Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 774 – 785
© TÜBİTAK
doi:10.3906/mat-1701-24

Strongly CM-semicommutative rings

Liang ZHAO[1,2*], **Jiaqun WEI**[1]
[1]Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, P. R. China
[2]School of Mathematics and Physics, Anhui University of Technology, Maanshan, P. R. China

Received: 09.01.2017 • **Accepted/Published Online:** 19.07.2017 • **Final Version:** 08.05.2018

Abstract: We study the strongly semicommutative properties relative to a monoid crossed product. The concept of strongly CM-semicommutative rings is introduced and investigated. Many results related to semicommutative properties over polynomial rings, skew polynomial rings, monoid rings, and skew monoid rings are extended and unified.

Key words: Monoid crossed products, strongly CM-semicommutative rings, strongly M-semicommutative rings

1. Introduction

Throughout, unless otherwise indicated, R denotes an associative ring with identity and M is a monoid. A ring R is said to be a semicommutative ring if, for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Semicommutative rings and related topics were investigated by many authors (see, for example, [2, 4, 5, 10], and [13]). It was shown in [2] that the polynomial rings over semicommutative rings need not be semicommutative. Strongly semicommutative rings were studied in [13]. A ring R is strongly semicommutative if $f(x)g(x) = 0$ implies $f(x)R[x]g(x) = 0$, where $f(x), g(x) \in R[x]$. More generally, recall from [10] that a ring R is called strongly M-semicommutative whenever $\alpha\beta = 0$ implies that $\alpha R[M]\beta = 0$, where $\alpha, \beta \in R[M]$. According to [1], a ring R is α-compatible if for any $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. It is clear that this happens only when the endomorphism α is injective. Krempa [6] introduced the notion of an α-rigid ring. An endomorphism α of a ring R is said to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, while a ring R is said to be α-rigid if there exists a rigid endomorphism α of R. By [1, Lemma 2.2], R is α-rigid if and only if R is α-compatible and reduced.

A monoid M is called a *u.p.-monoid* (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$, there exists an element $g \in M$ uniquely in the form of ab with $a \in A$ and $b \in B$. If there exists a monoid homomorphism $\omega : M \to Aut(R)$, we denote by $\omega_g(r)$ the image of r under $\omega(g)$ with $g \in M$ and $r \in R$. The monoid homomorphism $\omega : M \to Aut(R)$ defined by $\omega_g(r) = r$ for each $g \in M$ and $r \in R$ is called the trivial monoid homomorphism. If R is a ring and M is a monoid, then the crossed product $R * M$ over R consists of all finite sums $R * M = \{\sum r_g g | r_g \in R, g \in M\}$ with addition defined componentwise and multiplication defined by the distributive law and two rules that are called the twisting and the action explained below. Specifically, we have the twisting operation $gh = f(g,h)gh$ for every $g, h \in M$, where $f : M \times M \to U = U(R)$. For every $r \in R$ and $g \in M$, we have $gr = \omega_g(r)g$ with $\omega : M \to Aut(R)$. Note that the map ω is a weak action of M on R and f is a ω-cocycle (see [9]).

*Correspondence: lzhao78@gmail.com
2010 *AMS Mathematics Subject Classification:* Primary 16U80; Secondary 16S35, 13B02

A monoid crossed product is a quite general ring construction. Let $R * M$ be a monoid crossed product with twisting f and action ω. If the twisting f is trivial, that is $f(x,y) = 1$ for all $x, y \in M$, then $R * M$ is the skew monoid ring $R\sharp M$. If the action ω is trivial, i.e. $\omega_g = i_R$ with i_R the identity automorphism over R, then $R * M$ is the twisted monoid ring $R^\tau[M]$. If both the twisting f and the action ω are trivial, then $R * M$ is a monoid ring, denoted by $R[M]$ (see [3] and [11] for more details). For a ring R and a monoid M with $\omega : M \to Aut(R)$ a monoid homomorphism, we say that R is M-compatible (resp., M-rigid) if ω_g is compatible (resp., rigid) for any $g \in M$. According to [14], a ring R is called a CM-Armendariz ring if whenever $\alpha = a_1 g_1 + \cdots + a_n g_n$, $\beta = b_1 h_1 + \cdots + b_m h_m \in R * M$ satisfy $\alpha\beta = 0$, we have $a_i \omega_{g_i}(b_j) = 0$ for all i, j. If R is a CM-Armendariz ring with f trivial, then R is said to be a skew M-Armendariz ring. It is clear that M-Armendariz rings [7] are just those CM-Armendariz rings with both twisting and action trivial. In particular, if both the twisting f and action ω are trivial with $M = (\mathbb{N} \cup \{0\}, +)$, then R is CM-Armendariz if and only R is Armendariz [12].

