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A study on (strong) order-congruences in ordered semihypergroups

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Abstract: In this paper, we introduce the concepts of order-congruences and strong order-congruences on an ordered semihypergroup S, and obtain the relationship between strong order-congruences and pseudoorders on S. Furthermore, we characterize the (strong) order-congruences by the ρ-chains, where ρ is a (strong) congruence on S. Moreover, we give a method of constructing order-congruences, and prove that every hyperideal I of an ordered semihypergroup S is congruence class of one order-congruence on S if and only if I is convex. Finally, we define and study the strong order-congruence generated by a strong congruence. As an application of the results of this paper, we solve an open problem on ordered semihypergroups given by Davvaz et al.

Key words: Ordered semihypergroup, pseudoorder, (strong) order-congruence, ρ-chain

1. Introduction

In mathematics, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation " $\leq$ " such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. Ordered semigroups have several applications in the theory of sequential machines, formal languages, computer arithmetics, and error-correcting codes. As we know, congruences on ordered semigroups play an important role in studying the structures of ordered semigroups, for example, see [13–15, 22–24]. For any congruence ρ on an ordered semigroup S, in general, we do not know whether the quotient semigroup S/ρ is also an ordered semigroup. Even if S/ρ is an ordered semigroup, the order on S/ρ is not necessarily relative to the order on the original ordered semigroup S. As to the above-mentioned questions, Kehayopulu and Tsingelis [13, 14] introduced the concept of pseudoorder on an ordered semigroup S and proved that if σ is a pseudoorder on S, then there exists a congruence $\bar{\sigma}$ on S such that $S/\bar{\sigma}$ is an ordered semigroup. In the same papers a necessary and sufficient condition such that S is a subdirect product of some ordered semigroups was given and two isomorphism theorems of S were established. Since then, Xie [23] introduced the concept of regular congruences on an ordered semigroup S, and proved that ρ is a regular congruence on S if and only if there exists a pseudoorder σ on S such that $\rho = \sigma \cap \sigma^{-1}$.

On the other hand, algebraic hyperstructures, particularly hypergroups, were introduced by Marty [17] in 1934. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a set. Thus algebraic hyperstructures are a suitable

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generalization of classical algebraic structures. Surveys of hyperstructure theory can be found in the books by Corsini [4], Corsini and Leoreanu [5], Davvaz and Leoreanu-Fotea [8], and Vougiouklis [20]. In the hyperstructure theory, semihypergroups are the simplest algebraic hyperstructures that are a generalization of the concept of semigroups. At present, many researchers have studied different aspects of semihypergroups. For more details, the reader is referred to [1, 3, 6, 9, 10, 12, 16, 18, 25]. Especially, regular and strong regular relations on semihypergroups have been introduced and investigated in [4].

A theory of hyperstructures on ordered semigroups has been recently developed. In [11], Heidari and Davvaz applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. Later on, a lot of papers on ordered semihypergroups have been written; for instance, see [2, 7, 19, 26]. It is worth pointing out that Davvaz et al. [7] introduced the concept of a pseudoorder on an ordered semihypergroup, and extended some results in [13] on ordered semigroups to ordered semihypergroups. In particular, they posed an open problem about ordered semihypergroups: Is there a regular relation ρ on an ordered semihypergroup $(S,*,\leq)$ for which S/ρ is an ordered semihypergroup? As a further study, in this paper we define and study the order-congruences and strong order-congruences on an ordered semihypergroup, and extend some results in ordered semigroups to ordered semihypergroups. The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic notions and results from the hyperstructure theory. In Section 3, we introduce the concepts of order-congruences and strong order-congruences on an ordered semihypergroup S, and establish the relationship between strong order-congruences and pseudoorders on S. Moreover, we described the least pseudoorder containing a strong order-congruence on an ordered semihypergroup, and give out a homomorphism theorem of ordered semihypergroups by pseudoorders. In Section 4, we characterize the strong order-congruences (resp. order-congruences) by the ρ-chains, where ρ is a strong congruence (resp. congruence). Furthermore, we provide a method of constructing order-congruences, and prove that every hyperideal I of an ordered semihypergroup S is congruence class of one order-congruence on S if and only if I is convex. By this constructing method of order-congruences, we answer to the open problem given by Davvaz et al. in [7]. Finally, we define and discuss the strong order-congruence generated by a strong congruence.

2. Preliminaries and some notations

Recall that a *hypergroupoid* $(S,*)$ is a nonempty set S together with a hyperoperation, that is a map $*: S\times S\to P^{\star}(S)$, where $P^{\star}(S)$ denotes the set of all the nonempty subsets of S. The image of the pair (x,y) is denoted by $x*y$. If $x\in S$ and A,B are nonempty subsets of S, then $A*B$ is defined by $A*B=\bigcup\limits_{a\in A,b\in B} a*b$. Also $A*x$ is used for $A*\{x\}$ and $x*A$ for $\{x\}*A$. Generally, the singleton $\{x\}$ is identified by its element x.

We say that a hypergroupoid $(S,*)$ is a *semihypergroup* if the hyperoperation "$*$" is associative, that is, $(x*y)*z=x*(y*z)$ for all $x,y,z\in S$ (see [4]).

We now recall the notion of ordered semihypergroups from [11].

Definition 2.1 *An algebraic hyperstructure $(S,*,\leq)$ is called an ordered semihypergroup (also called po-semihypergroup in [11]) if $(S,*)$ is a semihypergroup and $(S,\leq)$ is a partially ordered set such that: for any $x,y,a\in S$, $x\leq y$ implies $a*x\leq a*y$ and $x*a\leq y*a$. Here, if $A,B\in P^{\star}(S)$, then we say that $A\leq B$ if for every $a\in A$ there exists $b\in B$ such that $a\leq b$.*

Clearly, every ordered semigroup is an ordered semihypergroup.

Let $(S, \leq)$ be a partially ordered set (or briefly poset). For $\emptyset \neq H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

A nonempty subset A of a poset S is called *convex* if $a \leq b \leq c$ implies $b \in A$ for all $a, c \in A, b \in S$. A nonempty subset B of a poset S is called *strongly convex* if $B = (B]$ (equivalently, $a \in S, b \in B$ and $a \leq b$ imply $a \in B$). Any strongly convex subset of S is clearly a convex subset; however, the converse does not hold in general.

By a *subsemihypergroup* of an ordered semihypergroup S we mean a nonempty subset A of S such that $A * A \subseteq A$. A nonempty subset A of a semihypergroup $(S, *)$ is called a *left* (resp. *right*) *hyperideal* of S if $S * A \subseteq A$ (resp. $A * S \subseteq A$). If A is both a left and a right hyperideal of S, then it is called a *hyperideal* of S. A nonempty subset A of an ordered semihypergroup $(S, *, \leq)$ is called an *ordered hyperideal* of S if A is a strongly convex hyperideal of S.

Let ρ be an equivalence relation on a semihypergroup $(S, *)$ or an ordered semigroup $(S, *, \leq)$. If A and B are nonempty subsets of S, then we write $A\overline{\rho}B$ to denote that for every $a \in A$, there exists $b \in B$ such that $a\rho b$ and for every $b \in B$ there exists $a \in A$ such that $a\rho b$. We write $A\overline{\overline{\rho}}B$ if for every $a \in A$ and for every $b \in B$ we have $a\rho b$. The equivalence relation ρ is called *congruence* (also called *regular relation* in [4, 7]) if for every $(x, y) \in S \times S$ the implication $x\rho y \Rightarrow a * x \ \overline{\rho}\ a * y$ and $x * a \ \overline{\rho}\ y * a$, for all $a \in S$, is valid. ρ is called *strong congruence* (also called *strongly regular relation* in [4, 7]) if for every $(x, y) \in S \times S$, from $x\rho y$, it follows that $a * x \ \overline{\overline{\rho}}\ a * y$ and $x * a \ \overline{\overline{\rho}}\ y * a$ for all $a \in S$.

