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Graphs of BCI/BCK-algebras

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Abstract: The aim of this paper is to study special graphs of BCI/BCK-algebras. In this paper, we introduce one kind of graph of BCI-algebras based on branches of X and two kinds of graphs of BCK-algebras based on ideal I. Then we study some of the essential properties of graph theory on the basis of those structures. In particular, we study the planar, outerplanar, toroidal, K-connected, chordal, K-partite, and Eulerian properties of graph theory.

Key words: BCI-algebra, chordal graph, connected graph, clique, K-partite graph, planar graph, girth

1. Introduction

In recent years, the study of the zero-divisor graphs of algebraic structures and ordered structures has received a lot of attention from researchers. The idea of the zero-divisor graph of a commutative ring with unity was introduced by Beck [8], who was particularly interested in the coloring of commutative rings with unity. He introduced the associated graph G to a ring R so that the vertex set of G contains elements of R and two vertices x and y of R are connected to each other if and only if $xy = 0$. Many mathematicians, such as Anderson and Naseer [5], Anderson and Livingston [6], Maimani and Pournaki [4, 18], Yassemi and Khashyarmanesh [2, 3, 21], and Torkzadeh and Ahadpanah [22], investigated the interplay between properties of the algebraic structure and graph theoretic properties. Motivated by these works, in this paper we study the associated graphs of BCI/BCK-algebras as two classes of abstract algebras, as introduced by Imai and Iseki [13, 14] in 1996. The associated graph $G(X)$ to BCI/BCK-algebras was well studied by Jun and Lee in [17] and Borzooei and Zahiri in [25]. We also introduce the zero-divisor graphs $\Gamma^I(X)$ and $\Gamma_I(X)$ associated with BCK-algebra regarding an ideal I, where the vertex set of graphs $\Gamma^I(X)$ and $\Gamma_I(X)$ are the set of elements of X and two distinct vertices x and y are adjacent in graph $\Gamma^I(X)$ if and only if $x*y \in I$ and $y*x \in I$, and two distinct vertices x and y are adjacent in graph $\Gamma_I(X)$ if and only if $x*y \in I$ or $y*x \in I$. In this article, we introduce the concepts of diameter and girth of graphs. We show that $\Gamma_I(X)$ and $\Gamma^I(X)$ must be connected with a diameter less than or equal 2, $gr(\Gamma_I(X)) := 3$. We also study properties of graphs $\Gamma^I(X), \Gamma_I(X)$ such as planar, outerplanar, K-connected, chordal, K-partite, and Eulerian.

2. Preliminaries of graph theory

Definition 2.1 *[10, 12, 15, 16] A graph $G = (V, E)$ is connected if any of vertices x, y of G are connected by a path in G; otherwise, the graph is disconnected. A graph G is called a complete graph on n vertices if*

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$\mid V(G) \mid = n$, $xy \in E(G)$, for any distinct element $x, y \in V(G)$, denoted by K_n. For any $T \subseteq V(G)$, the graph with vertex set $V(G) - T$ and edge set $E(G) - T'$ is denoted by $G - T$, where $T' = \{xy \in E(G); x \in T, y \in G\}$. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is called a star graph if there exists a vertex x in G in such a way that every vertex in G connected to x and other vertices in G do not connect to each other. In graph G with vertex set $V(G)$, the distance between two distinct vertices x and y, denoted by $d(x, y)$, is the length of the shortest path connecting x and y, if such a path exists; otherwise, we set $d(x, y) := \infty$. The diameter of graph G is $diam(G) := sup\{d(x, y),$ and x, y are distinct vertices of $G\}$. Also, the girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle in G if G has a cycle; otherwise, we get $gr(G) := \infty$. For a vertex x in graph G, the neighborhood of x is the set of vertices adjacent to x, denoted by $N_G(x)$, $N_G[x] = N_G(x) \cup \{x\}$, $deg(x) = \mid N_G(x) \mid$. A graph G is called regular of degree k when every vertex has precisely K neighbors. A cubic graph is a graph in which all vertices have degree three. In other words, a cubic graph is a 3-regular graph. A graph G is chordal if every cycle of length at least 4 has a chord, which is not part of the cycle but connects two vertices of the cycle. The greatest induced complete subgraph denotes a clique. A graph G is called K-partite when its vertex set can be partitioned into K-disjoint parts $X_1, X_2, \dots, X_k$, so that for $x, y \in X_i, i = 1, \dots k$, we have $xy \notin E(G)$, and for $x \in X_i, y \in X_j, i \neq j, i, j = 1, \dots k$, we have $xy \in E(G)$. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. Moreover, for distinct vertices x and y, we use the notation $x - y$ to show that x is connected to y. A subset A of the vertices is called an independent set if the induced subgraph on A has no edges. The maximum size of an independent set in a graph G is called the independence number of G and is denoted by $\alpha(G)$. Let G be a graph and let $P = \{V_1, \dots, V_k\}$ be a partition of the vertex set of G into nonempty classes. The quotient $\frac{G}{P}$ of G is the graph whose vertices are the sets $V_1, \dots, V_k$ and whose edges are the pairs $[V_i, V_j]$, $i \neq j$, so that there are $u_i \in V_i$, $u_j \in V_j$ with $[u_i, u_j] \in E(G)$. Let $P = (V, \leq)$ be a poset. If $x \leq y$ but $x \neq y$, then we write $x < y$. If x and y are in V, then y covers x in P if $x < y$ and there is no z in V with $x < z < y$. Two vertices of G are orthogonal, denoted by $x \perp y$, if x and y are adjacent to G and there is no vertex $z \in G$, which can be adjacent to both x and y. A graph G is called complemented if for each vertex x of G there is a vertex y of G, in such a way that $x \perp y$. A set S is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The dominating number $\gamma(G)$ is the minimum cardinality of a dominating set in G. An isomorphism of graphs G and H is a bijection between vertex sets of G and H, $f : V(G) \to V(H)$ so that any two vertices x and y of G are adjacent to G if and only if $f(x)$ and $f(y)$ are adjacent to H. If an isomorphism exists between two graphs, then the graphs are called isomorphic and we write $G \simeq H$.

