Artinian uncountable $\mathbb{F}_pG$-module (see [2]). Furthermore, if G is a Čarin group and $p \notin \Pi(G)$, then there also exists an Artinian uncountable $\mathbb{F}_pG$-module (see [2]). On the other hand, if G is an Abelian nonperiodic group such that $|G/G^p| \geq p^2$ and F is a field, then the problem of the description, with respect to representation theory, of an Artinian FG-module is a "wild" one when $p = char(F)$ (see [6]).

Let R be a ring. If A is a module over R and B is a nonzero R-submodule of A, then we call *proper* the factor-module A/B. We say that an R-module A is *just non-Artinian* if A is not Artinian as R-module, but each proper factor-module of A is Artinian as R-module.

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2010 *AMS Mathematics Subject Classification:* 16P20, 20C07
This work was supported by the "National Group for Algebraic and Geometric Structures and their Applications" (GNSAGA-INdAM) and the ADV-AGTA, project of which the second author is a member. The second author would like to thank Oles Honchar Dnipro National University for its hospitality while part of this work was being done.

In [3], where such modules were called *almost Artinian*, a description of the structure of just non-Artinian modules over a Dedekind domain was obtained. Later, in [4], the structure of a just non-Artinian module over a Noetherian domain of dimension 1 was obtained.

Just non-Artinian modules are partially connected with *just infinite modules*, that is, infinity modules whose proper factor-modules are finite (see [7, Part II]). In [10] FG-modules were considered, where F is a field, whose proper factor-modules have finite dimension over F. Another partial type of almost Artinian modules was considered in [12].

In this paper we will study just non-Artinian modules over group rings DG, where D is a Dedekind domain. As we have seen above, the description of Artinian DG-modules has been obtained for periodic Abelian-by-finite groups. Therefore, it is natural to study just non-Artinian DG-modules with the condition that G is a periodic Abelian-by-finite group.

Let A be a module over a ring R and put

$$Tor_R(A) = \{a \in A \mid Ann_R(a) \neq \langle 0 \rangle\}.$$

It is not hard to prove that, if R is an integral domain, then $Tor_R(A)$ is an R-submodule of A, which is called the *R-periodic part* of A. An R-module A is called *R-periodic* if $A = Tor_R(A)$, while it is called *R-torsion-free* if $Tor_R(A) = \langle 0 \rangle$.

As we will see later, the study of just non-Artinian DG-modules A splits into two parts: analyzing the case in which $Ann_D(A)$ is a maximal ideal of D and that in which A is D-torsion-free. In the first case, we can consider A as an FG-module where F is a field and we obtain the following result.

Theorem A *Let F be a field, G be a group such that $G/C_G(A)$ is a periodic Abelian-by-finite group, and A be an FG-module. If every proper factor-module of A is Artinian, then A is an Artinian FG-module.*

If R is a commutative ring, the *prime spectrum, $Spec(R)$*, or just the *spectrum*, of R is the set of all prime ideals of R. Let A be a module over R.

Let I be an ideal of R. Put

$$A_I = \{a \in A \mid aI^n = \langle 0 \rangle \text{ for some } n \in \mathbb{N}\}.$$

Clearly A_I is an R-submodule of A, and it is called the *I-component* of A. If $A = A_I$, then A is called an *I-module.*

We now define the *R-assassinator of A* as the set

$$Ass_R(A) = \{P \mid P \text{ is a nonzero prime ideal of } R \text{ such that } Ann_A(P) \neq \langle 0 \rangle\}.$$

If D is a Dedekind domain and A is a D-module, then

$$Tor_D(A) = \bigoplus_{P \in \pi} A_P$$

where $\pi = Ass_R(A)$ (see, for example, [9, Corollary 3.8]). Finally, note that in Dedekind domains every nonzero prime ideal is maximal.

We can now describe the structure of a D-torsion-free just non-Artinian DG-module A, when G is a periodic Abelian group (see Proposition 15). Let F be the field of quotients of D and K be an algebraic closure of F. We can find in $\mathbb{U}(K)$ and isomorphic copy G^* of $G/C_G(A)$, which must be a locally cyclic p'-group, with $p = char(D)$. Put $U = D[G^*]$. We consider K as a DG-module with the action defined by right multiplication of corresponding elements in $D[G^*]$. Let $T/U = Tor_D(K/U)$. There is a finite subset $\pi \subseteq Ass_D(K/U)$ such that A is isomorphic to some DG-submodule of T_π, where T_π/U is the π-component of T/U.

If G is a periodic Abelian-by-finite group, then the following theorem gives a reduction to the case of Abelian groups.