Lemma 2.2 *([4]) Let $(S, *)$ be a semihypergroup and ρ an equivalence relation on S. Then*

(i) If ρ is a congruence, then $(S/\rho, \otimes)$ is a semihypergroup with respect to the following hyperoperation: $(a)_\rho \otimes (b)_\rho = \bigcup\limits_{c \in a*b} (c)_\rho$, and it is called a factor semihypergroup.

(ii) If ρ is a strong congruence, then $(S/\rho, \otimes)$ is a semigroup with respect to the following operation: $(a)_\rho \otimes (b)_\rho = (c)_\rho$ for all $c \in a * b$, and it is called a factor semigroup.

Let I be a hyperideal of a semihypergroup $(S, *)$. The relation ρ_I on S is defined as follows:

$$\rho_I := \{(x, y) \in S\backslash I \times S\backslash I \mid x = y\} \cup (I \times I).$$

Clearly, ρ_I is an equivalence relation on S. Moreover, we have the following lemma.

Lemma 2.3 *Let $(S, *)$ be a semihypergroup and I a hyperideal of S. Then ρ_I is a congruence on S and it is called Rees congruence induced by I.*

Proof Let $x, y \in S$ and $x\rho_I y$. Then $x = y \in S\backslash I$ or $x, y \in I$. We consider the following cases:

Case 1. If $x = y \in S\backslash I$, then, for any $z \in S$, $x * z = y * z$. Hence $x * z \ \overline{\rho_I}\ y * z$.

Case 2. Let $x, y \in I$. Since I is a hyperideal of S, we have $x * z \subseteq I, y * z \subseteq I$ for any $z \in S$. Thus, for any $a \in x * z, b \in y * z$, we have $(a, b) \in I \times I \subseteq \rho_I$. Therefore, $x * z \ \overline{\rho_I}\ y * z$. □

Similarly, we can show that $z * x \ \overline{\rho_I}\ z * y$ for any $z \in S$. We have thus shown that ρ_I is a congruence on S.

Remark 2.4 *(1)* $S/\rho_I = \{\{x\} \mid x \in S\backslash I\} \cup \{I\}$.

(2) By Lemmas 2.2 and 2.3, $(S/\rho_I, \otimes_I)$ *forms a factor semihypergroup, which is called Rees factor semihypergroup (also called Rees quotient semihypergroup). Here the hyperoperation* $\otimes_I$ *on* S/ρ_I *is defined by* $(a)_{\rho_I} \otimes_I (b)_{\rho_I} = \bigcup_{c\in a*b} (c)_{\rho_I}$.

A relation ρ on an ordered semihypergroup $(S, *, \leq)$ is called pseudoorder if it satisfies the following conditions: (1) $\leq \subseteq \rho$, (2) $a\rho b$ and $b\rho c$ imply $a\rho c$, i.e. $\rho \circ \rho \subseteq \rho$ and (3) $a\rho b$ implies $a*c \,\overline{\overline{\rho}}\, b*c$ and $c*a \,\overline{\overline{\rho}}\, c*b$, for all $c \in S$ (see [7]).

Lemma 2.5 *Let* $(S, *, \leq)$ *be an ordered semihypergroup and* ρ *a pseudoorder on* S*. Then* $(S/\rho^*, \otimes, \preceq_\rho)$ *is an ordered semigroup, where* ρ^* $(= \rho \cap \rho^{-1})$ *is a strong congruence on* S*, and the order relation* $\preceq_\rho$ *is defined as follows:*

$$\preceq_\rho := \{((x)_{\rho^*}, (y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (x, y) \in \rho\}.$$

Let $(S, *, \leq)$ and $(T, \diamond, \preceq)$ be two ordered semihypergroups, $f : S \to T$ a mapping from S to T. f is called *isotone* if $x \leq y$ implies $f(x) \preceq f(y)$, for all $x, y \in S$. f is called *reverse isotone* if $x, y \in S$, $f(x) \preceq f(y)$ implies $x \leq y$. f is called *homomorphism* (resp. *strong homomorphism*) if it is isotone and satisfies $f(x) \diamond f(y) = \bigcup_{z\in x*y} f(z)$ (resp. $f(x) \diamond f(y) = f(z)$, $\forall z \in x * y$), for all $x, y \in S$. f is called *isomorphism* (resp. *strong isomorphism*) if it is homomorphism (resp. strong homomorphism), onto, and reverse isotone. The ordered semihypergroups S and T are called *strongly isomorphic*, in symbol $S \cong T$, if there exists a strong isomorphism between them.

Remark 2.6 *Let* S *and* T *be two ordered semihypergroups. Then*

(1) If f *is a strong homomorphism and reverse isotone mapping from* S *to* T*, then* $S \cong Im(f)$.

(2) In particular, if S *and* T *are both ordered semigroups, then, in this case, the concepts of strong isomorphisms of ordered semihypergroups and isomorphisms of ordered semigroups coincide.*

The reader is referred to [5, 21] for notation and terminology not defined in this paper.

3. Strong order-congruences and order-congruences on ordered semihypergroups

As we know, pseudoorders on ordered semihypergroups play an important role in studying the structures of ordered semihypergroups (see [7]). To investigate the properties of pseudoorders on ordered semihypergroups in detail, in this section we shall introduce the concepts of order-congruences and strong order-congruences on an ordered semihypergroup, and study the relationship between strong order-congruences and pseudoorders.

Definition 3.1 *Let* $(S, *, \leq)$ *be an ordered semihypergroup. A congruence (resp. strong congruence)* ρ *is called an order-congruence (resp. a strong order-congruence) if there exists an order relation "* $\preceq$ *" on* $(S/\rho, \otimes)$ *such that:*

(1) $(S/\rho, \otimes, \preceq)$ *is an ordered semihypergroup (resp. ordered semigroup), where the hyperoperation "* $\otimes$ *" is defined as one in Lemma 2.2.*

(2) The canonical epimorphism $\varphi : S \to S/\rho, x \mapsto (x)_\rho$ is isotone, that is, φ is a homomorphism (resp. strong homomorphism) from S onto S/ρ.

It is clear that the equality relation 1_S and the universal relation $S \times S$ on S are both order-congruences. In general, an example of order-congruence is given as follows:

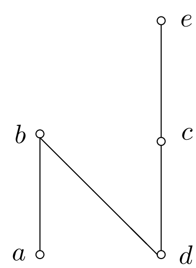
Example 3.2 *We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation "$*$" and the order "$\leq$":*

$*$	a	b	c	d	e
a	$\{b,d\}$	$\{b,d\}$	$\{d\}$	$\{d\}$	$\{d\}$
b	$\{b,d\}$	$\{b,d\}$	$\{d\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

$$\leq := \{(a,a), (a,b), (b,b), (c,c), (c,e), (d,b), (d,c), (d,e), (d,d), (e,e)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a,b), (d,b), (d,c), (c,e)\}.$$



Then $(S, *, \leq)$ is an ordered semihypergroup. Let ρ_1, ρ_2 be congruences on S defined as follows:

$$\rho_1 := \{(a,a), (b,b), (c,c), (d,d), (e,e), (d,e), (e,d)\},$$

$$\rho_2 := \{(a,a), (b,b), (c,c), (d,d), (e,e), (c,e), (e,c)\}.$$

Then $S/\rho_1 = \{\{a\}, \{b\}, \{c\}, \{d,e\}\}, S/\rho_2 = \{\{a\}, \{b\}, \{c,e\}, \{d\}\}$. Moreover, we have

(1) ρ_1 is not an order-congruence on S. In fact, if ρ_1 is an order-congruence on S, then there exists an order "$\preceq_1$" on S/ρ_1 such that $(S/\rho_1, \otimes_1, \preceq_1)$ is an ordered semihypergroup and the mapping $\varphi_1 : S \to S/\rho_1, x \mapsto (x)_{\rho_1}$ is isotone. Since $d \leq c$, we have $(d)_{\rho_1} \preceq_1 (c)_{\rho_1}$. Also, since $c \leq e$, we have $(c)_{\rho_1} \preceq_1 (e)_{\rho_1} = (d)_{\rho_1}$. Then $(d)_{\rho_1} = (c)_{\rho_1}$. Impossible.