Definition 2.2 *[7] We say that a graph G is K-colorable if we can assign the colors $\{1, \dots, k\}$ to the vertices in $V(G)$, in such a way that every vertex gets exactly one color and no edge in $E(G)$ has both its endpoints colored in the same color. We call this proper coloring, though sometimes we will just call it "coloring." If K is the smallest number so that G admits K-coloring, we say that the chromatic number of G is K and write $\chi(G) = K$. If graph G contains a clique with n elements, and every clique has at most n elements, we say that the clique number of G is n and write $\omega(G) = n$. Moreover, we have $\chi(G) \geq \omega(G)$.*

Definition 2.3 *[2] A walk or path graph has vertices $v_1, v_2, \dots, v_n$ and edges $e_1, e_2, \dots, e_{n-1}$ in such a way that edge e_k joins vertices v_k and v_{k+1}, denoted by P_n. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Graph G is planar if it can be drawn on the plane without edges having to cross. Proving that a graph is planar amounts to redrawing the edges in such a way* *that no edges will cross. The vertices may have to be moved around and the edges drawn in an indirect manner. Kuratowski's theorem says that a finite graph is planar if and only if it does not contain a subdivision of* k_5 *or* $k_{3,3}$. *The chromatic number of any planar graph is less than or equal to 4.*

Example 2.4 *[2] Figure 1 shows that the 3-cube and complete graph* K_4 *are planar.*

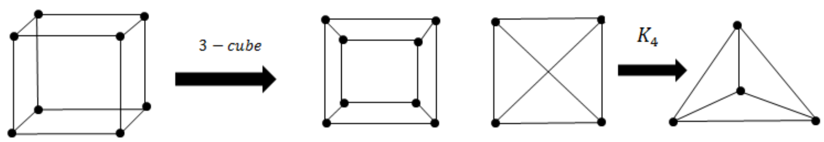

Figure 1.

Definition 2.5 *[10] Let* G *be a plane graph. The connected pieces of the plane that remain when the vertices and edges of* G *are removed are called the region of* G. *A "face" marks a region bounded by edges. An undirect graph is an outerplanar graph if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of* K_4 *or* $K_{2,3}$.

Example 2.6 *In Figure 2, graph* G *has 3 regions, as follows:*

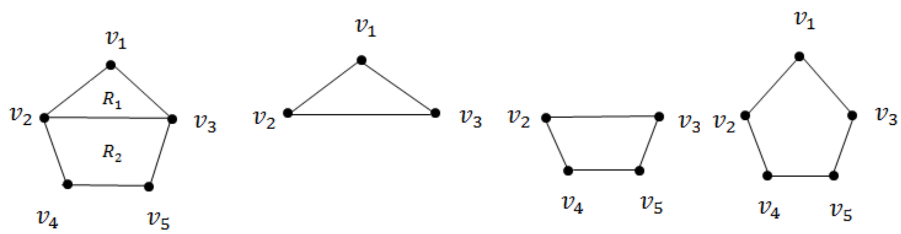

Figure 2.

Definition 2.7 *[1] Number* g *is called the genus of the surface if it is homeomorphic to a sphere with* g *handles, or equivalently holes. Besides, genus* g *of a graph* G *is meant to be the smallest genus of all surfaces so that graph* G *can be drawn on it without edge-crossing. The graphs of genus 0 are precisely the planar graphs since the genus of plane is zero. The graphs that can be drawn on a torus without edge-crossing are called toroidal. They have genus 1 since the genus of a torus is 1. The notation* $\gamma(G)$ *stands for the genus of a graph* G. *The complete graphs* K_7, K_5, K_6 *and complete bipartite graph* $K_{3,3}$ *are examples of toroidal graphs. A cubic graph with 14 vertices embedded on a torus is toroidal.*

Theorem 2.8 *[20] For positive integers* m *and* n, *we have:*

(i) $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$ *if* $n \geq 3$,

(ii) $\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$ *if* $m, n \geq 2$.

Definition 2.9 *[9] The adjacency matrix* A *is an* $n \times n$ *matrix where* $n = \mid G \mid$ *depicts which vertices are connected by an edge. If vertices* i *and* j *are adjacent, then* $a_{ij} = 1$. *If vertices* i *and* j *are not connected,* *then* $a_{ij} = 0$. *If* G *is a simple graph, then* $a_{ii} = 0$ *for all* i, *because there are no loops. It is also because simple implies undirected,* $a_{ij} = a_{ji}$ *for all* $i, j \in V$. *We also denote the characteristic polynomial of matrix* G *by* $\chi(G, \lambda)$, *which is* $\det(\lambda I - G)$. *An eigenvalue is a root of the characteristic polynomial associated with a matrix. The (ordinary) spectrum of a finite graph* G *is a set of eigenvalues together with their multiplicities. This set of all* n *eigenvalues of the* $n \times n$ *adjacency matrix is denoted as* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, *where* $\lambda_i \geq \lambda_j$, $\forall i < j$.

Theorem 2.10 *[9] Let* G *be the complete graph* K_n *on* n *vertices and* $\chi(G, \lambda)$ *be the characteristic polynomial of matrix* G. *Then the following statements hold:*

(i) $\chi(G, \lambda) = (\lambda - n + 1)(\lambda + 1)^n$

(ii) The spectrum of G *is* $\{(n-1)^1, (-1)^{n-1}\}$.

Definition 2.11 *[10] A closed walk in a graph* G *containing all the edges of* G *is called an Euler line in* G. *A graph containing an Euler line is called an Euler graph. We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected. The following problem, often referred to as the bridges of Konigsberg problem, was first solved by Euler in the 18th century. The problem was rather simple. The town of Konigsberg consists of four islands and seven bridges. Is it possible, by beginning anywhere and ending anywhere, to walk through the town by crossing all seven bridges but not crossing any bridge twice? Euler modeled the problem representing the four land areas by four vertices and the seven bridges by seven edges joining these vertices. This is illustrated in Figure 3. We see from graph* G *of the Konigsberg bridges that not all its vertices are of even degrees. Thus,* G *is not an Euler graph, implying that there is no closed walk in* G *containing all the edges of* G. *Thus, it is not possible to walk over each of the seven bridges exactly once and return to the starting point. Euler's theorem says that a connected graph* G *is Euler if and only if all vertices of* G *are of even degrees.*

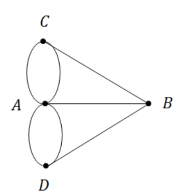

Figure 3.