Theorem B *Let D be a Dedekind domain with infinite spectrum, G be an Abelian-by-finite group with finite 0-rank, and A be a DG-module. Suppose that A is a just non-Artinian DG-module that is D-torsion-free and such that $C_G(A) = \langle 1 \rangle$.*
Let H be a normal subgroup of G of finite index and let X be a transversal to H in G such that $1 \in X$. Then A includes a DH-submodule T such that A/Tx is a just non-Artinian DH-module for every x in X. Moreover,

$$\bigcap_{x \in X} Tx = \langle 0 \rangle$$

and

$$\bigcap_{x \in X} x^{-1} C_H(A/T) x = \langle 1 \rangle,$$

so that A is isomorphic to some DH-submodule of

$$\bigoplus_{x \in X} A/Tx,$$

and H is isomorphic to some subgroup of

$$\underset{x \in X}{Dr} (H/(x^{-1} C_H(A/T) x)).$$

A more detailed description of the structure of just non-Artinian DG-modules, with G periodic Abelian, is given by the following theorems.

Let R be an integral domain and G be a group. An RG-module A is called *almost RG-irreducible* if the factor-module A/B is R-periodic for every nonzero RG-submodule B of A.

Theorem C *Let D be a Dedekind domain of characteristic p with infinite spectrum, G be a periodic Abelian group, and A be a just non-Artinian DG-module. Then the following conditions hold:*

(i) *$G/C_G(A)$ is a locally cyclic p'-group;*

(ii) *A is D-torsion-free;*

(iii) *A is almost DG-irreducible;*

(iv) *A includes a cyclic (and DG-Noetherian) DG-submodule C such that A/C is D-periodic and the set $Ass_D(A/C)$ is finite;*

(v) *A/C is DG-Artinian and it is a direct sum of finitely many monolithic DG-submodules;*

(vi) *there exists a finite subset* π *of* $Spec(D)$ *such that* $A \neq AP$ *for all* $P \notin \pi$;

(vii) *if* $P \notin \pi$ *and* $p > 0$, *then* A/AP *is a direct sum of simple non-isomorphic* DG*-modules.*

A Dedekind domain D is said to be a *Dedekind* Z*-module* (see, for example, [7, p. 133]) if $Spec(D)$ is infinite and the field D/P is locally finite for each $\langle 0\rangle \neq P \in Spec(D)$.

Theorem D *Let* D *be a Dedekind* Z*-domain of characteristic* 0, G *be a periodic Abelian group, and* A *be a just non-Artinian* DG*-module. Then the following conditions hold:*

(i) $G/C_G(A)$ *is locally cyclic;*

(ii) A *is* D*-torsion-free;*

(iii) A *is almost* DG*-irreducible;*

(iv) *there exists a finite subset* π *of* $Spec(D)$ *such that* $A \neq AP$ *for all* $P \notin \pi$;

(v) *if* B *is a nonzero submodule and* $P \in Ass_D(A/B)$, *then the Sylow* p*-subgroups of* $G/C_G(A)$ *are cyclic, where* $char(D/P) = p$;

(vi) A *includes a nonzero cyclic (and* DG*-Noetherian)* DG*-submodule* C *such that* A/C *is* D*-periodic and the set* $Ass_D(A/C)$ *is finite;*

(vii) *a Sylow* p*-subgroup of* $G/C_G(A_P/C)$, *with* $p = char(D/P)$ *and* $P \in Ass_D(A/C)$, *is cyclic;*

(viii) A_P/C *is a direct sum of finitely many monolith* DQ*-submodules, where* $Q/C_G(A_P/C)$ *is a Sylow* p'*-subgroup of* $G/C_G(A_P/C)$, *where* $p = char(D/P)$.

2. D-Periodic just non-Artinian DG-modules

The following lemmas with their consequences are standard and their proof can omitted.

Lemma 1 *Let* R *be a ring and* A *be a just non-Artinian* R*-module.*

(i) *If* B *is a nonzero submodule of* A, *then* B *is a just non-Artinian* R*-module and in particular* B *is not Artinian.*

(ii) *If* B *and* C *are nonzero submodules of* A, *then* $B \cap C \neq \langle 0\rangle$.

Lemma 2 *Let* R *be a ring and* A *be a just non-Artinian* R*-module. If* f *is a nonzero* R*-endomorphism of* A, *then* f *is a monomorphism.*

Corollary 3 *Let* R *be a ring and* A *be a just non-Artinian* R*-module. Then the ring* $End_R(A)$ *has no zero-divisors.*

Corollary 4 *Let* R *be a ring and* A *be a just non-Artinian* R*-module. Furthermore, let* $I = Ann_R(A)$ *and* C/I *be the center of* R/I. *Then* C/I *is an integral domain.*

Corollary 5 *Let R be a domain, G be a group, and A be a just non-Artinian RG-module. Then the center $Z(G/C_G(A))$ of $G/C_G(A)$ is isomorphic to the multiplicative group of some field. In particular, $Tor(Z(G/C_G(A)))$ is a locally cyclic p'-group, where $p = char(R)$.*