(2) ρ_2 is an order-congruence on S. In fact, let $S/\rho_2 = \{x, y, z, w\}$, where $x = \{a\}, y = \{b\}, z = \{c,e\}, w = \{d\}$.

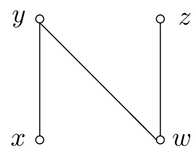
The hyperoperation "$\otimes_2$" and the order "$\preceq_2$" on S/ρ_2 are as follows:

$\otimes_2$	x	y	z	w
x	$\{y,w\}$	$\{y,w\}$	$\{w\}$	$\{w\}$
y	$\{y,w\}$	$\{y,w\}$	$\{w\}$	$\{w\}$
z	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$
w	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$

$$\preceq_2 := \{(x,x),(y,y),(z,z),(w,w),(x,y),(w,y),(w,z)\}.$$

We give the covering relation "$\prec$" and the figure of S/ρ_2 as follows:

$$\prec_2 = \{(x,y),(w,y),(w,z)\}.$$



Then $(S/\rho_2, \otimes_2, \preceq_2)$ is an ordered semihypergroup and the mapping $\varphi_2 : S \to S/\rho_2, x \mapsto (x)_{\rho_2}$ is isotone. Hence ρ_2 is an order-congruence on S.

Proposition 3.3 *Let $(S,*,\leq)$ be an ordered semihypergroup and ρ a pseudoorder on S. Then ρ^* is a strong order-congruence on S, where $\rho^* = \rho \cap \rho^{-1}$.*

Proof By Lemma 2.5, $(S/\rho^*, \otimes, \preceq_\rho)$ is an ordered semigroup, where the order relation $\preceq_\rho$ is defined as follows:

$$\preceq_\rho := \{((x)_{\rho^*},(y)_{\rho^*}) \in S/\rho^* \times S/\rho^* \mid (x,y) \in \rho\}.$$

Also, let $x, y \in S$ and $x \leq y$. Then, since ρ is a pseudoorder on S, $(x,y) \in \leq \subseteq \rho$, and thus $((x)_{\rho^*},(y)_{\rho^*}) \in \preceq_\rho$, i.e. $(x)_{\rho^*} \preceq_\rho (y)_{\rho^*}$. Therefore, ρ^* is a strong order-congruence on S. □

In order to establish the relationship between strong order-congruences and pseudoorders on an ordered semihypergroup, the following lemma is essential.

Lemma 3.4 *Let $(S,*,\leq)$ be an ordered semihypergroup and σ a relation on S. Then the following statements are equivalent:*

(1) σ is a pseudoorder on S.

(2) There exist an ordered semihypergroup $(T,\diamond,\preceq)$ and a strong homomorphism $\varphi : S \to T$ such that

$$\overrightarrow{ker}\varphi := \{(a,b) \in S \times S \mid \varphi(a) \preceq \varphi(b)\} = \sigma,$$

where $\overrightarrow{ker}\varphi$ is called the directed kernel of φ.

Proof (1) $\Rightarrow$ (2). Let σ be a pseudoorder on S. We denote by σ^* the strong congruence on S defined by

$$\sigma^* := \{(a,b) \in S \times S \mid (a,b) \in \sigma, (b,a) \in \sigma\} (= \sigma \cap \sigma^{-1}).$$

Then, by Lemma 2.5, the set $S/\sigma^* := \{(a)_{\sigma^*} \mid a \in S\}$ with the operation $(a)_{\sigma^*} \otimes (b)_{\sigma^*} = (c)_{\sigma^*}$, $\forall c \in a * b$, for all $a, b \in S$ and the order

$$\preceq_\sigma := \{((x)_{\sigma^*},(y)_{\sigma^*}) \in S/\sigma^* \times S/\sigma^* \mid (x,y) \in \sigma\}$$

is an ordered semigroup. Let $T = (S/\sigma^*, \otimes, \preceq_\sigma)$ and φ be the mapping of S onto S/σ^* defined by $\varphi : S \to S/\sigma^* \mid a \mapsto (a)_{\sigma^*}$. Then, by Proposition 3.3, φ is a strong homomorphism from S onto S/σ^* and clearly, $\overrightarrow{ker}\varphi = \sigma$.

(2) $\Rightarrow$ (1). Let $(S,*,\leq)$ be an ordered semihypergroup. If there exist an ordered semihypergroup $(T,\diamond,\preceq)$ and a strong homomorphism $\varphi : S \to T$ such that $\overrightarrow{ker}\varphi = \sigma$, then σ is a pseudoorder on S. Indeed, let $(a,b) \in \leq$. Then, by hypothesis, $\varphi(a) \preceq \varphi(b)$. Thus $(a,b) \in \overrightarrow{ker}\varphi = \sigma$, and we have $\leq \subseteq \sigma$. Moreover, let $(a,b) \in \sigma$ and $(b,c) \in \sigma$. Then $\varphi(a) \preceq \varphi(b) \preceq \varphi(c)$. Hence $\varphi(a) \preceq \varphi(c)$, i.e. $(a,c) \in \overrightarrow{ker}\varphi = \sigma$. Also, if $(a,b) \in \sigma$, then $\varphi(a) \preceq \varphi(b)$. Since $(T,\diamond,\preceq)$ is an ordered semihypergroup, for any $c \in S$ we have $\varphi(a) \diamond \varphi(c) \preceq \varphi(b) \diamond \varphi(c)$. Since φ is a strong homomorphism from S to T, for every $x \in a*c$ and $y \in b*c$, we have

$$\varphi(x) = \varphi(a) \diamond \varphi(c) \preceq \varphi(b) \diamond \varphi(c) = \varphi(y).$$

Then $(x,y) \in \overrightarrow{ker}\varphi = \sigma$, and thus $a*c \ \overline{\overline{\sigma}}\ b*c$. In the same way, it can be shown that $c*a \ \overline{\overline{\sigma}}\ c*b$. □

Theorem 3.5 *Let $(S,*,\leq)$ be an ordered semihypergroup and ρ a strong congruence on S. Then the following statements are equivalent:*

(1) ρ is a strong order-congruence on S.

(2) There exists a pseudoorder σ on S such that $\rho = \sigma \cap \sigma^{-1}$.

(3) There exist an ordered semihypergroup T and a strong homomorphism $\varphi : S \to T$ such that $\rho = ker(\varphi)$, where $ker\varphi = \{(a,b) \in S \times S \mid \varphi(a) = \varphi(b)\}$ is the kernel of φ.

Proof (1) $\Rightarrow$ (2). Let ρ be a strong order-congruence on S. Then there exist an order relation "$\preceq$" on the factor semigroup $(S/\rho,\otimes)$ such that $(S/\rho,\otimes,\preceq)$ is an ordered semigroup, and $\varphi :\ S \to S/\rho$ is a strong homomorphism. Let $\sigma = \overrightarrow{ker}\varphi$. By Lemma 3.4, σ is a pseudoorder on S and it is easy to check that $\rho = \sigma \cap \sigma^{-1}$.