Example 2.12 *Consider the graph shown in Figure 4. Clearly,* $v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_3e_6v_6e_7v_1$ *on the left is an Euler line, whereas the graph shown on the right is non-Eulerian.*

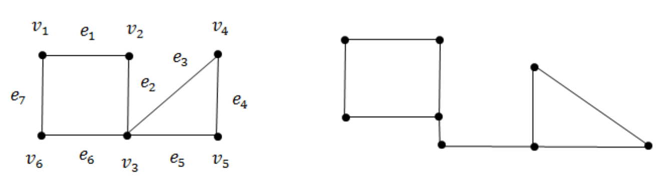

Figure 4.

Example 2.13 *Figure 5 has an Euler walk. On the other hand, the following pictures can be drawn on paper without ever lifting the pencil.*

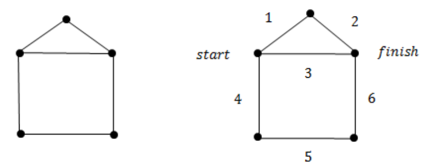

Figure 5.

Definition 2.14 *[23] A lattice is an algebra* $L = (L, \wedge, \vee)$ *that satisfies the following conditions for all* $a, b, c \in L$*:*

(i) $a \wedge a = a, a \vee a = a$,

(ii) $a \wedge b = b \wedge a, a \vee b = b \vee a$,

(iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c), a \vee (b \vee c) = (a \vee b) \vee c$,

(iv) $a \vee (a \wedge b) = a \wedge (a \vee b)$.

An equivalent definition of a lattice is as follows:

Definition 2.15 *[23] A lattice is a poset any two of whose elements have a greatest lower bound* $(g.l.b)$*, denoted by* $x \wedge y$*, and a least upper bound* $(l.u.b)$*, denoted by* $x \vee y$.

Definition 2.16 *[11] Let* $P = (X, \leq_p)$ *be a poset. Then a comparability graph (com-graph) of* $P = (X, \leq_P)$ *is the graph* $com(P) = (X, E_{com(P)})$*, where* $xy \in E_{com(P)}$ *if and only if* x *is comparable with* y *in* P.

Definition 2.17 *[11] An element* b *in a lattice* L *is left modular if for all* $a, c \in L$ *so that* $a \leq c$ *we have* $a \vee (b \wedge c) = (a \vee b) \wedge c$. L *is modular if every element is left modular. A poset or lattice is (upper) semimodular if, whenever two elements have a common lower cover, they have a common upper cover; (lower) semimodularity is defined dually.*

Let $G = (V, E)$ *be a graph with vertex set* V *and edge set* $E \subseteq V \times V$*. It is* n*-connected (where* $n \in N_0$*) if the restriction of* G *to the vertices* $V - C$ *is connected whenever* $C \subseteq V$ *has fewer than* n *elements. A chain* C *has rank* d *if its cardinality of* C *is* $d+1$*, in which case we write* $\mid C \mid= d+1$*. A poset is ranked at* d *if every maximal chain has rank* d.

Example 2.18 *Figure 6 shows a semimodular poset with no simplicial element.*

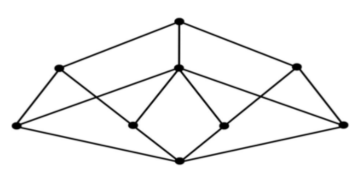

Figure 6.

Theorem 2.19 *[11] Let L be a (finite or infinite) semimodular lattice of rank d that is not a chain. Following this, the comparability graph of L is $(d+1)$-connected if and only if L has no simplicial element, where $z \in L$ is simplicial if the elements comparable to z form a chain.*

3. Introduction to BCI/BCK-algebras

Definition 3.1 *[19, 24] A BCI-algebra $(X, *, 0)$ is an algebra of type $(2,0)$ satisfying the following conditions:*

$(BCI1)$ $((x*y)*(x*z))*(z*y) = 0$,

$(BCI2)$ $x*0 = x$,

$(BCI3)$ $x*y = 0$, $y*x = 0$ *imply* $y = x$.

If X satisfies the following identity:

$$(\forall x \in X)(0 * x = 0),$$

then X is called a BCK-algebra. Any BCI/BCK-algebra X satisfies the following conditions:

(i) $(x*(x*y))*y = 0$,

(ii) $x*x = 0$,

(iii) $(x*y)*z = (x*z)*y$,

(iv) $x \le y$ *implies* $x*z \le y*z$ *and* $z*y \le z*x$*, for any* $z \in X$.

Moreover, the relation $\le$ was defined by $x \le y \leftrightarrow x*y = 0$, for any $x, y \in X$, which is a partial-order on X. Set $P = \{x \in X; 0*(0*x) = x\}$ is called the P-semisimple part of X. It is the set of all minimal elements of X with respect to the BCI-ordering of X. For any $a \in, P$ denotes $V(a) = \{x \in X; a*x = 0\}$, which is called the branch of X with respect to a. Also, $\{V(a); a \in P\}$ forms a partition for X. The element $x \in V(a)$ is called an a-atom if $x \neq a$ and $y*x = 0$ implies $y = a$ or $y = x$ for all $y \in X$. Briefly, 0-atom is called an atom. $(X, *, 0)$ is said to be commutative if it satisfies, for all x, y in X,

$$x*(x*y) = y*(y*x).$$

Besides, X is called associative if $(x*y)*z = x*(y*z)$ for any $x, y, z \in X$. In any associative BCI-algebra, $x*y = y*x$ and $0*x = x$, for any $x, y \in X$. A subset Y of X is called a subalgebra of X if the constant 0 of X is in Y, and $(Y, *, 0)$ itself forms a BCI-algebra.

Example 3.2 *[19, 24] Assume that $(X, \le)$ is a partially ordered set with the least element 0. Define operation * on X by*

$$x*y = \begin{cases} 0 & \text{if } x \le y, \\ x & \text{if } x \not\le y. \end{cases}$$

Therefore, $(X, *, 0)$ is a BCK-algebra.

Proposition 3.3 *[19, 24] Let $(X, *_1, 0)$ and $(Y, *_2, 0)$ be two BCK-algebras so that $X \bigcap Y = \{0\}$ and * be the binary operation on $X \bigcup Y$ as follows: for any $x, y \in X \bigcup Y$*

$$x * y = \begin{cases} x *_1 y & \text{if } x, y \in X, \\ x *_2 y & \text{if } x, y \in Y, \\ x & \text{otherwise.} \end{cases}$$

Therefore, $(X, *, 0)$ is a BCK-algebra.