Proof Put $Z/C_G(A) = Z(G/C_G(A))$. For each element $z \in Z$ define the mapping $\mu_z : A \longrightarrow A$ by the rule $\mu_z(a) = az$, $a \in A$. It is not hard to see that μ_z is an RG-automorphism of A. Consider now the function $f : Z \longrightarrow Aut_{RG}(A)$, defined by the rule $f(z) = \mu_z$, $z \in Z$. If $z, y \in Z$, then $zy = xyz$ for some element $x \in C_G(A)$. We then have

$$\mu_{zy}(a) = a(zy) = a(xyz) = (ay)z = \mu_z(ay) = \mu_z(\mu_y(a)) = (\mu_z \cdot \mu_y)(a),$$

so that $f(zy) = f(z) \cdot f(y)$. It follows that f is a homomorphism. Clearly, $Ker(f) = C_G(A)$ and the center C of $End_{RG}(A)$ includes $Im(f)$. By Corollary 3, $End_{RG}(A)$ has no zero-divisors. It follows that C is an integral domain. In this case C can be embedded in a field K. Thus, $Z(G/C_G(A))$ is isomorphic to a subgroup of the multiplicative group of K. Note that, if $char(R) > 0$, then

$$char(K) = char(C) = char(End_{RG}(A)) = char(R).$$

The last part of the statement follows from [5, Chapter 4, Proposition 4.1]. □

Corollary 6 *Let R be a ring, G be a group, and A be a just non-Artinian RG-module. If $gC_G(A)$ is a nontrivial element of $Z(G/C_G(A))$, then $C_A(g) = \langle 0 \rangle$.*

Proof Consider the mapping $\xi_g : A \longrightarrow A$, defined by the rule $\xi_g(a) = a(g-1)$, $a \in A$. The choice of g implies that ξ_g is an RG-endomorphism. Lemma 2 shows that ξ_g is an RG-monomorphism. In particular, $C_A(g) = Ann_A(g-1) = Ker(\xi_g) = \langle 0 \rangle$. □

Corollary 7 *Let R be a ring, G be a group, and A be a just non-Artinian RG-module. If p is a prime in $\Pi(Z(G/C_G(A)))$, then $A_p = \{a \in A \,|\, p^n a = 0 \text{ for some } n \in \mathbb{N}\} = \langle 0 \rangle$.*

Proof Let $gC_G(A)$ be a nontrivial p-element of $Z(G/C_G(A))$. Suppose that $A_p \neq \langle 0 \rangle$ and let $0 \neq b \in A_p$ and $B = b\mathbb{Z}\langle gC_G(A) \rangle$. Then the natural semidirect product $B \rtimes \langle gC_G(A) \rangle$ is a finite p-group. It follows that $\langle 0 \rangle \neq Z(B \rtimes \langle gC_G(A) \rangle) \cap B = C_B(g)$, and we obtain a contradiction with Corollary 6. □

Proposition 8 *Let D be a Dedekind domain with infinite spectrum, G be a locally (polycyclic-by-finite) group of finite 0-rank, and A be a just non-Artinian DG-module. Then either A is a P-module for some maximal ideal P of D or A is D-torsion-free.*

Proof Suppose that $T = Tor_D(A) \neq \langle 0 \rangle$. If we assume that $Ass_D(A)$ contains two distinct maximal ideals P and Q, then $A_P \neq \langle 0 \rangle$ and $A_Q \neq \langle 0 \rangle$. Clearly, A_P and A_Q are DG-submodules of A and they have trivial intersection, contradicting Lemma 1(ii). This contradiction shows that $Ass_D(A) = \{P\}$.

Suppose now that $T \neq A$. Then A/T is an Artinian DG-module that is D-torsion-free. Being Artinian, A/T includes a nonzero simple DG-submodule M/T. On the other hand, [7, Corollary 1.16] shows that $Ann_D(M/T) \neq \langle 0 \rangle$, from which it follows that M is D-periodic, a contradiction that proves the result. □

Let D be a Dedekind domain and A be a D-module. Suppose that A is a P-module for some maximal ideal P of D. Put

$$\Omega_{P,\,n}(A) = \{a \in A \,|\, Ann_D(a) \geq P^n\}.$$

Clearly, $\Omega_{P,\,n}(A)$ is a D-submodule of A, $\Omega_{P,\,n}(A) \leq \Omega_{P,\,n+1}(A)$ for each $n \in \mathbb{N}$ and

$$A = \bigcup_{n\in\mathbb{N}} \Omega_{P,\,n}(A).$$

Lemma 9 *Let D be a Dedekind domain, G be a group, and A be a just non-Artinian DG-module. If A is a P-module for some maximal ideal P of D, then $A = \Omega_{P,\,1}(A)$.*

Proof Suppose the contrary, and then $A_1 = \Omega_{P,\,1}(A) \neq \Omega_{P,\,2}(A) = A_2$. We have $A_2P \leq A_1$. Since P cannot be $\langle 0\rangle$, we can choose an element $y \in P\backslash P^2$. Then $A_2P = A_2y$ [9, Proposition 6.2]. Consider the mapping $\rho : A_2 \longrightarrow A_1$ defined by the rule $\rho(a) = ay$, $a \in A_2$. Clearly, this mapping is a DG-endomorphism. Since $A_2 \neq A_1$, ρ is nonzero, therefore, by Lemma 2, ρ it is a monomorphism. On the other hand, $Ker(\rho) = A_1 \neq \langle 0\rangle$. This contradiction shows that $A = A_1 = A_2$. □

In the hypotheses of the above lemma, the equality $A = \Omega_{P,\,1}(A)$ means that $Ann_D(A) = P$ is a maximal ideal of D. In this case, we can consider A as an FG-module, where $F = D/P$.