(2) $\Rightarrow$ (3). For a pseudoorder σ on S, by Lemma 3.4, there exist an ordered semihypergroup T and a strong homomorphism $\varphi :\ S \to T$ such that $\sigma = \overrightarrow{ker}\varphi$. Then we have

$$ker\varphi = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1} = \sigma \cap \sigma^{-1} = \rho.$$

(3) $\Rightarrow$ (1). By hypothesis and Lemma 3.4, $\overrightarrow{ker}\varphi$ is a pseudoorder on S. Then, by Lemma 2.5, $\rho = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1}$ is a strong congruence on S. Thus, by the proof of Lemma 3.4, ρ is a strong order-congruence on S. □

Lemma 3.6 *(1) For a strong order-congruence ρ on S, since the order "$\preceq$" such that $(S/\rho,\otimes,\preceq)$ is an ordered semigroup is not unique in general, we have the pseudoorder σ containing ρ such that $\rho = \sigma \cap \sigma^{-1}$ is not unique.*

(2) If σ is a pseudoorder on an ordered semihypergroup S, then $\rho = \sigma \cap \sigma^{-1}$ is the greatest strong order-congruence on S contained in σ. In fact, if ρ_1 is a strong order-congruence on S contained in σ, then $\rho_1 = \rho_1 \cap \rho_1^{-1} \subseteq \sigma \cap \sigma^{-1} = \rho$.

Theorem 3.7 *Let ρ be a strong order-congruence on an ordered semihypergroup $(S,*,\leq)$. Then the least pseudoorder σ containing ρ is the transitive closure of relations $\leq \circ \rho$ (resp. $\rho \circ \leq$), that is,*

$$\sigma = \bigcup_{n=1}^{\infty} (\leq \circ \rho)^n = \bigcup_{n=1}^{\infty} (\rho \circ \leq)^n.$$

Proof

(1) Let $\sigma_1 = \bigcup\limits_{n=1}^{\infty} (\leq \circ\rho)^n$. Clearly, $\rho \subseteq \leq \circ\rho \subseteq \sigma_1$. Similarly, since $\leq \subseteq \leq \circ\, \rho$, we have $\leq \subseteq \sigma_1$.

(2) If $(a,b) \in \sigma_1$, $(b,c) \in \sigma_1$, then there exist $m,n \in Z^+$ such that $(a,b) \in (\leq \circ\rho)^m$ and $(b,c) \in (\leq \circ\rho)^n$, where Z^+ denotes the set of positive integers. Thus $(a,c) \in (\leq \circ\rho)^{m+n} \subseteq \sigma_1$, i.e. σ_1 is transitive.

(3) Let $(a,b) \in \sigma_1$ and $c \in S$. Then there exists $n \in Z^+$ such that $(a,b) \in (\leq \circ\rho)^n$, that is, there exist $a_1, b_1, a_2, b_2, \ldots, a_n \in S$ such that

$$a \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_n \rho b.$$

Since $(S, *, \leq)$ is an ordered semihypergroup and ρ is a strong congruence on S, we have

$$a * c \leq a_1 * c \;\overline{\overline{\rho}}\; b_1 * c \leq a_2 * c \;\overline{\overline{\rho}}\; b_2 * c \leq \cdots \leq a_n * c \;\overline{\overline{\rho}}\; b * c.$$

Then, for any $x \in a*c, y \in b*c$, there exist $x_i \in a_i * c \ (i = 1,2,\ldots,n), y_j \in b_j * c \ (j = 1,2,\ldots,n-1)$ such that

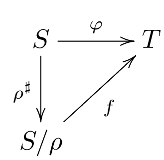
$$x \leq x_1 \rho y_1 \leq x_2 \rho y_2 \leq \cdots \leq x_n \rho y.$$

It thus implies that $(x,y) \in (\leq \circ\rho)^n \subseteq \sigma_1$, and we obtain that $a * c \;\overline{\overline{\sigma_1}}\; b * c$. Similar to the above way, it can be shown that $c * a \;\overline{\overline{\sigma_1}}\; c * b$. Thus $\bigcup\limits_{n=1}^{\infty} (\leq \circ\rho)^n$ is a pseudoorder on S containing ρ.

Furthermore, since σ is transitive, and $\rho \subseteq \sigma, \leq \subseteq \sigma$, we have $\bigcup\limits_{n=1}^{\infty} (\leq \circ\rho)^n \subseteq \sigma$. Thus, by hypothesis, $\sigma = \bigcup\limits_{n=1}^{\infty} (\leq \circ\rho)^n$. In the same way, we can conclude that $\sigma = \bigcup\limits_{n=1}^{\infty} (\rho \circ \leq)^n$. □

Let σ be a pseudoorder on an ordered semihypergroup $(S, *, \leq)$. Then, by Theorem 3.5, $\rho = \sigma \cap \sigma^{-1}$ is a strong order-congruence on S. We denote by $\rho^\sharp$ the canonical epimorphism from S onto S/ρ, i.e. $\rho^\sharp : S \to S/\rho \mid x \mapsto (x)_\rho$, which is a strong homomorphism. In the following, we give out a homomorphism theorem of ordered semihypergroups by pseudoorders, which is a generalization of Theorem 1 in [14]. In fact, in Theorem 3.8, if our ordered semihypergroup is an ordered semigroup, i.e. the hyperoperation is an ordinary binary operation, we shall obtain Theorem 1 in [14].

Theorem 3.8 *Let $(S, *, \leq)$ and $(T, \diamond, \preceq)$ be two ordered semihypergroups, $\varphi : S \to T$ a strong homomorphism. Then: If σ is a pseudoorder on S such that $\sigma \subseteq \overrightarrow{ker}\varphi$, then there exists the unique strong homomorphism $f : S/\rho \to T \mid (a)_\rho \mapsto \varphi(a)$ such that the diagram*



commutes, where $\rho = \sigma \cap \sigma^{-1}$. Moreover, $Im(\varphi) = Im(f)$. Conversely, if σ is a pseudoorder on S for which there exists a strong homomorphism $f : (S/\rho, \otimes, \preceq_\sigma) \to (T, \diamond, \preceq)$ $(\rho = \sigma \cap \sigma^{-1})$ such that the above diagram commutes, then $\sigma \subseteq \overrightarrow{ker}\varphi$.

Proof Let σ be a pseudoorder on S such that $\sigma \subseteq \overrightarrow{ker}\varphi$, $f : S/\rho \to T \mid (a)_\rho \mapsto \varphi(a)$. Then

(1) f is well defined. Indeed, if $(a)_\rho = (b)_\rho$, then $(a,b) \in \rho \subseteq \sigma$. Since $\sigma \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(a), \varphi(b)) \in \preceq$. Furthermore, since $(b,a) \in \sigma \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(b), \varphi(a)) \in \preceq$. Therefore, $\varphi(a) = \varphi(b)$.