Definition 3.4 *[19, 24] A subset I is called an ideal of X if it satisfies the following conditions:*

(i) $0 \in I$,

(ii) $(\forall x, y \in X)$ $(x * y \in I, y \in I \to x \in I)$.

An ideal P of X is prime if $x * (x * y) \in P$ implies $x \in P$ or $y \in P$.

An ideal I of X is implicative if:

(i) $0 \in I$,

(ii) $(x * (y * x)) * z \in I$ *and* $z \in I$ *imply* $x \in I$, *for all* $x, y, z \in X$.

An ideal I is maximal if it is a proper ideal of X and not a proper subset of any proper ideal of X.

*An ideal I of X is called closed if it is closed under the * multiplication on X. (i.e. I is a subalgebra of X). Moreover, an ideal I of X is closed if and only if $0 * x \in I$ for all $x \in I$.*

Theorem 3.5 *[19, 24] Let X_1, X_2 be two BCK-algebras, $X_1 \bigcap X_2 = \{0\}, X = X_1 \bigcup X_2$. Then I_1, I_2 are two ideals of X_1, X_2, respectively, if and only if $I = I_1 \bigcup I_2$ is an ideal of X.*

Theorem 3.6 *[19, 24] Let I be an ideal of X. Then the following are equivalent:*

(i) I is maximal and implicative.

(ii) I is maximal and positive implicative.

(iii) If $x, y \notin I$ then $x * y \in I$ and $y * x \in I$ for all $x, y \in X$.

Theorem 3.7 *[19, 24] Let I be an ideal of X. If $x * y \in I, y * z \in I$ then $x * z \in I$.*

Definition 3.8 *[19, 24] Let I be an ideal of BCK-algebra X and define relation $\equiv_I$ on X as follows:*

$$x \equiv_I y \leftrightarrow x * y \in I, y * x \in I \text{ for all } x, y \in X.$$

It is easily verified that $\equiv_I$ is a congruence relation. Let X/I be a set of congruence classes of $\equiv_I$, i.e. $X/I := \{[x]_I \mid x \in x\}$, where $[x]_I := \{y \in X \mid x \equiv_I y\}$. Therefore, $(X/I, *, [0]_I)$ is a BCK-algebra, $[x]_I * [y]_I = [x * y]_I$.

Note: A BCK-algebra X is said to be bounded if there exists $e \in X$ in such a way that $x \leq e$ for any $x \in X$, and the element e is said to be the unit of X. In a bounded BCK-algebra, we denote $e * x$ by $N(x)$.

4. Graph of BCI-algebras based on branches of BCI-algebras

In this section, we associate a new graph with BCI-algebras X, which is denoted by $G(X)$. This definition is based on branches of X. We also explain some properties of this graph. From now on, we let X be a BCI-algebra, unless otherwise stated.

Definition 4.1 *[17, 25] Let $G(X)$ be the associated graph to X. For every $x, y \in X$, we denote*

$$L(\{x,y\}) = \{z \in X; z \leq x, z \leq y\}.$$

Therefore, for graph $G(X)$, whose vertices are just the element of X, and for distinct $x, y \in X$, there is an edge connecting x and y if and only if there is $a \in P$ in so that $L(\{x,y\}) = \{a\}$, where P is the semisimple part of X.

Note: $G(V(a))$ is a graph that gains $G(X)$ when vertex set $G(X)$ is restricted to $V(a)$. In this section, we characterize the zero-divisor graph $G(X)$.

Theorem 4.2 *$G(V(a))$ can contain a subgraph isomorphic to C_3, C_4, but it cannot contain a subgraph isomorphic to $C_n, n \geq 5$.*

Proof Let $V(a)$ contain subposet Q_1, shown in Figure 7. Therefore, $G(V(a))$ contains subgraph C_3 on vertices $\{a, u, v\}$ and subgraph C_4 on vertices $\{x_1, x_2, x_3, x_4\}$. Therefore, $G(V(a))$ has a subgraph isomorphic to C_3, C_4. Let $G(V(a))$ be isomorphic to $C_n, n \geq 5$. Consider a n-cycle $a_1 - a_2 - a_3 - \ldots - a_n - a_1$ with $n \geq 5$. Supposing that $L(\{a_2, a_4\}) = \{x\}$, then $x \leq a_2, x \leq a_4$ gives $L(\{x, a_3\}) = \{a\}, L(\{x, a_5\}) = \{a\}$. Therefore, x is a common neighbor of a_3 and a_5. We note that if $x = a_4$, then $a_4 \leq a_2$, and hence $L(\{a_4, a_1\}) = \{a\}$, which is a contradiction. This shows that $L(\{a_2, a_4\})$ does not exist. Hence, $G(V(a))$ cannot be an n-cycle for any $n \geq 5$. □

Theorem 4.3 *Let $G(X)$ be an associated graph to X. Then $gr(G(X)) \leq 4$.*

Proof Straightforward by Theorem 4.2. □

Theorem 4.4 *$G(V(a))$ is chordal if and only if $V(a)$ does not contain $Q_1, Q_2, Q_3,$ Q_4 as a subposet, where Q_1, Q_2, Q_3, Q_4 are given in Figure 7.*

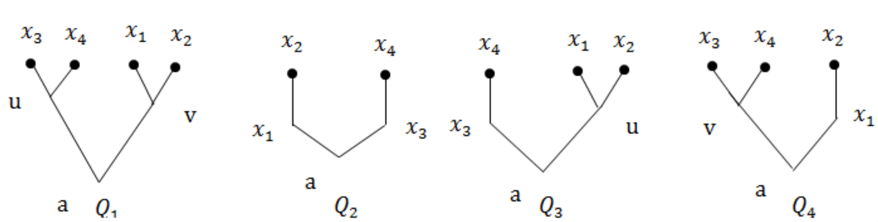

Figure 7.