Lemma 10 *Let F be a field, G be a group such that $G/C_G(A)$ is a periodic Abelian group, and A be an FG-module. Then A cannot be a just non-Artinian FG-module.*

Proof Let A be a just non-Artinian FG-module. Without loss of generality we may assume that $C_G(A) = \langle 1\rangle$. Then, by Corollary 5, G is locally cyclic. Thus, G has an ascending series of cyclic subgroups

$$\langle 1\rangle \leq \langle g_1\rangle \leq \ldots \leq \langle g_n\rangle \leq \langle g_{n+1}\rangle \leq \ldots \bigcup_{n\in\mathbb{N}} \langle g_n\rangle = G.$$

Let $\langle x\rangle$ be an infinite cyclic group and $J = F\langle x\rangle$ be the group algebra of $\langle x\rangle$ over the field F. Letting $j \in \mathbb{N}$, we can consider A as a JG-module if we define the action of x on A by the rule $ax = ag_j$, $a \in A$. Since g_j has a finite order, A is a periodic J-module. Since J is a principal ideal domain with infinite spectrum, we can apply Proposition 8 and Lemma 9 and get that $Ann_J(A) = M$ is a maximal ideal of J. Therefore,

$$A = \bigoplus_{\lambda\in\Lambda} B_\lambda,$$

where $B_\lambda \simeq_J J/M$, $\lambda \in \Lambda$.

Let $0 \neq b \in A$, $B_j = bF\langle g_j\rangle$, $j \in \mathbb{N}$, and $B = bFG$. We have that B_j is a simple $F\langle g_j\rangle$-module [8, Proposition 4.5] for every $j \in \mathbb{N}$ and that

$$B = \bigcup_{j\in\mathbb{N}} B_j.$$

Let C be a nonzero FG-submodule of B. Then there is a positive integer k such that $C \cap B_k \neq \langle 0\rangle$. Then $C \cap B_k$ is a nonzero $F\langle g_k\rangle$-submodule of B_k. The fact that B_k is a simple $F\langle g_k\rangle$-module implies that

$C \cap B_k = B_k$, i.e. $B_k \leq C$. Therefore, $B_j \leq C$ for each $j \geq k$. Thus, $B = C$, or, in other words, B is a simple FG-submodule of A. Since A/B is an Artinian FG-module, A must be FG-Artinian, and we obtain a contradiction that proves the result. □

Let R be a ring, and let A be an R-module. The submodule $Soc_R(A)$ generated by all the minimal R-submodules of A is called the *R-socle* of A. If A has no such minimal submodules, we define $Soc_R(A) = \langle 0 \rangle$. We note that $Soc_R(A)$ is a direct sum of simple R-submodules.

Starting from the socle we define the *upper socular series of A* as

$$\langle 0 \rangle = S_0 < S_1 < \dots S_\alpha < S_{\alpha+1} < \dots S_\gamma$$

where $S_1 = Soc_R(A)$, $S_{\alpha+1}/S_\alpha = Soc_R(A/S_\alpha)$ for any ordinal α,

$$S_\lambda = \bigcup_{\beta<\lambda} S_\beta$$

for all limit ordinals λ, and $Soc_R(A/S_\gamma) = \langle 0 \rangle$. The ordinal γ is called the *socular height of A*.

Proposition 11 *Let R be a commutative ring, G be an Abelian-by-finite group, and A be an Artinian RG-module. If A is finitely generated, then A has a finite RG-composition series.*

Proof Take an arbitrary element $b \in A$ and consider the cyclic RG-submodule bRG. Let H be an Abelian normal subgroup of G having finite index. Denote by T a transversal to H in G. Then

$$RG = \bigoplus_{x \in T} x(RH).$$

It follows that

$$bRG = \sum_{x \in T} (bx)RH.$$

Since A is RH-Artinian (see, for example, [8, Theorem 5.2]), also

$$(bx)RH \simeq RH/Ann_{RH}(bx)$$

is RH-Artinian. However, RH is a commutative ring, and so $RH/Ann_{RH}(bx)$ is Noetherian [11, Theorem 8.44]. Therefore, bRG is both RH-Noetherian and RH-Artinian, since T is finite. Hence, bRG is both RG-Noetherian and RG-Artinian, and so it has a finite RG-composition series.