(2) f is a strong homomorphism and $\varphi = f \circ \rho^\sharp$. In fact: By Lemma 3.4, there exist an order relation "$\preceq_\sigma$" on the factor semigroup $(S/\rho, \otimes)$ such that $(S/\rho, \otimes, \preceq_\sigma)$ is an ordered semigroup and the canonical epimorphism $\rho^\sharp$ is a strong homomorphism. Moreover, we have

$$\begin{aligned}(a)_\rho \preceq_\sigma (b)_\rho \quad &\Rightarrow \quad (a,b) \in \sigma \subseteq \overrightarrow{ker}\varphi \\ &\Rightarrow \quad \varphi(a) \preceq \varphi(b) \\ &\Rightarrow \quad f((a)_\rho) \preceq f((b)_\rho).\end{aligned}$$

Also, let $(a)_\rho, (b)_\rho \in S/\rho$. For any $(c)_\rho \in (a)_\rho \otimes (b)_\rho$, we have $c \in a * b$. Since φ is a strong homomorphism from S to T, we have

$$f((a)_\rho) \diamond f((b)_\rho) = \varphi(a) \diamond \varphi(b) = \varphi(c) = f((c)_\rho).$$

Furthermore, for any $a \in S$, $(f \circ \rho^\sharp)(a) = f((a)_\rho) = \varphi(a)$, and thus $\varphi = f \circ \rho^\sharp$.

We claim that f is a unique strong homomorphism from S/ρ to T. To prove our claim, let g be a strong homomorphism from S/ρ to T such that $\varphi = g \circ \rho^\sharp$. Then

$$f((a)_\rho) = \varphi(a) = (g \circ \rho^\sharp)(a) = g((a)_\rho).$$

Moreover, $Im(f) = \{f((a)_\rho) \mid a \in S\} = \{\varphi(a) \mid a \in S\} = Im(\varphi)$.

Conversely, let σ be a pseudoorder on S, $f : S/\rho \to T$ is a strong homomorphism, and $\varphi = f \circ \rho^\sharp$. Then $\sigma \subseteq \overrightarrow{ker}\varphi$. Indeed, by hypothesis, we have

$$\begin{aligned}(a,b) \in \sigma \quad &\Leftrightarrow \quad (a)_\rho \preceq_\sigma (b)_\rho \Rightarrow f((a)_\rho) \preceq f((b)_\rho) \\ &\Rightarrow \quad (f \circ \rho^\sharp)(a) \preceq (f \circ \rho^\sharp)(b) \\ &\Rightarrow \quad \varphi(a) \preceq \varphi(b) \Rightarrow (a,b) \in \overrightarrow{ker}\varphi,\end{aligned}$$

where the order $\preceq_\sigma$ on S/ρ is defined as in the proof of Lemma 3.4, that is

$$\preceq_\sigma := \{((x)_\rho, (y)_\rho) \in S/\rho \times S/\rho \mid (x,y) \in \sigma\}.$$

□

Corollary 3.9 *Let $(S, *, \leq)$ and $(T, \diamond, \preceq)$ be two ordered semihypergroups and $\varphi : S \to T$ a strong homomorphism. Then $S/ker\varphi \cong Im(\varphi)$, where $ker\varphi$ is the kernel of φ.*

Proof Let $\sigma = \overrightarrow{ker}\varphi$ and $\rho = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1}$. Then, by Theorems 3.5 and 3.8, ρ is a strong order-congruence on S and $f : S/\rho \to T \mid (a)_\rho \mapsto \varphi(a)$ is a strong homomorphism. Moreover, f is inverse isotone. In fact, let $(a)_\rho, (b)_\rho$ be two elements of S/ρ such that $f((a)_\rho) \preceq f((b)_\rho)$. Then $\varphi(a) \preceq \varphi(b)$, and we have $(a,b) \in \overrightarrow{ker}\varphi$. Thus, by Lemma 3.4, $((a)_\rho, (b)_\rho) \in \preceq_\sigma$, i.e. $(a)_\rho \preceq_\sigma (b)_\rho$. Clearly, $\rho = ker\varphi$. By Remark 2.6(1), $S/ker\varphi \cong Im(f)$. Also, by Theorem 3.8, $Im(f) = Im(\varphi)$. Therefore, $S/ker\varphi \cong Im(\varphi)$. □

Note that if S and T are both ordered semigroups, then Corollary 3.9 coincides with Corollary in [14].

4. Characterizations of (strong) order-congruences on ordered semihypergroups

In the above section, we have characterized the strong order-congruences by the properties of pseudoorders on an ordered semihypergroup. In the current section, we shall give out other characterizations of (strong) order-congruences on ordered semihypergroups. In order to prove the main results in this section, we first introduce the following concept.

Definition 4.1 *Let $(S, *, \leq)$ be an ordered semihypergroup and ρ an equivalence relation on S. A finite sequence of the form $(x, a_1, b_1, a_2, b_2, \ldots, a_{n-1},\ b_{n-1}, a_n, y)$ of elements in S is called a ρ-chain if*

(1) $(a_1, b_1) \in \rho, (a_2, b_2) \in \rho, \ldots, (a_{n-1}, b_{n-1}) \in \rho, (a_n, y) \in \rho$;

(2) $x \leq a_1, b_1 \leq a_2, b_2 \leq a_3, \ldots, b_{n-2} \leq a_{n-1}, b_{n-1} \leq a_n$.

Briefly we write

$$x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_n \rho y.$$

The number n is called the length, x and y initial and terminal elements, respectively, of the ρ-chain. A ρ-chain is called close if its initial and teriminal elements are equal, i.e. $x = y$.

We denote by $\rho^{C_{xy}}$ the set of all ρ-chains with x as the initial and y as the terminal elements in the sequel.

Lemma 4.2 *Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a congruence on S. Then the following statements are true:*

(1) $(x, y) \in (\leq \circ \rho)^n$ *if and only if there exists a ρ-chain with length n in $\rho^{C_{xy}}$, i.e. $\rho^{C_{xy}} \neq \emptyset$.*

(2) For any $z \in S$, if $\rho^{C_{xy}} \neq \emptyset$ for some $x, y \in S$, then for every $u \in x * z$, there exists $v \in y * z$ such that $\rho^{C_{uv}} \neq \emptyset$.

(3) For any $z \in S$, if $\rho^{C_{xy}} \neq \emptyset$ for some $x, y \in S$, then for every $u' \in z * x$, there exists $v' \in z * y$ such that $\rho^{C_{u'v'}} \neq \emptyset$.

Proof

(1) The proof is straightforward by Definition 4.1 and we omit it.

(2) Let $(x, a_1, b_1, a_2, b_2, \ldots, a_{n-1},\ b_{n-1}, a_n, y) \in \rho^{C_{xy}}$ and $z \in S$. Then

$$x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_{n-1} \rho b_{n-1} \leq a_n \rho b.$$

Since $(S, *, \leq)$ is an ordered semihypergroup and ρ is a congruence on S, we have

$$x * z \leq a_1 * z\ \overline{\rho}\ b_1 * z \leq a_2 * z\ \overline{\rho}\ b_2 * z \leq \cdots \leq a_{n-1} * z\ \overline{\rho}\ b_{n-1} * z \leq a_n * z\ \overline{\rho}\ y * z.$$

Then, for any $u \in x * z$, there exist $x_i \in a_i * z\ (i = 1, 2, \ldots, n), y_j \in b_j * z\ (j = 1, 2, \ldots, n-1),\ v \in y * z$ such that

$$u \leq x_1 \rho y_1 \leq x_2 \rho y_2 \leq \cdots \leq x_{n-1} \rho y_{n-1} \leq x_n \rho v.$$

It thus implies that $(u, x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1}, x_n, v) \in \rho^{C_{uv}}$, i.e. $\rho^{C_{uv}} \neq \emptyset$.

(3) It is similar to that of (2) and we omit it. □

Lemma 4.3 *Let* $(S,*,\leq)$ *be an ordered semihypergroup and* ρ *a strong congruence on* S. *If* $\rho^{C_{xy}} \neq \emptyset$ *for some* $x,y \in S$, *then, for any* $z \in S$, *we have* $\rho^{C_{uv}} \neq \emptyset$ *and* $\rho^{C_{u'v'}} \neq \emptyset$ *for every* $u \in x*z, v \in y*z, u' \in z*x, v' \in z*y$.