Proof Let $V(a)$ contain Q_1 as a subposet. Then $L(\{x_3, x_4\}) = \{v, a\}$, $L(\{x_1, x_2\}) = \{u, a\}, L(\{x_3, x_1\} = \{a\}, L(\{x_4, x_1\}) = \{a\}, L(\{x_2, x_4\}) = \{a\}$, $L(\{x_2, x_3\} = \{a\}$. Then $x_3x_1, x_4x_1, x_2x_4, x_2x_3 \in E(G(Q_1))$. Therefore, $G(Q_1)$ is not chordal, and similarly $G(Q_2), G(Q_3), G(Q_4)$ is not chordal. Conversely, we know by Theorem 4.2 that $G(V(a))$ cannot be isomorphic to $C_n, n \geq 5$. Therefore, we should study chordality $G(V(a))$ in the case $n = 4$. Also, we have $G(V(a))$, which is chordal only in the case that $V(a)$ does not contain Q_1, Q_2, Q_3, Q_4 as a subposet. □

Example 4.5 *Let $X = \{0, 1, a, u, v, x_1, x_2, x_3, x_4\}$. Define binary operation $*$ on X based on Table 1:*

$*$	0	1	a	u	v	x_1	x_2	x_3	x_4
0	0	0	a	a	a	a	a	a	a
1	1	0	a	a	a	a	a	a	a
a	a	a	0	0	0	0	0	0	0
u	u	a	1	0	1	1	1	0	0
v	v	a	1	1	0	0	0	1	1
x_1	x_1	a	1	1	1	0	1	1	1
x_2	x_2	a	1	1	1	1	0	1	1
x_3	x_3	a	1	1	1	1	1	0	1
x_4	x_4	a	1	1	1	1	1	1	0

Table 1.

Therefore, subposet $(X, *, 0)$, shown in Figure 8, is a BCI-algebra, $P = \{0, a\}$, $V(0) = \{0, 1\}$, $V(a) = \{a, u, v, x_1, x_2, x_3, x_4\}$. Moreover, $L(\{x_1, x_3\}) = \{a\}, L(\{x_3, x_2\}) = \{a\}, L(\{x_2, x_4\}) = \{a\}, L(\{x_3, x_4\}) = \{v, a\}$, $L(\{x_1, x_2\}) = \{u, a\}$ and so $E(G(V(a))) = \{x_1x_3, x_3x_2, x_2x_4, x_1x_4\}$. Therefore, graph $G(V(a))$ is not chordal.

Example 4.6 *Let $X = \{0, 1, a, b, c\}$. Define binary operation * based on Table 2:*

*	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	1
c	c	a	1	1	0

Table 2.

Then $(X, *, 0)$ is a BCI-algebra, $P = \{0, a\}, V(0) = \{0, 1\}, V(a) = \{a, b, c\}$. Moreover, $L(\{0, 1\}) = \{0\}, L(\{a, b\}) = L(\{a, c\}) = L(\{b, c\}) = \{a\}$ and so $E(G(X)) = \{10, ac, bc, ab\}$. Therefore, graph $G(X)$ shown in Figure 9 is chordal.

Theorem 4.7 *Let $A = \{a_1, \ldots, a_n \mid a_i$ be a-atom, $i = 1 \ldots n\}$ and $A_i = \{x \mid x \geq a_i\}$, $\mid A_i \mid= c_i$. Then $G(V(a))$ has an induced subgraph isomorphic to graph n-partite $K_{c_1, c_2, \ldots, c_n}$.*

Proof If $B = \{A_1, A_2, \ldots, A_n\}$, then every element A_i adjacent to any element A_j for all $i \neq j, i, j = 1, \ldots, n$ because if $x_i \in A_i, x_j \in A_j$, then $L(\{x_i, x_j\}) = \{a\}$, and also because elements of A_i are not adjacent to all

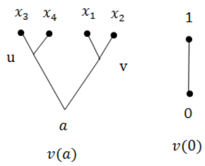

Figure 8.

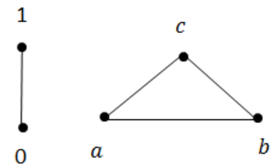

Figure 9.

$i = 1, \ldots, n$. If $a, b \in A_i$, then $L(\{a, b\}) = \{a_i\} \neq \{a\}$. The resulting graph $G(V(a))$ is an n-partite graph $K_{c_1, c_2, \ldots, c_n}$ with partitions $B = \{A_1, A_2, \ldots, A_n\}$. □

Notation: Denote by $Atom(X)$ the set of all atoms of a BCK-algebra X.

Theorem 4.8 *If X_1, X_2 are BCK-algebra, $\mid Atom(X_1) \mid = \mid Atom(X_2) \mid = 1, \mid X_1 \mid = m+1, \mid X_2 \mid = n+1$ and $X \cong X_1 \times X_2$. Then $G(X)$ is the complete bipartite graph $K_{m,n}$.*

Proof If $X_1 = \{0, x_1, \ldots, x_m\}, X_2 = \{0, y_1, y_2, \ldots, y_n\}$, then the pairs of the form $(x_i, 0)$ and $(0, y_j)$ are all adjacent. Moreover, no pairs of the form $(x_i, 0), (x_k, 0)$ are adjacent because $L(\{x_i, x_k\}) \neq \{0\}$. The resulting graph is a complete bipartite graph with partitions $A = \{(x_1, 0), \ldots, (x_m, 0)\}$ and $B = \{(0, y_1), \ldots, (0, y_n)\}$. □

Theorem 4.9 *If X_1, X_2 are BCK-algebras, $\mid Atom(X_1) \mid = \mid Atom(X_2) \mid = 1, \mid X_1 \mid = m+1, \mid X_2 \mid = n+1$ and $X \cong X_1 \times X_2$, then $diam(G(X)) \leq 2$.*

Proof Based on Theorem 4.8, $G(X)$ is a complete bipartite graph, so $diam(G(X)) \leq 2$. □

Theorem 4.10 *Let $A = \{z \mid z$ be a-atom $\}$, $\mid A \mid = n$. Define equivalence classes on $V(a) - \{a\}$ as follows:*

$$[x] = \{y \mid y \geq x \text{ or } y \leq x, y \neq a\}.$$

Denote $P = \{[x] \mid x \in V(a) - \{a\}\}$. Thus, $\frac{G(V(a)-\{a\})}{P}$ is isomorphic to K_n.