In this paper, we investigate a common generalization of strongly semicommutative properties over polynomial rings, skew polynomial rings, monoid rings, and skew monoid rings. The main idea is to study the strongly semicommutative properties relative to a monoid crossed product. The new class of strongly CM-semicommutative rings defined for a monoid crossed product is introduced and studied. Some well-known results on this subject are generalized and unified. If R is an M-rigid ring and M a monoid with action $\omega : M \to Aut(R)$, we show that the ring $T_3(R)$ is skew strongly M-semicommutative, where $|M| \geq 2$. We also study the relationship between the strongly CM-semicommutative property of a ring R and that of its subrings induced by a central idempotent (see Proposition 2.12). Let I be an ω-invariant ideal of R and M a u.p.-monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. It is proved that if R/I is strongly CM-semicommutative and I is an M-rigid ideal (as a ring without identity), then R is strongly CM-semicommutative.

2. Strongly CM-semicommutative rings

In this section, we study the strongly semicommutative properties relative to a monoid crossed product. The notion of strongly CM-semicommutative rings is introduced and studied. Some constructions of this class of rings are also given.

We begin with the following definition:

Definition 2.1 *Let R be a ring and M a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. We call R a strongly CM-semicommutative ring, i.e. R is strongly semicommutative with respect to the monoid crossed product $R * M$ if whenever $\alpha = a_1 g_1 + \cdots + a_n g_n$, $\beta = b_1 h_1 + \cdots + b_m h_m \in R * M$ satisfy $\alpha\beta = 0$, then $\alpha(R * M)\beta = 0$.*

It is clear that a ring R is a strongly M-semicommutative ring if and only if it is a strongly CM-semicommutative ring with both twisting and action trivial. If $M = (\mathbb{N} \cup \{0\}, +)$ and both the twisting f and action ω are trivial, then the class of strongly CM-semicommutative rings is precisely the class of strongly semicommutative rings. Some other variants of strongly CM-semicommutative rings can be obtained when specialized to special M, f, and ω.

In particular, we give the following two special classes of strongly CM-semicommutative rings, which are closely related to some well-known results.

Remark 2.2 *Let R be a ring and M a monoid with twisting $f : M\times M \to U(R)$ and action $\omega : M \to Aut(R)$. Then:*

(1) *If R is strongly CM-semicommutative with f trivial, then we call R a skew strongly M-semicommutative ring.*

(2) *If R is strongly CM-semicommutative with ω trivial, then R is called a strongly TM-semicommutative (i.e. twisted M-semicommutative) ring.*

It is a well-known fact that if a ring R is a reduced ring, then its polynomial ring $R[x]$ is reduced. The next lemma extends this result.

Lemma 2.3 *Let M be a u.p.-monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If R is an M-rigid ring, then $R * M$ is reduced.*

Proof Let $\alpha = a_1g_1 + \cdots + a_ng_n \in R * M$ such that $\alpha^2 = 0$. Then R is a CM-Armendariz ring by [14, Proposition 2.2] and this implies that $a_i\omega_{g_i}(a_j) = 0$ for all i, j. Since every M-rigid ring is M-compatible and reduced, we conclude that $a_i = 0$ for all $1 \le i \le n$. It follows that $\alpha = 0$, and hence $R * M$ is reduced. □

For a ring R, let

$$T_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in R \right\}.$$

Let M be a monoid with $\omega : M \to Aut(R)$ a monoid homomorphism. For every $g \in M$, ω can be extended to a monoid homomorphism $\bar{\omega}$ from M to $Aut(T_3(R))$ defined by $\bar{\omega}_g((a_{ij})) = (\omega_g(a_{ij}))$.

Lemma 2.4 *[14, Proposition 2.8] Let R be an M-rigid ring and M a monoid with action $\omega : M \to Aut(R)$, where $|M| \ge 2$. Then R is skew M-Armendariz if and only if $T_3(R)$ is skew M-Armendariz.*

Proposition 2.5 *Let R be an M-rigid ring and M a monoid with action $\omega : M \to Aut(R)$, where $|M| \ge 2$. Then $T_3(R)$ is skew strongly M-semicommutative.*

Proof Assume that $\alpha = A_1g_1 + A_2g_2 + \cdots + A_ng_n$, $\beta = B_1h_1 + B_2h_2 + \cdots + B_mh_m \in T_3(R)\sharp M$ such that $\alpha\beta = 0$. Since R is M-rigid, R is skew M-Armendariz by [14, Proposition 2.2], and hence $T_3(R)$ is skew M-Armendariz by Lemma 2.4. This implies that $A_i\omega_{g_i}(B_j) = 0$. Since R is M-rigid, $T_3(R)$ is an M-compatible ring by [1, Example 1.2]. It follows that $A_iB_j = 0$ for all i, j. This implies that $A_iT_3(R)B_j = 0$ for all i, j by [5, Proposition 1.2]. Then $A_i\bar{\omega}_{g_i}(T_3(R))B_j = 0$ since $T_3(R)$ is M-compatible, and hence $A_i(T_3(R)\sharp M)B_j = 0$. Therefore, $\alpha(T_3(R)\sharp M)\beta = 0$ and thus $T_3(R)$ is skew strongly M-semicommutative. □

Corollary 2.6 *[10, Proposition 2.1] Let M be a monoid with $|M| \ge 2$ and R a reduced M-Armendariz ring. Then $T_3(R)$ is a strongly M-semicommutative ring.*

Recall that a ring R is a strongly M-reversible ring if $\alpha\beta = 0$ implies $\beta\alpha = 0$ for all $\alpha, \beta \in R[M]$. More generally, we say that a ring R is a strongly CM-reversible ring if whenever $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m \in R * M$ satisfy $\alpha\beta = 0$, then $\beta\alpha = 0$.