Let $a_1, \dots, a_n$ be elements of A such that $A = a_1RG + \dots + a_nRG$. By what we have proved above, each a_jRG has a finite RG-composition series, for $1 \leq j \leq n$. It follows that A has a finite RG-composition series. □

Corollary 12 *Let R be a commutative ring, G be an Abelian-by-finite group, and A be an Artinian RG-module. Then A has socular height at most ω, the first infinite ordinal.*

Proof Let

$$\langle 0 \rangle = S_0 \leq S_1 \leq \dots S_\alpha \leq S_{\alpha+1} \leq S_\gamma$$

be the upper socular series of A. If a is an arbitrary element of A, then by Proposition 11, the cyclic RG-submodule aRG has a finite RG-composition series of length, say, n. Then, clearly, $aRG \leq S_n$ and $A = \bigcup S_n = S_\omega$, where n, in the union, ranges over all natural numbers. □

Proof of Theorem A — Without loss of generality we may assume that $C_G(A) = \langle 1 \rangle$. Let H be a normal Abelian subgroup of G of finite index and let T be a transversal to H in G. Let $0 \neq b \in A$ and $B = bFG$. If B is FG-Artinian, then, since A/B is FG-Artinian, also A is FG-Artinian. Thus, we may assume that B is not FG-Artinian, and, in particular, B is not a simple FG-module. Then B includes a nonzero proper RG-submodule C. Being Artinian and finitely generated, B/C has a finite FG-composition series by Proposition 11. Thus, B is FG-Noetherian and hence also FH-Noetherian (see [8, Theorem 5.3]).

Put $\mathfrak{D}$ to be the set of all RH-submodules of B such that B/U has no finite FH-composition series. This set is not empty, since $\langle 0 \rangle \in \mathfrak{D}$. Since B is a Noetherian FH-module, then we can choose a maximal element in $\mathfrak{D}$, say M. Therefore, every proper FH-factor-module of B/M has finite FH-composition series, i.e. B/M is a just non-Artinian FH-module. Then, by Lemma 10, we get a contradiction that proves the statement. □

3. D-Torsion-free just non-Artinian DG-modules

Let D be a Dedekind domain with infinite spectrum, G be a periodic Abelian-by-finite group, and A be a just non-Artinian DG-module. Then Proposition 8, Lemma 9, and Theorem A show that A must be D-torsion-free. Therefore, we will now study the D-torsion-free case. Theorem B will allow us to reduce to the case in which G is periodic Abelian.

Proof of Theorem B — Let $0 \neq c \in A$ and $C = cDG$. Then C is a just non-Artinian DG-module. Take a nonzero DG-submodule of C, say B. Then C/B is a DG-Artinian module and by Proposition 11 it follows that C/B has a finite DG-composition series. Therefore, C is DG-Noetherian. Put

$$M = \{U \,|\, U \text{ is a } DH\text{-submodule of } C \text{ such that } C/U \text{ has no finite } DH\text{-composition series}\}.$$

Since C is not DG-Artinian, it cannot be also DH-Artinian, so $\langle 0 \rangle \in M$. Hence, M has a maximal element, say B. Clearly, the intersection

$$\bigcap_{x \in X} Bx$$

is a DG-submodule and so it must be $\langle 0 \rangle$.

Let $T/B = Tor_D(A/B)$. Clearly T is a DH-submodule of A. Suppose that $T_0 = T \cap C \neq B$, and then C/T_0 has a finite DH-composition series and therefore C/T_0 is D-periodic. It follows that C/B is D-periodic. We have $C/Bx = Cx/Bx \simeq_D C/B$; thus, C/Bx is D-periodic for each $x \in X$ and, by Remak's theorem, we obtain that C is D-periodic, a contradiction. This contradiction proves that $T \cap C = B$. It follows that T/B is Artinian as a DH-module and, furthermore,

$$(C + T)/T \simeq_{DH} C/(C \cap T) = C/B.$$

Letting

$$T_1 = \bigcap_{x \in X} Tx,$$

then T_1 is a DG-submodule. Suppose that $T_1 \neq \langle 0 \rangle$. Then A/T is Artinian as DH-module. Since T/B is Artinian as DH-module, the DH-module A/B must be Artinian. Then A/Bx is also Artinian as DH-module and by Remak's theorem we obtain a contradiction. Therefore, $T_1 = \langle 0 \rangle$.

Let E/T be a nonzero DH-submodule of A/T. If $K/T = (E/T) \cap ((C+T)/T)$ is nonzero, then using the isomorphism $(C+T)/T \simeq_{DH} C/B$, we obtain that $(T+C)/K$ has a finite DH-composition series. Since $A/(C+T)$ is Artinian as DH-module, A/K is Artinian as DH-module. Then

$$A/E \simeq (A/K)/(E/K)$$

is Artinian as DH-module.

Suppose now that the intersection $(E/T) \cap ((C+T)/T) = \langle 0 \rangle$. It follows that E/T is Artinian as DH-module and so it is D-periodic. On the other hand, the choice of T yields that A/T is D-torsion-free. This last contradiction proves that A/T is a just non-Artinian DH-module.