Proof Let $x, y \neq a$, $u \in [x]$. Then there is an element a_1 so that a_1 is a-atom and $x \geq a_1$. Therefore, $u \geq a_1$. In the same way that $v \in [y]$ follows there exists a_2 such that a_2 is a-atom, $v \geq a_2$. Therefore, $L(\{u, v\}) = \{a\}$ since there is not an ordered relation between a_1, a_2. Then $uv \in E(G(V(a) - \{a\})$. Therefore, by Definition 2.1 of quotient graph $[x][y] \in E(\frac{G(V(a)-\{a\})}{P})$, since $L(\{a, y\}) = \{a\}$ we have $[a][y] \in E(\frac{G(V(a)-\{a\})}{P})$. Then $\frac{G(V(a)-\{a\})}{P}$ is isomorphic to K_n. □

Theorem 4.11 *Let $A = \{x;\ x$ be a-atom $\}$, $\mid A \mid = n$. Then $G(V(a))$ contains a clique such that the vertices of the clique are set A, element a. Therefore, clique number of $G(V(a)) \geq n+1$ and $\chi(G(V(a))) \geq n+1$.*

Proof Based on Definition 2.2 of the chromatic number and the clique number of the graph, inequality $\chi(G) \geq \omega(G)$ completes the proof. □

Theorem 4.12 *Let $A = \{x \in X \mid x$ be a-atom $\}$, $\mid A \mid \geq 4$. Then $G(V(a))$ is not planar.*

Proof $G(V(a))$ contains an induced subgraph isomorphic to K_5 by Theorem 4.11. Therefore, according to Kuratowski's theorem, $G(V(a))$ is not planar. □

Theorem 4.13 *Let b and b' be vertices of $G(V(a))$. If $b \leq b'$, then $deg(b') \leq deg(b)$.*

Proof We know if $b \leq b'$ then $L(\{x,b\}) \subseteq L(\{x,b'\})$. Thus, $L(\{x,b'\}) = \{a\}$ implies $L(\{x,b\}) = \{a\}$. On the other hand, $xb' \in E(G(V(a)))$ implies $xb \in E(G(V(a)))$ and then $deg(b') \leq deg(b)$. □

5. Graphs of BCI/BCK-algebras based on ideal

In this section, we apply a notion of BCK-algebras that uses the concept of graph and equivalence relation $\equiv_I$ for BCK-algebras. Conditions for the associated graphs of BCK-algebras being planar, outerplanar, toroidal, chordal, and Eulerian are also provided and several examples are displayed.

Henceforth, we let X be a bounded BCK-algebra.

Definition 5.1 *Let I be an ideal of X. Therefore, we have:*

(i) $\Gamma^I(X) = (X, E^I)$ *is a graph with vertices X and edges E^I, where $xy \in E^I$ if and only if $x * y \in I$ and $y * x \in I$, for any $x, y \in X$.*

(ii) $\Gamma_I(X) = (X, E_I)$ *is a graph with vertices X and edges E_I, where $xy \in E_I$ if and only if $x * y \in I$ or $y * x \in I$, for any $x, y \in X$.*

Example 5.2 *Let $X = \{0,1,2,3\}$ and the operation * be defined by Table 3:*

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	3	3	0

Table 3.

Therefore, $(X, *, 0)$ is a BCK-algebra. It is easy to verify that $\{0,1,2\}$ is an ideal of X. In Figure 10, we can see the graphs $\Gamma^I(X), \Gamma_I(X)$.

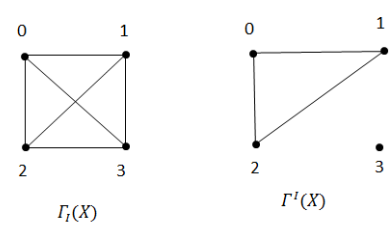

Figure 10.

Theorem 5.3 *Let I be an ideal of X. Then:*

(i) $\Gamma_I(X)$ *is regular if and only if it is complete.*

(ii) If $\Gamma^I(X)$ *is regular, then it is a complete graph on I.*

Proof

(i) ($\Longrightarrow$) Let $\Gamma_I(X)$ be a regular graph. Since $0 * x = 0 \in I$, for any $x \in X$, then $deg(0) =| X | -1$. Now, since $\Gamma_I(X)$ is regular, then for any $x \in X$, $deg(x) = deg(0)$, and so for any $x \in X$, $deg(x) =| X | -1$. This means that $\Gamma_I(X)$ is a complete graph.

($\Longleftarrow$) It is clear that any complete graph is regular.

(ii) Let $\Gamma^I(X)$ be a regular graph. Since $0*x = 0 \in I$ and $x*0 = x \in I$ for any $x \in I$, then $deg(0) =| I | -1$. Now, since $\Gamma^I(X)$ is regular, then for any $x \in I$, $deg(x) = deg(0) =| I | -1$. Thus, $\Gamma^I(X)$ is complete on ideal I.

□

Theorem 5.4 *Let I be an ideal of X. Then:*

(i) If X is a chain, then $\Gamma_I(X)$ is complete.

(ii) If I is closed, then $\Gamma^I(X)$ and $\Gamma_I(X)$ are complete on I.

Proof

(i) If X is a chain, then for any $x, y \in X$, $x \leq y$, or $y \leq x$ and so $x * y = 0 \in I$ or $y * x = 0 \in I$. Therefore, $deg(x) =| X | -1$, for any $x \in X$ and so $\Gamma_I(X)$ is complete.

(ii) Let I be a closed ideal. Then for any $x, y \in I$, $x * y \in I$ and $y * x \in I$, and so for any $x, y \in I$, in graph $\Gamma^I(X)$, we have $deg(x) =| I | -1$. Hence, $\Gamma^I(X)$ is complete on I. It is clear that $\Gamma_I(X)$ is complete on I, too.

□

Example 5.5 *Let N be set of nonnegative integers and $m * n = \{m - n, 0\}$, for any $m, n \in N$. Therefore, $(N, *, 0)$ is a BCK-algebra. For all $m, n \in N$, we have $m < n$ or $m \geq n$. Then $m * n = 0$ or $n * m = 0$. Thus, $nm \in E(\Gamma_{\{0\}}(N))$. Therefore, $\Gamma_{\{0\}}(N)$ is a complete graph.*

Example 5.6 *Let $X = \{0, a, b, c, d\}$. Define the binary operation * on X by Table 4:*

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	0
c	c	c	c	0	c
d	d	d	b	b	0

Table 4.

Therefore, $(X, *, 0)$ is a BCK-algebra that is not a chain and $\Gamma_{\{0\}}(X)$, given by Figure 11, is not complete.

Remark 5.7 *Let I be an ideal of X. If $\Gamma^I(X)$ is a complete graph, then $I = X$.*

*	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Table 5.