Lemma 2.7 *Let R be a ring and M a u.p.-monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If R is an M-rigid ring, then R is strongly CM-reversible.*

Proof Suppose that $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m \in R * M$ such that $\alpha\beta = 0$. Then we have $(\beta\alpha)^2 = (\beta\alpha)(\beta\alpha) = \beta(\alpha\beta)\alpha = 0$, and thus $\beta\alpha = 0$ since $R * M$ is reduced by Lemma 2.3. This implies that R is strongly CM-reversible. □

We have the following proposition immediately.

Proposition 2.8 *Let R be an M-rigid ring and M a u.p.-monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Then R is a strongly CM-semicommutative ring.*

Proof Assume that $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m \in R * M$ such that $\alpha\beta = 0$. Because R is a strongly CM-reversible ring by Lemma 2.7, we deduce that $\beta\alpha = 0$. This implies that

$$(\alpha(R * M)\beta)^2 = (\alpha(R * M)\beta)(\alpha(R * M)\beta) = \alpha(R * M)(\beta\alpha)(R * M)\beta = 0.$$

Since $R * M$ is a reduced ring by Lemma 2.3, we get $\alpha(R * M)\beta = 0$. Therefore, R is strongly CM-semicommutative. □

Let R be a ring and M a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. The restrictions of f and ω on an ideal N of M are denoted by $\bar{f}|_{N\times N}$ and $\bar{\omega}|_N$, respectively.

Proposition 2.9 *Let R be an M-rigid ring and M a u.p.-monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If R is a strongly CN-semicommutative ring for an ideal N of M, then R is strongly CM-semicommutative.*

Proof Assume that $\alpha = a_1g_1 + \cdots + a_ng_n$, $0 \neq \beta = b_1h_1 + \cdots + b_mh_m \in R * M$ such that $\alpha\beta = 0$. If we take $g \in N$, then

$$gg_1, gg_2, \cdots, gg_n, h_1g, h_2g, \cdots, h_mg \in N.$$

Since every u.p.-monoid is a cancellative monoid, we get $gg_i \neq gg_j$ and $h_ig \neq h_jg$ whenever $i \neq j$. Let $\alpha_1 = \sum_{i=1}^{n} a_igg_i$, $\beta_1 = \sum_{j=1}^{m} b_jh_jg$. Then α_1, $\beta_1 \in R * N$. In the following, we freely use the fact that $\omega_{g_i}(R)f(g_i, h_j) = Rf(g_i, h_j) = R$ for any $g_i, h_j \in M$. Since R is an M-rigid ring and $\alpha\beta = 0$, we get

$$\alpha_1\beta_1 = \Big(\sum_{i=1}^{n} a_igg_i\Big)\Big(\sum_{j=1}^{m} b_jh_jg\Big) = \sum_{i,j} a_i\omega_{gg_i}(b_j)f(gg_i, h_jg)gg_ih_jg = 0.$$

Now we claim that $\alpha\gamma\beta = 0$ for any $\gamma = c_1t_1 + c_2t_2 + \cdots + c_kt_k \in R * M$. Because N is an ideal of M, it is clear that $\gamma_1 = c_1t_1g + c_2t_2g + \cdots + c_kt_kg \in R * N$. Then

$$\alpha_1\gamma_1\beta_1 = \sum_{i,j,k} a_i\omega_{gg_i}(c_k)f(gg_i, t_kg)\omega_{gg_it_kg}(b_j)f(gg_it_kg, h_jg)gg_it_kgh_jg = 0$$

since R is a strongly CN-semicommutative ring. This implies that

$$a_i\omega_{gg_i}(c_k)f(gg_i,gt_k)\omega_{gg_igt_k}(b_j)f(gg_igt_k,h_jg)=0$$

for each i,j,k. Therefore, we get $a_i\omega_{g_i}(c_k)f(gg_i,gt_k)\omega_{g_it_k}(b_j)=0$ for each i,j,k since R is an M-rigid ring. Then $a_i\omega_{g_i}(c_k)\omega_{g_it_k}(b_j)=0$ for each i,j,k. It follows that

$$\alpha\gamma\beta=\sum_{i,j,k}a_i\omega_{g_i}(c_k)f(g_i,t_k)\omega_{g_it_k}(b_j)f(g_it_k,h_jg)g_it_kh_j=0.$$

Then we have $\alpha(R*M)\beta=0$, and the result follows. □

Let I be an ideal of R and $\omega: M\to Aut(R)$ a monoid homomorphism. An ideal I of R is said to be an ω-invariant ideal of R in case $\omega_g(I)\subseteq I$ for every $g\in M$. Note that $\bar{\omega}: M\to Aut(R/I)$ defined by

$$\bar{\omega}_g(r+I)=\omega_g(r)+I$$

is a monoid homomorphism. Moreover, it is easy to see that the twisting $f: M\times M\to U(R)$ induces a twisting $\bar{f}: M\times M\to U(R/I)$ given by

$$\bar{f}(x,y)=f(x,y)+I.$$

Moreover, for every $\alpha=\sum_{i=1}^{n}a_ig_i$ in $R*M$, we denote $\bar{\alpha}=\sum_{i=1}^{n}\bar{a}_ig_i$ in $(R/I)*M\cong(R*M)/(I*M)$, where $\bar{a}_i=a_i+I$ for $1\le i\le n$. It can be easily checked that the map $\mu: R*M\to(R/I)*M$ defined by $\mu(\alpha)=\bar{\alpha}$ is a ring homomorphism.