Finally, $C_H(A/Bx) = x^{-1}C_H(A/B)x$ for every $x \in X$, so

$$\bigcap_{x \in X} x^{-1}C_H(A/B)x = \bigcap_{x \in X} C_H(A/Bx) = C_H(A) = \langle 1 \rangle,$$

and Remak's theorem gives the last embedding. □

Proposition 13 *Let D be a Dedekind domain with infinite spectrum, G be a periodic Abelian group, and A be a just non-Artinian DG-module. Then A is D-torsion-free.*

Proof Suppose that A is not D-torsion-free. Then Proposition 8 and Lemma 9 show that $Ann_D(A) = P$ is a maximal ideal of D. Then we can consider A as an FG-module, where $F = D/P$. Now we can apply Lemma 10 and obtain a contradiction. □

Lemma 14 *Let D be a Dedekind domain with infinite spectrum, G be a group such that $G/C_G(A)$ is locally (polycyclic-by-finite) with finite 0-rank, and A be a just non-Artinian DG-module. Then A is almost DG-irreducible.*

Proof Let $0 \neq b \in A$ and $B = bDG$. Then B is a nonzero DG-submodule of A, so that A/B is an Artinian DG-module and hence D-periodic [7, Corollary 1.16]. This shows that A is almost DG-irreducible. □

Proposition 15 *Let D be a Dedekind domain with infinite spectrum, G be a group such that $G/C_G(A)$ is periodic Abelian, and A be a just non-Artinian DG-module. Furthermore, let F be the field of quotients of D and K be the algebraic closure of F. Then there exists a subgroup G^* of the multiplicative group $\mathbb{U}(K)$ of K such that $G^* \simeq G/C_G(A)$. There exists also a DG-submodule $V \simeq_{DG} A$ of K, containing $D[G^*]$, and such that $V/D[G^*]$ is contained in the π-component of $K/D[G^*]$, where π is a finite subset of $Ass_D(K/D[G^*])$.*

Proof Without loss of generality we may assume that $C_G(A) = \langle 1 \rangle$. Using Proposition 13 we have that A is D-torsion-free, so we can suppose that A is contained in the FG-module $B = A \otimes_D F$. Then, for every nonzero FG-submodule C of B, the intersection $C \cap A$ is also nonzero and therefore, by Lemma 14, $A/C \cap A$ is D-periodic. Hence, $B/A \cap C$ is D-periodic, since also B/A is D-periodic. On the other hand, C is an FG-submodule of B and so $B = C$, which shows that B is a simple FG-module.

For each element $x \in FG$ define the mapping $\mu_x : B \longrightarrow B$ by the rule $\mu_x(b) = bx$, $b \in B$. It is not hard to see that μ_x is an FG-endomorphism of B. Consider now the mapping $\Phi : FG \longrightarrow End_{FG}(B)$, defined by the rule $\Phi(x) = \mu_x$, $x \in FG$. It is easy to see that Φ is a ring homomorphism. Put $E = End_{FG}(B)$ and let $0 \neq a \in A$ and $\gamma \in E$. Since B is a simple FG-module, $B = aFG$. Then we have $\gamma(a) = ay$ for some element $y \in FG$. If $b \in B$, then $b = az$ for some element $z \in FG$. Thus,

$$\gamma(b) = \gamma(az) = (ay)z = (az)y = by,$$

which shows that $\gamma = \mu_y$ and $E = \Phi(FG)$.

Let Γ be the restriction of Φ to G. Then $Im(\Gamma) \leq \mathbb{U}(E)$, so we can consider Γ as a monomorphism of G in $\mathbb{U}(E)$. Then the equality $E = \Phi(FG)$ shows that $E = F[\Gamma(G)]$. The fact that B is a simple FG-module implies by Schur's lemma that E is a division ring and hence a field, since it is commutative. Furthermore, since G is periodic, each element μ_g, $g \in G$, is algebraic over F, so that E is an algebraic field extension of F.

Define now the mapping $\Theta : B \longrightarrow E$ by the following rule: if $b \in B$, then $b = ay$ for some $y \in FG$; put $\Theta(b) = \mu_y$. We first show that this mapping is well defined. Indeed, let $b = az$, $z \in FG$. Then $0 = ay - az = a(y - z)$, so that $y - z \in Ann_{FG}(a)$. Since G is Abelian, $Ann_{FG}(a) = Ann_{FG}(B)$. Thus, $0 = \mu_{y-z} = \mu_y - \mu_z$, which shows that $\mu_y = \mu_z$. Thus, the application is well defined.

If $au = d \in B$, $u \in FG$, then $b + d = ay + au = a(y + u)$ and

$$\Theta(b + d) = \mu_{y+u} = \mu_y + \mu_u = \Theta(b) + \Theta(d).$$

If $v \in FG$, then $bv = (ay)v = a(yv)$ and $\Theta(bv) = \mu_{yv} = \mu_y v = \Theta(b)v$. Thus, Θ is an FG-homomorphism. The equality $E = \Phi(FG)$ shows that Θ is also an epimorphism. Finally, the equality $\mu_y = 0$ implies that $y \in Ann_{FG}(B)$, from which it follows that $ay = 0$. Thus, Θ is an FG-isomorphism.