Proof The proof is similar to that of Lemma 4.2 with a slight modification. □

Lemma 4.4 *Let* S *be an ordered semihypergroup and* ρ *a (strong) congruence on* S. *If* $(x,y) \in \rho$, $(k,z) \in \rho$, *then* $\rho^{C_{xk}} \neq \emptyset$ *if and only if* $\rho^{C_{yz}} \neq \emptyset$.

Proof $(\Longrightarrow)$. If $\rho^{C_{xk}} \neq \emptyset$, by Lemma 4.2(1), there exists $n \in Z^+$ such that $(x,k) \in (\leq \circ\rho)^n$. Since $(x,y) \in \rho$, $(z,k) \in \rho$, we have

$$y \leq y\rho x(\leq \circ\rho)^n k \leq k\rho z,$$

which implies that $(y,z) \in (\leq \circ\rho)^{n+2}$. By Lemma 4.2(1), we have $\rho^{C_{yz}} \neq \emptyset$. □

$(\Longleftarrow)$. Similar to the proof of necessity and we omit it.

Now we shall give a characterization of order-congruences on an ordered semihypergroup.

Theorem 4.5 *Let* $(S,*,\leq)$ *be an ordered semihypergroup and* ρ *a congruence on* S. *Then* ρ *is an order-congruence on* S *if and only if every close* ρ*-chain is contained in a single equivalent class of* ρ.

Proof Let ρ be an order-congruence on S. Then there exists an order $\preceq$ on the factor semihypergroup $(S/\rho,\otimes)$ such that $(S/\rho,\otimes,\preceq)$ is an ordered semihypergroup and $\varphi : S \to S/\rho$ is a homomorphism. For any $x \in S$, and every close ρ-chain $(x,a_1,b_1,\dots,a_n,x)$ in $\rho^{C_{xx}}$, we have

$$x \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \cdots \leq a_n\rho x.$$

Then,

$$\varphi(x) \preceq \varphi(a_1) = \varphi(b_1) \preceq \varphi(a_2) = \varphi(b_2) \preceq \cdots \preceq \varphi(a_n) = \varphi(x).$$

It implies that $\varphi(x) = \varphi(a_1) = \varphi(b_1) = \varphi(a_2) = \varphi(b_2) = \cdots = \varphi(a_n)$. Consequently, $(x,a_1,b_1,\dots,a_n,x)$ is contained in a single ρ-class.

Conversely, since ρ is a congruence on S, by Lemma 2.2, $(S/\rho,\otimes)$ is a semihypergroup. We define a relation "$\preceq$" on the factor semihypergroup $(S/\rho,\otimes)$ as follows:

$$\preceq := \{((x)_\rho,(y)_\rho) \mid \rho^{C_{xy}} \neq \emptyset\}.$$

(1) $\preceq$ is well defined. In fact, let $x_1,y_1 \in S$ be such that $(x)_\rho = (x_1)_\rho$, $(y)_\rho = (y_1)_\rho$. If $(x)_\rho \preceq (y)_\rho$, then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.4, we have $\rho^{C_{x_1y_1}} \neq \emptyset$, and $(x_1)_\rho \preceq (y_1)_\rho$.

(2) $\preceq$ is an ordered relation on S/ρ.

(α) $\preceq$ is reflexive. In fact, since for any $x \in S$, $x \leq x\rho x$, and we have $\rho^{C_{xx}} \neq \emptyset$, i.e. $((x)_\rho,(x)_\rho) \in \preceq$.

(β) $\preceq$ is transitive. Indeed, let $((x)_\rho,(y)_\rho) \in \preceq$, $((y)_\rho,(z)_\rho) \in \preceq$. Then we have $\rho^{C_{xy}} \neq \emptyset$, $\rho^{C_{yz}} \neq \emptyset$. By Lemma 4.2(1), there exist $m,n \in Z^+$ such that $(x,y) \in (\leq \circ\rho)^m$, $(y,z) \in (\leq \circ\rho)^n$. Then we have

$$(x,z) \in (\leq \circ\rho)^m \circ (\leq \circ\rho)^n = (\leq \circ\rho)^{m+n},$$

i.e. $\rho^{C_{xz}} \neq \emptyset$. Thus $((x)_\rho,(z)_\rho) \in \preceq$.

(γ) $\preceq$ is anti-symmetric. In fact, if $((x)_\rho, (y)_\rho) \in \preceq$, $((y)_\rho, (x)_\rho) \in \preceq$, then $\rho^{C_{xy}} \neq \emptyset$, $\rho^{C_{yx}} \neq \emptyset$. Similar to the above proof, it can be obtained that $\rho^{C_{xx}} \neq \emptyset$, i.e. there exists a close ρ-chain in ${}_\rho C_{xx}$ containing x and y. By hypothesis, $(x)_\rho = (y)_\rho$.

(3) $(S/\rho, \otimes, \preceq)$ is an ordered semihypergroup. Indeed, let $(x)_\rho \preceq (y)_\rho$ and $(z)_\rho \in S/\rho$. Then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.2(2), for every $u \in x * z$, there exists $v \in y * z$ such that $\rho^{C_{uv}} \neq \emptyset$, i.e. $(u)_\rho \preceq (v)_\rho$. Thus

$$(x)_\rho \otimes (z)_\rho = \bigcup_{u \in x*z} (u)_\rho \preceq \bigcup_{v \in y*z} (v)_\rho = (y)_\rho \otimes (z)_\rho.$$

Similarly, it can be shown that $(z)_\rho \otimes (x)_\rho \preceq (z)_\rho \otimes (y)_\rho$.

(4) The mapping $\varphi : S \to S/\rho \mid x \mapsto (x)_\rho$ is isotone. In fact, let $x, y \in S$ be such that $x \leq y$. Then $(x, y) \in \leq \circ \rho$, we have $\rho^{C_{xy}} \neq \emptyset$, i.e. $(x)_\rho \preceq (y)_\rho$.

Therefore, ρ is an order-congruence on S. □

Similarly, strong order-congruences on an ordered semihypergroup can be characterized as follows:

Theorem 4.6 *Let $(S, *, \leq)$ be an ordered semihypergroup and ρ a strong congruence on S. Then ρ is a strong order-congruence on S if and only if every close ρ-chain is contained in a single equivalent class of ρ.*

Proof The proof is similar to that of Theorem 4.5 with suitable modification by using Lemma 4.3. □

By Theorem 4.5, we immediately obtain the following corollary:

Corollary 4.7 *If ρ is an order-congruence on an ordered semihypergroup S, then every ρ-class in S is convex.*

Proof Let ρ be an order-congruence on S and I a congruence class of ρ. If $x \leq y \leq z$ and $x, z \in I$, then $(x)_\rho = (z)_\rho$. Thus we have $x \leq y \rho y \leq z \rho x$. Hence (x, y, y, z, x) is a close ρ-chain and by Theorem 4.5 we have $(x)_\rho = (y)_\rho = (z)_\rho$. It thus follows that $y \in I$, and I is convex. □

Furthermore, we have the following theorem.

Theorem 4.8 *Let $(S, *, \leq)$ be an ordered semihypergroup and I a hyperideal of S. Then I is a congruence class of one order-congruence on S if and only if I is convex.*

Proof The proof is straightforward by Corollary 4.7.