Let I be any proper ideal of a ring R. One may suspect that if I (as a ring without identity) and R/I are strongly CM-semicommutative, then R is strongly CM-semicommutative. However, the following example erases this possibility.

Example 2.10 *Let D be a division ring and M a u.p.-monoid with twisting $f: M\times M\to U(D)$ and action $\omega: M\to Aut\ (D)$. Let*

$$R=\left\{\begin{pmatrix}a&b&c\\0&a&d\\0&0&a\end{pmatrix}|a,b,c,d\in D\right\},$$

$$I=\left\{\begin{pmatrix}0&0&D\\0&0&0\\0&0&0\end{pmatrix}\right\}.$$

Then R is a ring and I is a nonzero ω-invariant proper ideal of the ring R. Clearly, R is not strongly CM-semicommutative (an easy example is that of both twisting f and action ω being trivial with $M=(\mathbb{N}\cup\{0\},+)$).

Moreover, I is a strongly CM-semicommutative ideal of R since D is a domain. Now we claim that R/I is a strongly CM-semicommutative ring. In fact, if

$$\alpha=\sum_{i=1}^{n}\begin{pmatrix}a_i&b_i&0\\0&a_i&d_i\\0&0&a_i\end{pmatrix}g_i,\quad \beta=\sum_{j=1}^{n}\begin{pmatrix}u_j&v_j&0\\0&u_j&w_j\\0&0&u_j\end{pmatrix}g_j$$

are elements in $(R/I)*M$ *such that* $\alpha\beta=0$*, then we have*

$$\begin{pmatrix} \sum_{i=1}^{n} a_ig_i & \sum_{i=1}^{n} b_ig_i & 0 \\ 0 & \sum_{i=1}^{n} a_ig_i & \sum_{i=1}^{n} d_ig_i \\ 0 & 0 & \sum_{i=1}^{n} a_ig_i \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{m} u_jh_j & \sum_{j=1}^{m} v_jh_j & 0 \\ 0 & \sum_{j=1}^{m} u_jh_j & \sum_{j=1}^{m} w_jh_j \\ 0 & 0 & \sum_{j=1}^{m} u_jh_j \end{pmatrix} = 0.$$

This implies that

$$\big(\sum_{i=1}^{n} a_ig_i\big)\big(\sum_{j=1}^{m} u_jh_j\big) = \sum_{i,j} a_i\omega_{g_i}(u_j)f(g_i,h_j)g_ih_j = 0.$$

Since D *is a division ring, it is easy to see that* D *is an* M*-rigid ring and thus* D *is* CM*-Armendariz. It follows that* $a_i\omega_{g_i}(u_j)=0$*, and hence* $a_iu_j=0$ *since* D *is an* M*-rigid ring. Then we have*

$$\sum_{i=1}^{n} a_ig_i = 0 \text{ or } \sum_{j=1}^{m} u_jh_j = 0.$$

Because D *is a division ring, it is clear that* $\alpha((R/I)*M)\beta=0$*. This shows that* R/I *is strongly* CM*-semicommutative, as desired.*

However, we can give an affirmative answer as in the following proposition.

Proposition 2.11 *Let* I *be an* ω*-invariant ideal of* R *and* M *a u.p.-monoid. If* R/I *is strongly* CM*-semicommutative and* I *is an* M*-rigid ideal, then* R *is strongly* CM*-semicommutative.*

Proof Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ and $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m$ be elements in $R*M$ such that $\alpha\beta = 0$. Then we have $\alpha(R*M)\beta \subseteq I*M$. Sine I is an M-rigid ideal and M is a u.p.-monoid, $I*M$ is reduced by Lemma 2.3. Furthermore, since I is an ω-invariant ideal, we have

$$\beta(I*M)\alpha \subseteq I*M,\ (\beta(I*M)\alpha)^2 = 0.$$

Because $I*M$ is reduced, this implies that $\beta(I*M)\alpha = 0$. Therefore, we have

$$\big((\alpha(R*M)\beta)(I*M)\big)^2 = \alpha(R*M)\big(\beta(I*M)\alpha\big)(R*M)\beta(I*M) = 0.$$

It follows that $\alpha(R*M)\beta(I*M) = 0$, and thus we have

$$(\alpha(R*M)\beta)^2 \subseteq \alpha(R*M)\beta(I*M) = 0$$

since $\alpha(R*M)\beta \subseteq I*M$, proving $(\alpha(R*M)\beta)^2 = 0$. Therefore, we have $\alpha(R*M)\beta = 0$ and the result follows. □

The next proposition gives the relationship between the strongly CM-semicommutative property of a ring R and that of its subrings induced by a central idempotent.