It follows that $A \simeq_{DG} \Theta(A) \leq E$. Putting $A_1 = aDG$, we then have $\Theta(A_1) = D[\Gamma(G)]$. Since A is almost DG-irreducible, $\Theta(A)/D[\Gamma(G)]$ is also D-periodic. It follows that $Ass_D(\Theta(A)/D[\Gamma(G)]) = \pi$ is finite. Thus, $\Theta(A)/D[\Gamma(G)]$ is contained in the π-component of $E/D[\Gamma(G)]$. □

The proof of Proposition 15 shows that cyclic submodules play an important role in the structure of just non-Artinian DG-modules. Now we will consider some details of their structure.

Lemma 16 *Let D be a Dedekind domain, G be a group, and A be a just non-Artinian DG-module. Suppose that A is D-torsion-free. Let $\langle 0 \rangle \neq P \in Spec(D)$ and suppose that A/AP is a cyclic nontrivial module and that $G/C_G(A/AP)$ is a periodic abelian p'-group of finite special rank, where $p = char(D/P)$. Then*

$$A/AP = S_1 \oplus \ldots \oplus S_n,$$

where S_j, S_i are simple nonisomorphic DG-modules for $1 \leq i \neq j \leq n$.

Proof Since A is D-torsion-free, then $AP \neq \langle 0 \rangle$. It follows that A/AP is an artinian DG-module. Then $A/AP = U_1/AP \oplus \ldots \oplus U_n/AP$, where U_j/AP is a simple DG-module for $1 \leq j \leq n$ (see [8, Theorem 12.8]). Suppose that there are distinct indexes l and m such that

$$U_l/AP^k \simeq_{DG} U_m/AP.$$

Put

$$V/AP = \bigoplus_{j \notin \{l,m\}} U_j/AP,$$

then $A/V = W_l \oplus W_m$, where W_l and W_m are simple isomorphic DG-modules. Letting a be an element of A such that $A = aDG$, then $A/V = (a+V)DG$ and so it is simple by [8, Proposition 4.5], a contradiction. □

Proof of Theorem C — The first assertion follows from Corollary 5, the second point from Proposition 13, and the third from Lemma 14. We now prove (iv). Let $0 \neq c \in A$ and $C = cDG$. It is worth noting that C is DG-Noetherian: in fact, every proper quotient is an Artinian DG-module, and so even DG-Noetherian. By (iii) A/C is D-periodic, and so

$$A/C = \bigoplus_{P \in \eta} A_P/C,$$

where $\eta = Ass_D(A/C)$ (see, for example, [9, Corollary 3.8]). Clearly, every P-component A_P/C is a DG-submodule. Therefore, it follows that η must be finite.

The fifth point follows from [8, Theorem 12.8]. Let us move on to prove (vi). By [7, Theorem 1.15] there exists a finite subset σ of $Spec(D)$ such that $C \neq CP$ for all $P \notin \sigma$. Put $\pi = \sigma \cup \eta$ and take $P \in Spec(D) \backslash \pi$. Then $C \neq CP$ and C/CP must coincide with the P-component of A/CP, giving the decomposition $A/CP = C/CP \oplus E/CP$, where $P \notin Ass_D(E/CP)$. Then $(E/CP)P = E/CP$ (see, for example [9, Lemma 6.7]) and it follows that $(A/CP)P = E/CP$. On the other hand,

$$(A/CP)P = (AP + CP)/CP = AP/CP.$$

Thus, we obtain (vi).

Now

$$A/AP \simeq (A/CP)/(AP/CP) = (A/CP)/((A/CP)P) = (A/CP)/(E/CP) \simeq C/CP.$$

Using now Lemma 16, we obtain the last point. □

Let D be a Dedekind domain. As we saw in the introduction, D is said to be a Dedekind Z-module if $Spec(D)$ is infinite and the field D/P is locally finite for each $\langle 0 \rangle \neq P \in Spec(D)$. Clearly, $\mathbb{Z}$ is a Dedekind Z-domain.

Put

$$Spchar(D) = \{char(D/P) \mid P \in Spec(D)\}.$$

We note that, if D is a Dedekind Z-domain of characteristic 0, then $Spchar(D)$ is infinite. Indeed, if $Spchar(D)$ is finite, then we can find an infinite subset π of $Spec(D)$ and a prime p such that $char(D/P) = p$ for all $P \in \pi$. Since D is a Dedekind domain, then

$$\bigcap_{P \in \pi} P = \langle 0 \rangle.$$

It follows that D can be embedded in $\underset{P \in \pi}{Cr}\, D/P$. It is clear that every element of the Cartesian product has characteristic p. Since D is isomorphic to a subring of $\underset{P \in \pi}{Cr}\, D/P$, we obtain a contradiction.