Conversely, let ρ_I be the Rees congruence induced by I on S. By Remark 2.4(1), I is a congruence class of ρ_I. Now we define a relation " $\preceq_I$ " on the factor semihypergroup $(S/\rho_I, \otimes)$ as follows:

$$(x)_{\rho_I} \preceq_I (y)_{\rho_I} \Leftrightarrow (x \leq y) \text{ or } (x \leq a, a' \leq y \text{ for some } a, a' \in I).$$

We claim that ρ_I is an order-congruence on S. To prove our claim, we first show that $\preceq_I$ is order relation on S/ρ_I, i.e. $\preceq_I$ is reflexive, anti-symmetric, and transitive.

(1) Let $(x)_{\rho_I}$ be any element of S/ρ_I. Then, since $x \leq x$, we have $(x)_{\rho_I} \preceq_I (x)_{\rho_I}$.

(2) Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $(y)_{\rho_I} \preceq_I (x)_{\rho_I}$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$, and $y \leq x$ or $y \leq b, b' \leq x$ for some $b, b' \in I$. We consider the following four cases:

Case 1. If $x \leq y$ and $y \leq x$, then $x = y$, and thus $(x)_{\rho_I} = (y)_{\rho_I}$.

Case 2. If $x \leq y$ and $y \leq b, b' \leq x$ for some $b, b' \in I$, then $b' \leq x \leq y \leq b$. Since I is convex and $b, b' \in I$, we have $x, y \in I$. Thus $(x)_{\rho_I} = (y)_{\rho_I} = I$.

Case 3. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq x$. Similar to the proof of Case 2, we have $(x)_{\rho_I} = (y)_{\rho_I}$.

Case 4. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq b, b' \leq x$ for some $b, b' \in I$. Then $b' \leq x \leq a$ and $a' \leq y \leq b$. Since I is convex, we have $x, y \in I$. Thus $(x)_{\rho_I} = (y)_{\rho_I}$.

(3) Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $(y)_{\rho_I} \preceq_I (z)_{\rho_I}$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$, and $y \leq z$ or $y \leq b, b' \leq z$ for some $b, b' \in I$. There are four cases to be considered:

Case 1. If $x \leq y$ and $y \leq z$, then $x \leq z$, and thus $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$.

Case 2. If $x \leq y$ and $y \leq b, b' \leq z$ for some $b, b' \in I$, then $x \leq y \leq b$ and $b' \leq z$. By the definition of $\preceq_I$, $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Case 3. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq z$. Analogous to the proof of Case 2, we have $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Case 4. Let $x \leq a, a' \leq y$ for some $a, a' \in I$ and $y \leq b, b' \leq z$ for some $b, b' \in I$. Then $x \leq a$ and $b' \leq z$. Hence $(x)_{\rho_I} \preceq_I (z)_{\rho_I}$.

Now we show that $(S/\rho_I, \otimes_I, \preceq_I)$ is an ordered semihypergroup. Let $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$ and $z \in S$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$. We consider the following two cases:

Case 1. If $x \leq y$, then $x * z \leq y * z$. Thus for every $u \in x * z$, there exists $v \in y * z$ such that $u \leq v$, and we have $(u)_{\rho_I} \preceq_I (v)_{\rho_I}$. Thus

$$(x)_{\rho_I} \otimes_I (z)_{\rho_I} = \bigcup_{u \in x*z} (u)_{\rho_I} \preceq_I \bigcup_{v \in y*z} (v)_{\rho_I} = (y)_{\rho_I} \otimes_I (z)_{\rho_I}.$$

Case 2. Let $x \leq a, a' \leq y$ for some $a, a' \in I$. Then $x * z \leq a * z, a' * z \leq y * z$. Thus for every $u \in x * z$, there exists $b \in a * z$ such that $u \leq b$, and for some $b' \in a' * z$ there exists $v \in y * z$ such that $b' \leq v$. Since I is a hyperideal of S and $a, a' \in I$, we have $b \in a * z \subseteq I, b' \in a' * z \subseteq I$. On the other hand, $u \leq b, b' \leq v$ for some $b, b' \in I$. Hence $(u)_{\rho_I} \preceq_I (v)_{\rho_I}$, and thus $(x)_{\rho_I} \otimes_I (z)_{\rho_I} \preceq_I (y)_{\rho_I} \otimes_I (z)_{\rho_I}$.

Similar to the above way, we can show that $(z)_{\rho_I} \otimes_I (x)_{\rho_I} \preceq_I (z)_{\rho_I} \otimes_I (y)_{\rho_I}$. Therefore, $(S/\rho_I, \otimes_I, \preceq_I)$ is an ordered semihypergroup.

Furthermore, by the definition of $\preceq_I$, it can be obtained that the canonical epimorphism $\varphi : S \to S/\rho_I, x \mapsto (x)_{\rho_I}$ is isotone. Thus ρ_I is an order-congruence on S. The proof is completed. □

By the proof of above theorem, we immediately obtain the following corollary:

Corollary 4.9 *Let $(S, *, \leq)$ be an ordered semihypergroup and I an ordered hyperideal of S. Then $(S/\rho_I, \otimes_I, \preceq_I)$ forms an ordered semihypergroup and the Rees congruence ρ_I induced by I on S is an order-congruence, where the order relation "$\preceq_I$" on S/ρ_I is defined as follows:*

$$(x)_{\rho_I} \preceq_I (y)_{\rho_I} \Leftrightarrow (x \leq y) \text{ or } (x \leq a, a' \leq y \text{ for some } a, a' \in I).$$

As an application of Corollary 4.9, we can give an answer to the open problem given by Davvaz et al. in [7].

Open Problem Is there a congruence relation (also called regular relation in [7]) ρ on an ordered semihypergroup $(S, *, \leq)$ for which S/ρ is an ordered semihypergroup?

To solve the above problem, we need only show that ρ_I is not a strong congruence on S in general. We illustrate it by the following example.

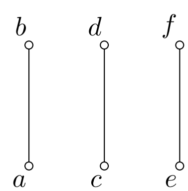
Example 4.9 *We consider a set* $S := \{a, b, c, d, e, f\}$ *with the following hyperoperation "* $*$ *" and the order "* $\leq$ *":*

$*$	a	b	c	d	e	f
a	$\{a\}$	$\{a, b\}$	$\{c\}$	$\{c, d\}$	$\{e\}$	$\{e, f\}$
b	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$
c	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{e\}$	$\{e, f\}$	$\{c\}$	$\{c, d\}$	$\{e\}$	$\{e, f\}$
f	$\{f\}$	$\{f\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d), (e, e), (e, f), (f, f)\}.$$

We give the covering relation " $\prec$ *" and the figure of* S *as follows:*

$$\prec = \{(a, b), (c, d), (e, f)\}.$$



Then $(S, \circ, \leq)$ *is an ordered semihypergroup. Let* $I = \{c, d\}$*. One can easily verify that* I *is an ordered hyperideal of* S*. Then* $\rho_I = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (c, d), (d, c)\}$*. By Lemma 2.3,* ρ_I *is a congruence on* S*. However, we claim that* ρ_I *is not a strong congruence on* S*. In fact, since* $(e, e) \in \rho_I$*, while* $e * b \ \overline{\overline{\rho_I}} \ e * b$ *does not hold.*

As a generalization of Proposition 2.7 in [23], we have the following theorem. The following theorem can be proved using similar techniques as in the proof of Theorem 4.8.