Proposition 2.12 *Let e be a central idempotent of R such that $\omega_g(e) = e$ for each $g \in M$. Then R is strongly CM-semicommutative if and only if eR and $(1-e)R$ are strongly CM-semicommutative.*

Proof If R is strongly CM-semicommutative, it is easy to see that eR and $(1-e)R$ are strongly CM-semicommutative. Assume that eR and $(1-e)R$ are strongly CM-semicommutative. Let $\alpha, \beta \in R*M$ such that $\alpha\beta = 0$. Then $e\alpha, e\beta \in eR*M$ and $(1-e)\alpha, (1-e)\beta \in (1-e)R*M$. Because e is a central idempotent of R and $\omega_g(e) = e$ for each $g \in M$, we have

$$e\alpha e\beta = 0,\ (1-e)\alpha(1-e)\beta = 0.$$

It suffices to show that $\alpha(R*M)\beta = 0$. Since e is a central idempotent of R and eR and $(1-e)R$ are strongly CM-semicommutative, we have

$$e\alpha(eR*M)e\beta e = 0,\ (1-e)\alpha\big((1-e)R*M\big)(1-e)\beta(1-e) = 0.$$

This implies that

$$\begin{aligned} 0 = \alpha(R*M)\beta &= e\alpha(R*M)\beta + (1-e)\alpha(R*M)\beta \\ &= e\alpha e(R*M)e\beta e + (1-e)\alpha(1-e)(R*M)(1-e)\beta(1-e) \\ &= e\alpha(eR*M)e\beta e + (1-e)\alpha\big((1-e)R*M\big)(1-e)\beta(1-e). \end{aligned}$$

Therefore, R is strongly CM-semicommutative. □

Let R be an algebra over a commutative ring S. Recall that the Dorroh extension D of R by S is the ring $R \times S$ with operations

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2), \text{ and}$$
$$(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2),$$

where $r_i \in R$ and $s_i \in S$. Let M be a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If there are action $\bar{\omega} : M \to Aut(S)$ and twisting $\bar{f} : M \times M \to U(S)$, then we have the ring $S*M$. For any

$$\sigma = \sum_{i=1}^{n} s_ig_i,\ \tau = \sum_{k=1}^{s} c_kt_k \in S*M \text{ and } \alpha = \sum_{j=1}^{m} a_jh_j \in R*M,$$

we have the following:

$$\sigma\alpha = \sum_{i+j=l} s_i\omega_{g_i}(a_j)f(g_i, h_j)g_ih_j,$$

$$\sigma\tau = (\sum_{i=1}^{n} s_ig_i)(\sum_{k=1}^{s} c_kt_k) = \sum_{i+k=l} s_i\bar{\omega}_{g_i}(c_k)\bar{f}(g_i, t_k)g_it_k.$$

Proposition 2.13 *Let R be an algebra over a commutative ring S and D the Dorroh extension of R by S. If R is strongly CM-semicommutative and S is a domain, then D is strongly CM-semicommutative.*

Proof Assume that

$$\alpha = (\alpha_1, \alpha_2) = \sum_{i=1}^{n} (a_i, s_i)g_i = (\sum_{i=1}^{n} a_ig_i, \sum_{i=1}^{n} s_ig_i),$$

$$\beta=(\beta_1,\beta_2)=\sum_{j=1}^{m}(b_j,t_j)h_j=(\sum_{j=1}^{m}b_jh_j,\sum_{j=1}^{m}t_jh_j)$$

are elements in $D{*}M$ such that $\alpha\beta=0$. By definition, we have $(\alpha_1,\alpha_2)(\beta_1,\beta_2)=(\alpha_1\beta_1+\alpha_2\beta_1+\beta_2\alpha_1,\alpha_2\beta_2)=0$. It follows that $\alpha_1\beta_1+\alpha_2\beta_1+\beta_2\alpha_1=0$ and $\alpha_2\beta_2=0$. Then we have

$$\alpha_2\beta_2=(\sum_{i=1}^{n}s_ig_i)(\sum_{j=1}^{m}t_jh_j)=\sum_{i,j}s_i\bar{\omega}_{g_i}(t_j)\bar{f}(g_i,h_j)g_ih_j=0.$$

This implies that $s_i\bar{\omega}_{g_i}(t_j)=0$. Since S is a domain, $s_i=0$ or $\bar{\omega}_{g_i}(t_j)=0$, and thus $s_i=0$ or $t_j=0$ because $\bar{\omega}$ is an automorphism of S. Therefore, we have $\alpha_2=0$ or $\beta_2=0$. If $\alpha_2=0$, then $\alpha_1\beta_1+\beta_2\alpha_1=\alpha_1(\beta_1+\beta_2)=0$. For any $\gamma=(\gamma_1,\gamma_2)\in D*M$, it suffices to show that $\alpha\gamma\beta=0$. In fact, since R is strongly CM-semicommutative, we have $\alpha_1(\gamma_1+\gamma_2)(\beta_1+\beta_2)=0$. This implies that

$$\alpha\gamma\beta=(\alpha_1\gamma_1\beta_1+\alpha_1\gamma_1\beta_2+\alpha_1\gamma_2\beta_1+\alpha_1\gamma_2\beta_2+\alpha_2\gamma_1\beta_1+\alpha_2\gamma_2\beta_1+\beta_2\alpha_2\gamma_1,\alpha_2\gamma_2\beta_2)=0.$$