Example 5.8 *Let $X = \{0, a, b, 1\}$. Define the binary operation * on X by Table 5:*

Therefore, $(X, *, 0)$ is a BCK-algebra and $I = \{0\} \neq X$, and $\Gamma_{\{0\}}(X)$, given by Figure 12, is complete.

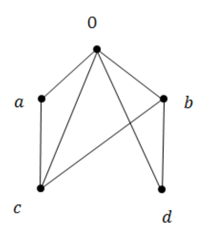

Figure 11.

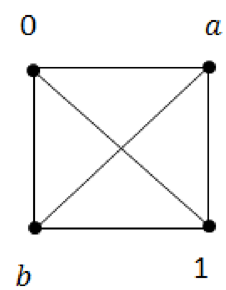

Figure 12.

Theorem 5.9 *Let I be an ideal of X. Then there is not $m, n \in N$ in such a way that $\Gamma^I(X)$ is isomorphic to $K_{m,n}$.*

Proof Let there be $m, n \in N$ so that $\Gamma^I(X)$ is isomorphic to $K_{m,n}$. Then there are the sets $A = \{x_1, \ldots, x_m\}, B = \{y_1, \ldots, y_n\}$ in such a way that $x_i * y_j \in I, y_j * x_i \in I$, for all $i = 1, \ldots, m, j = 1, \ldots, n$. Then by transitive property $*$, we have $x_i * x_k \in I, y_j * y_l \in I$, for all $i, k \in \{1, \ldots, m\}, j, l \in \{1, \ldots, n\}$, which is a contradiction to $K_{m,n}$ being a complete bipartite graph, completing the proof. □

Theorem 5.10 *Let I be an ideal of X. Then $\Gamma_I(X)$ is connected, $gr(\Gamma_I(X)) := 3$, $diam(\Gamma_I(X)) \leq 2$.*

Proof We know $0 * x = 0 \in I, x * 1 = 0 \in I$ for all $x \in X$. Thus, by Definition 5.1 of graph $\Gamma_I(X)$ 0,1, connected to every element $x \in X$, $\Gamma_I(X)$ is connected, $gr(\Gamma_I(X)) := 3$, $diam(\Gamma_I(X)) \leq 2$. □

Theorem 5.11 *Let I be an ideal of X. Then $0, 1$ are not orthogonal in the graph $\Gamma_I(X)$ and $\Gamma_I(X)$ is not complemented.*

Proof According to Theorem 5.10, every vertex in the graph $\Gamma_I(X)$ is connected to both 0 and 1. Thus, 0,1 are not orthogonal. Moreover, there are no vertices v, w in $\Gamma_I(X)$ in such a way that v, w are orthogonal. Therefore, $\Gamma_I(X)$ is not complemented □

Theorem 5.12 *Let I be an ideal of X. Then $S_1 = \{0\}$ and $S_2 = \{1\}$ are two dominating sets in graph $\Gamma_I(X)$. Therefore, $\gamma(\Gamma_I(X)) = 1$.*

Proof Straightforward by Definition 2.1 of the dominating set and by Theorem 5.10. □

Theorem 5.13 *Let X be a chain and I an ideal of X. Then the following statements hold:*

(i) $\Gamma_I(X)$ *is a planar graph if and only if* $| X | \leq 4$.

(ii) $\Gamma_I(X)$ *is an outerplanar graph if and only if* $| X | \leq 3$.

(iii) $\Gamma_I(X)$ *is a toroidal graph if and only if* $| X | \leq 7$.

Proof

(i) According to Theorem 5.4(i), $\Gamma_I(X)$ is a complete graph. If $| X | \geq 5$, then $\Gamma_I(X)$ has a subgraph isomorphic to K_5, and by Kuratowski's theorem $\Gamma_I(X)$ is not planar. Conversely, we know that K_5 has five vertices. Thus, if $\Gamma_I(X)$ is not planar, then $\Gamma_I(X)$ has at least five vertices, which is contrary to $| X | \leq 4$, completing the proof.

(ii) According to Theorem 5.4(i), $\Gamma_I(X)$ is a complete graph if $| X | \geq 4$, and then $\Gamma_I(X)$ has a subgraph isomorphic to K_4, and by Definition 2.5, $\Gamma_I(X)$ is not outerplanar. Conversely, we know that K_4 has four vertices; hence, if $\Gamma_I(X)$ is not outerplanar, then $\Gamma_I(X)$ has at least four vertices, which is contrary to $| X | \leq 3$, completing the proof.

(iii) According to Theorem 5.4(i), $\Gamma_I(X)$ is a complete graph, if $| X | \geq 8$, and then $\Gamma_I(X)$ has a subgraph isomorphic to K_8, and by Theorem 2.8, $\Gamma_I(X)$ is not toroidal. Conversely, we know that K_8 has eight vertices; hence, if $\Gamma_I(X)$ is not toroidal, then $\Gamma_I(X)$ has at least eight vertices, which is contrary to $| X | \leq 7$, completing the proof.

□

Example 5.14 *Let $Y = \{0, 1, 2, 3, 4\}, X = \{0, a, b, 1\}$. Define the binary operations $''*_1''$ and $''*_2''$ on Y and X, respectively, by Tables 6 and 7:*

$*_1$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Table 6.

$*_2$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	1	1	0

Table 7.

(i) It is easy to see that $(Y, *_1, 0)$ is a BCK-algebra. Moreover, Y is not a chain, $\mid Y \mid \leq 5$, $\Gamma_{\{0\}}(Y)$ is planar.

(ii) It is easy to see that $(X, *_2, 0)$ is a BCK-algebra. Moreover, X is not a chain, $\mid X \mid \leq 4$, $\Gamma_{\{0\}}(X)$ is outerplanar.

Proposition 5.15 *Let I be an ideal of X. Then $\omega(\Gamma_I(X)) \geq max\{\mid A \mid;\ A$ is a chain in $X\}$.*

Proof Let A be a chain in X. Then for all $x, y \in A$, we have $x \leq y$ or $y \leq x$. In other words, $x * y = 0 \in I$ or $y * x = 0 \in I$. Thus, $xy \in E(\Gamma_I(X))$ by Definition 5.1 of graph $\Gamma_I(X)$, since $\omega(\Gamma_I(X))$ is the length of greatest induced complete subgraph in the graph $\Gamma_I(X)$. Therefore, we have $\omega(\Gamma_I(X)) \geq max\{\mid A \mid;\ A$ is a chain in X$\}$. □

Proposition 5.16 *Let $I = \{0, a\}$ be an ideal of X, where $a \in Atom(X), A = \{y \in X; y$ covers $a\}$. Then the following statements hold:*

(i) If $\mid A \mid \geq 3$, then $\Gamma_I(X)$ is not planar.