Proposition 17 *Let R be a ring, G be a nilpotent group, and A be an Artinian RG-module. If the additive group of A is a bounded Abelian p-group for some prime p, then each Sylow p-subgroup $P/C_G(A)$ of $G/C_G(A)$ is bounded.*

Proof We can assume that $C_G(A) = \langle 1 \rangle$. We only need to prove that the center of P is bounded, so we may suppose that G is an Abelian p-group. Therefore, $C_A(H)$ is an RG-submodule of A for each subgroup H of G. The fact that A is an Artinian RG-module implies that G includes a finite subgroup K such that $C_A(K) = C_A(J)$ for every finite p-subgroup $J \geq K$. It follows that $C_A(K) = C_A(G)$. Since the natural semidirect product $A \rtimes K$ is nilpotent (see, for example, [1]), A has a finite upper K-central series

$$\langle 0 \rangle = C_0 \leq C_1 \leq \ldots \leq C_n = A,$$

where $C_1 = C_A(K)$, $C_{j+1}/C_j = C_{A/C_j}(K/C_j)$, $1 \leq j \leq n-1$. Since G is Abelian, each term of this series is an RG-submodule of A. Let $a \in C_2 \backslash C_1$ and $g \in G$. Then $a(g-1) \in C_2$. If x is an arbitrary element of K, then $(a(g-1))(x-1) \in C_1$ and we have

$$(a(g-1))(x-1) = a((x-1)(g-1)) = (a(x-1))(g-1) = 0$$

because $a(x-1) \in C_1 = C_A(G)$. It follows that $a(g-1) \in C_A(x)$. Since it is valid for each element $x \in K$, it follows that $a(g-1) \in C_A(K) = C_A(G)$, or, in other words, that the factor C_2/C_1 is G-central. Using similar arguments, after finitely many steps, we obtain that the series $\{C_j \,|\, 0 \leq j \leq n\}$ is G-central. Since A is an Abelian p-group of exponent, say, p^c, and the center of $A \rtimes G$ is contained in A, then $g^{p^{c(n-1)}} = 1$ for each element $g \in G$. □

Lemma 18 *Let D be a Dedekind domain of characteristic 0, P be a maximal ideal of D such that $char(D/P) = p > 0$, G be a periodic Abelian group, and A be a just non-Artinian DG-module that is D-torsion-free. Suppose that A includes a nonzero DG-submodule B such that A/B is D-periodic and $P \in Ass_D(A/B)$. Then the Sylow p-subgroups of $G/C_G(A)$ are cyclic.*

Proof Without loss of generality we may assume that $C_G(A) = \langle 1 \rangle$. Suppose by contradiction that a Sylow p-subgroup P of G is infinite. Since A/B is D-periodic and $P \in Ass_D(A/B)$, we have that

$$A/B = (A/B)_P \oplus V/B$$

where $P \notin Ass_D(V/B)$ and A/V is a nontrivial P-module. Let n be a positive integer. Since $char(D/P) = p$, it follows that $\Omega_{P,1}(A/V)$ is an elementary Abelian p-group and $\Omega_{P,n}(A/V)$ is a bounded p-group. Using Proposition 17 we have that $P/C_P(\Omega_{P,n}(A/V))$ must be bounded. By Corollary 5, we find that P is locally cyclic; hence, $P = C_P(\Omega_{P,n}(A/V))$, and so $P = C_P(A/V)$.

Put $A_1/V = \Omega_{P,1}(A/V)$, $A_m/V = \Omega_{P,m}(A/V)$ for $m \in \mathbb{N}$, and fix a positive integer n. As we have seen, A_1/V is an elementary Abelian p-group and therefore $p^n A_1 \leq V$; in particular, $p^n A_1 \neq A_1$. Clearly $p^n A_1$ is a nontrivial DG-submodule of A. Therefore, using Proposition 17, we find that $P = C_P(A_m/p^n A_1)$ for each natural number m. Thus, $P = C_P(A/p^n A_1)$. The fact that $char(D) = 0$ implies that A is $\mathbb{Z}$-torsion-free. Then $p^n A_1 \simeq A_1$ and so $p^{n+1} A_1 < p^n A_1$. This means that

$$\bigcap_{n \in \mathbb{N}} p^n A_1 = \langle 0 \rangle$$

and $P = C_P(A)$, a contradiction. □

Proof of Theorem D — Theorem C proves (i), (ii), (iii), (iv), and (vi). Lemma 18 proves (v). To prove (vii) use Lemma 18, the fact that $C_G(A_P) \leq C_G(A_P/C)$, and point (i). The last part of the theorem will now be proved. The Sylow p'-subgroup $Q/C_G(A_P/C)$ of $G/C_G(A_P/C)$ has finite index. Then A_P/C is an Artinian DQ-module (see, for example, [8, Theorem 5.2]). The conclusion follows from [8, Theorem 12.8]. □

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