Similarly, if $\beta_2=0$, then we have $\alpha_1\beta_1+\alpha_2\beta_1=(\alpha_1+\alpha_2)\beta_1=0$. For any $\delta=(\delta_1,\delta_2)\in D*M$, since R is strongly CM-semicommutative, we have $(\alpha_1+\alpha_2)(\delta_1+\delta_2)\beta_1=0$ since $\delta_1+\delta_2\in R*M$. This implies that

$$(\alpha_1\delta_1\beta_1+\alpha_1\delta_1\beta_2+\alpha_1\delta_2\beta_1+\beta_2\delta_2\alpha_1+\alpha_2\delta_1\beta_1+\alpha_2\delta_2\beta_1+\beta_2\alpha_2\delta_1,\alpha_2\delta_2\beta_2)=0.$$

It follows that $\alpha\delta\beta=(\alpha_1,\alpha_2)(\delta_1,\delta_2)(\beta_1,\beta_2)=0$. This implies that D is a strongly CM-semicommutative ring. □

Let $\triangle$ be a multiplicative monoid consisting of central regular elements of R. Then it is easy to see that $\triangle^{-1}R=\{u^{-1}a|u\in\triangle,a\in R\}$ is a ring. Let M be a monoid with $\omega:M\to Aut(R)$ a monoid homomorphism. If $\omega_g(\triangle)\subseteq\triangle$ for every $g\in M$, then ω can be extended to $\bar{\omega}:M\to Aut(\triangle^{-1}R)$ defined by

$$\bar{\omega}_g(u^{-1}a)=\omega_g(u)^{-1}\omega_g(a).$$

Note that if $f:M\times M\to U(R)$ is a twisted function, then f is also a twisted function from $M\times M$ to $\triangle^{-1}R$ since $U(R)\subseteq U(\triangle^{-1}R)$.

Proposition 2.14 *Let M be a cancellative monoid with twisting $f:M\times M\to U(R)$ and action $\omega:M\to Aut(R)$. Then R is strongly CM-semicommutative if and only if $\triangle^{-1}R$ is strongly CM-semicommutative.*

Proof It suffices to show the necessity. Suppose that R is a strongly CM-semicommutative ring. Let $\alpha=\sum_{i=1}^{m}u_i^{-1}a_ig_i$, $\beta=\sum_{j=1}^{n}v_j^{-1}b_jh_j\in\triangle^{-1}R*M$ such that $\alpha\beta=0$. Since $\triangle$ is a multiplicative monoid consisting of central regular elements of R, we have

$$0=\alpha\beta=\big(\sum_{i=1}^{m}u_i^{-1}a_ig_i\big)\big(\sum_{j=1}^{n}v_j^{-1}b_jh_j\big)=\sum_{k=i+j}u_i^{-1}a_i\omega_{g_i}\big(v_j^{-1}b_j\big)f(g_i,h_j)g_ih_j$$

$$=\sum_{k=i+j}a_i\omega_{g_i}(b_j)(u_i\omega_{g_i}(v_j))^{-1}f(g_i,h_j)g_ih_j.$$

Let $\tilde{\alpha} = \sum_{i=1}^{m} a_i g_i$, $\tilde{\beta} = \sum_{j=1}^{n} b_j h_j$. Then we have $\tilde{\alpha}, \tilde{\beta} \in R*M$, and thus we get $\tilde{\alpha}\tilde{\beta} = \sum_{k=i+j} a_i \omega_{g_i}(b_j) f(g_i, h_j) g_i h_j = 0$. Since R is strongly CM-semicommutative, we have

$$\tilde{\alpha}\tilde{\gamma}\tilde{\beta} = \sum_{i+j+k=l} a_i \omega_{g_i}(c_k) f(g_i, p_k) \omega_{g_i p_k}(b_j) f(g_i p_k, h_j) g_i p_k h_j = 0$$

for any $\tilde{\gamma} = \sum_{k=1}^{t} c_k p_k \in R * M$, where $l = 3, \cdots, m+n+t$. Therefore, for any $\gamma = \sum_{k=1}^{t} \eta_k^{-1} c_k p_k \in \triangle^{-1} R * M$, we have

$$\begin{aligned} 0 = \alpha\gamma\beta &= \Big(\sum_{i=1}^{m} u_i^{-1} a_i g_i\Big)\Big(\sum_{k=1}^{t} \eta_k^{-1} c_k p_k\Big)\Big(\sum_{j=1}^{n} v_j^{-1} b_j h_j\Big) \\ &= \sum_{i+j+k=l} a_i \omega_{g_i}(c_k) f(g_i, p_k) \omega_{g_i p_k}(b_j) \big(u_i \omega_{g_i}(\eta_k) \omega_{g_i p_k}(\nu_j)\big)^{-1} f(g_i p_k, h_j) g_i p_k h_j \end{aligned}$$

since $\triangle$ is a multiplicative monoid consisting of central regular elements of R and all u_i, v_j and $\eta_k \in \triangle$ for all i, j, k. This implies that $\triangle^{-1} R$ is strongly CM-semicommutative. □