Theorem 4.10 *Let* $(S, *, \leq)$ *be an ordered semihypergroup and* I *an ordered hyperideal of* S*. We define a relation "* $\preceq_1$ *" on* S/ρ_I $(= \{\{x\} \mid x \in S \backslash I\} \cup \{I\})$ *as follows:*

$$\preceq_1 := \{(I, \{x\}) \mid x \in S \backslash I\} \cup \{(\{x\}, \{y\}) \mid x, y \in S \backslash I, x \leq y\} \cup \{(I, I)\}.$$

Then $(S/\rho_I, \otimes_I, \preceq_1)$ *is an ordered semihypergroup, and* ρ_I *is an order-congruence on* S*.*

Proposition 4.11 *Let* $(S, *, \leq)$ *be an ordered semihypergroup and* I *an ordered hyperideal of* S*. Then the order relations in Corollary 4.9 and Theorem 4.11 are different. Moreover,* $\preceq_I \subseteq \preceq_1$ *.*

Proof Let $(x)_{\rho_I}, (y)_{\rho_I} \in S/\rho_I$ and $(x)_{\rho_I} \preceq_I (y)_{\rho_I}$. Then $x \leq y$ or $x \leq a, a' \leq y$ for some $a, a' \in I$. Since $(I] = I$, we have $x \leq y$ or $x \in I$ and $a' \leq y$ for some $a' \in I$. The first case implies $(x)_{\rho_I} \preceq_1 (y)_{\rho_I}$, and the second case implies $(x)_{\rho_I} \preceq_1 (a')_{\rho_I} \preceq_1 (y)_{\rho_I}$, i.e. $(x)_{\rho_I} \preceq_1 (y)_{\rho_I}$. Thus $\preceq_I \subseteq \preceq_1$. □

The following example shows that $\preceq_I \subsetneqq \preceq_1$ in general.

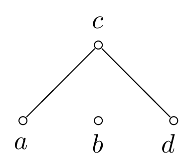
Example 4.12 *We consider a set* $S := \{a, b, c, d\}$ *with the following hyperoperation* "$*$" *and the order* "$\leq$":

$*$	a	b	c	d
a	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a\}$
b	$\{a, d\}$	$\{b\}$	$\{a, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{c\}$	$\{a, d\}$
d	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$

$$\leq := \{(a, a), (a, c), (b, b), (c, c), (d, c), (d, d)\}.$$

We give the covering relation "$\prec$" *and the figure of* S *as follows:*

$$\prec = \{((a, c), (d, c)\}.$$



Then $(S, *, \leq)$ *is an ordered semihypergroup. It is easy to check that* $I = \{a, d\}$ *is an ordered hyperideal of* S. *Since* $a \not\leq b$ *and there does not exist* $x \in I$ *such that* $x \leq b$, *we have* $(a)_{\rho_I} \not\preceq_I (b)_{\rho_I}$. *However, by the definition of* $\preceq_1$, *we have* $(a)_{\rho_I} \preceq_1 (b)_{\rho_I}$.

In the following we shall consider the strong order-congruence generated by a strong congruence on an ordered semihypergroup.

Definition 4.13 *Let* ρ *be a strong congruence on an ordered semihypergroup* S. *A strong order-congruence* σ *is called the strong order-congruence generated by* ρ *on* S, *if* σ *satisfies the following conditions:*

(1) $\rho \subseteq \sigma$.

(2) If there is a strong order-congruence η *on* S *such that* $\rho \subseteq \eta$, *then* $\sigma \subseteq \eta$.

Theorem 4.14 *Let* ρ *be a strong congruence on an ordered semihypergroup* $(S, *, \leq)$. *Then*

(1) If we define a relation $\underline{\rho}$ *on* S *as follows:*

$$(x, y \in S)\ (x, y) \in \underline{\rho} \text{ if and only if } \rho^{C_{xy}} \neq \emptyset,$$

then $\underline{\rho}$ *is a pseudoorder on* S.

(2) R_ρ *is a relation on* S *defined as follows:*

$$(x, y \in S)\ (x, y) \in R_\rho \Longleftrightarrow (x, y) \in \underline{\rho} \text{ and } (y, x) \in \underline{\rho}.$$

Then R_ρ *is the strong order-congruence generated by* ρ *on* S.

Proof

(1) Let $x, y \in S$ such that $x \leq y$. Then there is a ρ-chain from x to y: (x, y, y), i.e. $\rho^{C_{xy}} \neq \emptyset$. Thus $x \leq y$ implies $x\underline{\rho}y$, and we have $\leq \subseteq \underline{\rho}$. Suppose that $(x, y) \in \underline{\rho}$ and $(y, z) \in \underline{\rho}$. Then there exist $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{n-1},\ c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_m \in S$ such that

$$x \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \cdots \leq a_{n-1}\rho b_{n-1} \leq a_n\rho y,$$

$$y \leq c_1\rho d_1 \leq c_2\rho d_2 \leq \cdots \leq c_{m-1}\rho d_{m-1} \leq d_m\rho z.$$

Thus, $x \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \cdots \leq a_{n-1}\rho b_{n-1} \leq a_n\rho y \leq c_1\rho d_1 \leq c_2\rho d_2 \leq \cdots \leq c_{m-1}\rho d_{m-1} \leq d_m\rho z$, which is a ρ-chain from x to z. Hence $(x, z) \in \underline{\rho}$ and $\underline{\rho}$ is transitive. Furthermore, let $(x, y) \in \underline{\rho}$ and $z \in S$. Then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.3, for every $u \in x * z, v \in y * z, u' \in z * x, v' \in z * y$, we have $\rho^{C_{uv}} \neq \emptyset$ and $\rho^{C_{u'v'}} \neq \emptyset$, which imply that $(u, v) \in \underline{\rho}$ and $(u', v') \in \underline{\rho}$. It thus follows that $x * z \;\overline{\overline{\underline{\rho}}}\; y * z$ and $z * x \;\overline{\overline{\underline{\rho}}}\; z * y$. Therefore, $\underline{\rho}$ is a pseudoorder on S.

(2) By (1), $\underline{\rho}$ is a pseudoorder on S. Since $R_\rho = \underline{\rho} \cap \underline{\rho}^{-1}$, by Proposition 3.3 R_ρ is a strong order-congruence on S. We claim that R_ρ is the strong order-congruence generated by ρ on S. To prove our claim, let $(x, y) \in \rho$. Since ρ is a strong congruence on S, we have $(y, x) \in \rho$. Consequently, $(x, y) \in R_\rho$. Hence $\rho \subseteq R_\rho$. Furthermore, suppose that η is a strong order-congruence on S and $\rho \subseteq \eta$. Then $R_\rho \subseteq \eta$. Indeed, let $(x, y) \in R_\rho$. Then $(x, y) \in \underline{\rho}$ and $(y, x) \in \underline{\rho}$. By definition of $\underline{\rho}$, there exist $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{n-1}$, $c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_{m-1} \in S$ such that

$$x \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \cdots \leq a_{n-1}\rho b_{n-1} \leq a_n\rho y,$$

$$y \leq c_1\rho d_1 \leq c_2\rho d_2 \leq \cdots \leq c_{m-1}\rho d_{m-1} \leq d_m\rho x.$$

Thus, by $\rho \subseteq \eta$, we have $x \leq a_1\eta b_1 \leq a_2\eta b_2 \leq \cdots \leq a_{n-1}\eta b_{n-1} \leq a_n\eta y \leq c_1\eta d_1 \leq c_2\eta d_2 \leq \cdots \leq c_{m-1}\eta d_{m-1} \leq d_m\eta x$. Since η is a strong order-congruence on S, by Theorem 4.6 we can conclude that the closed η-chain $(x, a_1,\ b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, a_n, y,\ c_1, d_1, c_2, d_2, \ldots, c_{m-1}, d_{m-1}, d_m, x)$ is contained in a single equivalence class of η. In particular, we have $(x, y) \in \eta$. Therefore, R_ρ is the strong order-congruence generated by ρ on S.

□

By Theorem 4.15, we immediately obtain the following corollary:

Corollary 4.15 *Every strong congruence of an ordered semihypergroup S is contained in a strong order-congruence of S.*

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