Corollary 2.15 *Let M be a cancellative monoid with monoid homomorphism $\omega : M \to Aut(R)$. Then R is skew strongly M-semicommutative if and only if $\triangle^{-1} R$ is skew strongly M-semicommutative.*

Corollary 2.16 *Let M be a cancellative monoid. Then R is strongly M-semicommutative if and only if $\triangle^{-1} R$ is strongly M-semicommutative.*

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers. Denote it by $R[x; x^{-1}]$.

Corollary 2.17 *Let R be a reduced ring and M a monoid. Then $R[x]$ is strongly M-semicommutative if and only if $R[x; x^{-1}]$ is strongly M-semicommutative.*

Proof Let $\triangle = \{1, x, x^2, \cdots\}$. Then clearly $\triangle$ is a multipicatively closed subset of $R[x]$. Since $R[x; x^{-1}] \cong \triangle^{-1} R[x]$, it follows that $R[x; x^{-1}]$ is strongly M-semicommutative by Proposition 2.14. □

The next construction is due to Nagata [8]. Let R be a commutative ring, M be an R-module, and α be an endomorphism of R. Given $R \oplus M$, we have a (possibly noncommutative) ring structure with the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, \alpha(r_1) m_2 + r_2 m_1),$$

where $r_i \in R$ and $m_i \in M$. We shall call this extension the skew-trivial extension of R by M and α. Let $\tau = a_1 g_1 + a_2 g_2 + \cdots + a_n g_n \in R * M$. If α is an endomorphism of R, in the following we denote by

$$\alpha(\tau) = \alpha(a_1) g_1 + \alpha(a_2) g_2 + \cdots + \alpha(a_n) g_n$$

the image of τ under α.

Proposition 2.18 *Let R be a commutative domain and M a u.p.-monoid with twisting $f: M \times M \to U(R)$ and action $\omega: M \to Aut(R)$. If α is an injective endomorphism of R, then the skew-trivial extension $R \oplus R$ of R by R and α is strongly CM-semicommutative.*

Proof Suppose that $(\mu_1, \mu_2) = \sum_{i=1}^{n}(a_i, b_i)g_i, (\nu_1, \nu_2) = \sum_{j=1}^{m}(c_j, d_j)h_j \in (R \oplus R) * M$ such that $(\mu_1, \mu_2)(\nu_1, \nu_2) = 0$. For any $(\omega_1, \omega_2) \in (R \oplus R) * M$, it suffices to show that $(\mu_1, \mu_2)(\omega_1, \omega_2)(\nu_1, \nu_2) = 0$. Then we have

$$\mu_1\nu_1 = 0,\ \alpha(\mu_1)\nu_2 + \nu_1\mu_2 = 0.$$

Since R is a commutative domain, we have $\mu_1 = 0$ or $\nu_1 = 0$. If $\mu_1 = 0$, then we have $\nu_1\mu_2 = 0$. Note that R is a strongly CM-semicommutative ring by Proposition 2.8 since R is an α-rigid ring and α is an injective endomorphism of R. This implies that $\nu_1\omega_1\mu_2 = 0$. Therefore, we have

$$\begin{aligned}(\mu_1, \mu_2)(\omega_1, \omega_2)(\nu_1, \nu_2) &= (\mu_1\omega_1\nu_1, \alpha(\mu_1)\alpha(\omega_1)\nu_2 + \nu_1\alpha(\mu_1)\omega_2 + \nu_1\omega_1\mu_2) \\ &= (\mu_1\omega_1\nu_1, \nu_1\omega_1\mu_2) = 0,\end{aligned}$$

proving $R \oplus R$ is strongly CM-semicommutative. If $\nu_1 = 0$, then $\alpha(\mu_1)\nu_2 = 0$. It follows that $\alpha(\mu_1) = 0$ (and thus $\mu_1 = 0$ since α is injective) or $\nu_2 = 0$ since R is a domain. In this case, it is easy to see that $(\mu_1, \mu_2)(\omega_1, \omega_2)(\nu_1, \nu_2) = 0$. This also shows that $R \oplus R$ is strongly CM-semicommutative. □

Corollary 2.19 *Let R be a commutative domain and M a u.p.-monoid with twisting $f: M \times M \to U(R)$. If α is an injective endomorphism of R, then $R \oplus R$ is strongly TM-semicommutative.*

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

Let R be a commutative ring. It is clear that if $\alpha \equiv I_R$, then the skew-trivial extension of R by M and α is just the usual trivial extension of R by M.

Corollary 2.20 *If R is a commutative domain, then the trivial extension $T(R, R)$ of R by R is strongly M-semicommutative.*

More generally, we have the following:

Proposition 2.21 *Let R be a ring and M a u.p.-monoid with twisting $f: M \times M \to U(R)$ and action $\omega: M \to Aut(R)$. If R is an M-rigid ring, then $T(R, R)$ is strongly CM-semicommutative.*

Proof Let $\alpha = (\alpha_1, \alpha_2),\ \beta = (\beta_1, \beta_2) \in T(R, R) * M$ such that $\alpha\beta = 0$. Then we have

$$\alpha_1\beta_1 = 0,\ \alpha_1\beta_2 + \alpha_2\beta_1 = 0.$$

We claim that $\alpha\gamma\beta = 0$ for any $\gamma = (\gamma_1, \gamma_2) \in T(R, R) * M$. Since $R * M$ is a reduced ring by Lemma 2.3, it follows that $\beta_1\alpha_1 = 0$. Multiplying

$$\alpha_1\beta_2 + \alpha_2\beta_1 = 0$$

by β_1 on the left, we obtain $\beta_1\alpha_2\beta_1 = 0$. This implies that $(\alpha_2\beta_1)^2 = 0$, and hence $\alpha_2\beta_1 = 0$. Therefore, $\alpha_1\beta_2 = 0$. Since R is strongly CM-semicommutative by Proposition 2.8, we get $\alpha_1(R * M)\beta_1 = 0$, $\alpha_2(R * M)\beta_1 = 0$ and $\alpha_1(R * M)\beta_2 = 0$. This implies that

$$\alpha\gamma\beta = (\alpha_1\gamma_1\beta_1, \alpha_1\gamma_1\beta_2 + \alpha_1\gamma_2\beta_1 + \alpha_2\gamma_1\beta_1) = 0.$$

Therefore, $T(R, R)$ is strongly CM-semicommutative. □

The next proposition gives the condition under which a semicommutative ring is strongly CM-semicommutative.

Proposition 2.22 *Let R be an M-compatible CM-Armendariz ring. If R is semicommutative, then R is strongly CM-semicommutative.*

Proof Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R * M$ such that $\alpha\beta = 0$. Since R is a CM-Armendariz ring, we get $a_i\omega_{g_i}(b_j) = 0$ for all i, j. This implies that $a_i\omega_{g_it_k}(b_j) = 0$ for all i, j and $t_k \in M$ since R is M-compatible. Because R is a semicommutative ring, we have $a_iR\omega_{g_it_k}(b_j) = 0$ for all i, j and $t_k \in M$. Let $\gamma = c_1t_1 + c_2t_2 + \cdots + c_st_s$ be any element in $R * M$. Since $\omega_{g_i}(c_k)f(g_i, t_k) = R$, we have $\alpha\gamma\beta = \sum_{i,j,k} a_i\omega_{g_i}(c_k)f(g_i, t_k)\omega_{g_it_k}(b_j)f(g_it_k, h_j)g_it_kh_j = 0$. This implies that R is a strongly CM-semicommutative ring. □

Corollary 2.23 *Let R be an M-Armendariz ring. If R is a semicommutative ring, then R is strongly M-semicommutative.*

Acknowledgments

The authors wish to express their sincere thanks to the referee for the helpful comments and suggestions. The first author was supported by the Postdoctoral Science Foundation of China (No. 2017M611851), the Jiangsu Planned Projects for Postdoctoral Research Funds (No. 1601151C), and the Provincial Natural Science Foundation of Anhui Province of China (No. KJ2017A040). The second author was supported by the National Science Foundation of China (Grant No. 11771212). Both authors were supported by a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

References

[1] Hashemi E, Moussavi A. Polynomial extensions of quasi-baer rings. Acta Math Hungar 2005; 107: 207-224.

[2] Huh C, Lee Y, Smoktunowicz A. Armendariz rings and semicommutative rings. Comm Algebra 2002; 30: 751-761.

[3] Kelarev AV. Ring Constructions and Applications. Singapore: World Scientific Publishing, 2002.

[4] Kim NK, Lee Y. Armendariz rings and reduced rings. J Algebra 2000; 223: 477-488.

[5] Kim NK, Lee Y. Extensions of reversible rings. J Pure Appl Algebra 2003; 185: 207-223.

[6] Krempa J. Some examples of reduced rings. Algebra Colloq 1996; 3: 289-300.

[7] Liu ZK. Armendariz rings relative to a monoid. Comm Algebra 2005; 33: 649-661.

[8] Nagata M. Local Rings. New York, NY, USA: Interscience, 1962.

[9] Nastasescu C, Oystaeyen FV. Methods of Graded Rings. New York, NY, USA: Springer-Verlag, 2004.

[10] Nikmehr MJ. Strongly semicommutative rings relative to a monoid. Ukrainian Math J 2014; 66: 1715-1730.

[11] Passman DS. Infinite Crossed Products. New York, NY, USA: Academic Press, 1989.

[12] Rege MB, Chhawchharia S. Armendariz rings. Proc Japan Acad Ser A Math Sci 1997; 73: 14-17.

[13] Yang G, Du RJ. Rings over which polynomial rings are semicommutative. Vietnam J Math 2009; 37: 527-535.

[14] Zhao L, Zhou YQ. Generalized Armendariz properties of crossed product type. Glasgow Math J 2016; 58: 313-323.