(ii) If $\mid A \mid \geq 2$, then $\Gamma_I(X)$ is not outerplanar.

(iii) If $\mid A \mid \geq 7$, then $\Gamma_I(X)$ is not toroidal.

Proof

(i) Since $\mid A \mid \geq 3$, we can choose the subset $A' = \{x_1, x_2, x_3\}$ of the set A. It is clear that for all $i = 1, 2, 3, a * x_i = 0, 0 * x_i = 0, x_i * 1 = 0$. Thus, the graph of $\Gamma_I(X)$ on $A' \bigcup \{0, a, 1\}$ has a subgraph isomorphic to $K_{3,3}$ and thus by Kuratowski's theorem $\Gamma_I(X)$ is not planar.

(ii) Since $\mid A \mid \geq 2$, we can choose the subset $A' = \{x_1, x_2\}$ of the set A. It is clear that for all $i = 1, 2, a * x_i = 0, 0 * x_i = 0, x_i * 1 = 0$. Thus, the graph $\Gamma_I(X)$ on $A' \bigcup \{0, a, 1\}$ has a subgraph isomorphic to $K_{2,3}$ and thus by Definition 2.5, $\Gamma_I(X)$ is not outerplanar.

(iii) Since $\mid A \mid \geq 7$, we can choose the subset $A' = \{x_1, \ldots, x_7\}$ of the set A. It is clear that for all $i = 1, \ldots, 7, a * x_i = 0, 0 * x_i = 0, x_i * 1 = 0$. Thus, the graph $\Gamma_I(X)$ on $A' \bigcup \{0, a, 1\}$ has a subgraph isomorphic to $K_{3,7}$ and thus by Theorem 2.8, $\Gamma_I(X)$ is not toroidal.

□

Proposition 5.17 *Let I be a closed ideal of X. Then $\omega(\Gamma^I(X)) \geq \mid I \mid$.*

Proof We have for all $x, y \in I$, $x * y \in I, y * x \in I$, since I is a closed ideal. Thus, by Definition 5.1 of graph $\Gamma^I(X)$, $xy \in E(\Gamma^I(X))$. Thus, $\Gamma^I(X)$ contains a clique in such a way that the vertex set of the clique is elements of I. Thus, $\omega(\Gamma^I(X)) \geq \mid I \mid$. □

Theorem 5.18 *Let $I = \{0\}$ be an ideal of X. Then $Com(X) = \Gamma_{\{0\}}(X)$, where $Com(X)$ is a comparability graph of X.*

Proof Let $x, y \in X, xy \in E(\Gamma_{\{0\}}(X))$. Therefore, by Definition 5.1 of graph $\Gamma_I(X)$, $x * y = 0$ or $y * x = 0$. Thus, $x \leq y$ or $y \leq x$. Thus, $xy \in E(Com(X))$. The converse is clear. □

Theorem 5.19 *Let X be semimodular of rank d that is not a chain. Then $\Gamma_{\{0\}}(X)$ is $(d+1)$-connected if and only if X has no simplicial element, where $z \in X$ is simplicial if the elements comparable to z form a chain.*

Proof Straightforward by Theorems 2.17 and 5.18. □

Theorem 5.20 *Let I be an ideal of X. Then $I = \{0\}$ if and only if simple graph $\Gamma^I(X)$ is empty; that is, $E^I = \emptyset$.*

Proof Let $xy \in E(\Gamma^I(X))$, $x, y \in X$. Then $x * y \in I = \{0\}, y * x \in I = \{0\}$, so $x = y$. Therefore, $\Gamma^I(X)$ is an empty graph. Conversely, let $\Gamma^I(X)$ be an empty graph. Therefore, if for all $x, y \in X, xy \in E(\Gamma^I(X))$, then $x = y$. In other words, if for all $x, y \in X, x * y \in I, y * x \in I$, then $x = y$. Thus, $x * y = x * x = 0 \in I$. Thus, $I = \{0\}$, completing the proof. □

Proposition 5.21 *Let I be an ideal of X. If $1 \in I$, then the following statements hold:*

(i) $\Gamma^I(X)$ *is planar if and only if* $|\ X\ | \leq 4$.

(ii) $\Gamma^I(X)$ *is outerplanar if and only if* $|\ X\ | \leq 3$.

(iii) $\Gamma^I(X)$ *is toroidal if and only if* $|\ X\ | \leq 7$.

Proof

(i) If $1 \in I$, then $I = X$. Hence, $\Gamma^I(X)$ is a complete graph, if $|\ X\ | > 4$, and then $\Gamma^I(X)$ has induced subgraph isomorphic to K_5, so by Kuratowski's theorem $\Gamma^I(X)$ is not planar. Conversely, we know that K_5 has five vertices; hence, if $\Gamma^I(X)$ is not planar, then $\Gamma^I(X)$ has at least five vertices, which is contrary to $|\ X\ | \leq 4$, completing the proof.

(ii) If $1 \in I$, then $I = X$. Therefore, $\Gamma^I(X)$ is a complete graph, if $|\ X\ | > 3$, and then $\Gamma^I(X)$ has induced subgraph isomorphic to K_4, so by Definition 2.5, $\Gamma^I(X)$ is not outerplanar. Conversely, we know that K_4 has four vertices; hence, if $\Gamma^I(X)$ is not outerplanar, then $\Gamma^I(X)$ has at least four vertices, which is contrary to $|\ X\ | \leq 3$, completing the proof.

(iii) If $1 \in I$, then $I = X$. Therefore, $\Gamma^I(X)$ is a complete graph, if $|\ X\ | > 7$, and then $\Gamma^I(X)$ has induced subgraph isomorphic to K_8, so by Theorem 2.8, $\Gamma^I(X)$ is not toroidal. Conversely, we know that K_8 has eight vertices; hence, if $\Gamma^I(X)$ is not toroidal, then $\Gamma^I(X)$ has at least eight vertices, which is contrary to $|\ X\ | \leq 8$, completing the proof.

□

Theorem 5.22 *Let I be an ideal of X. Then the following statements hold:*

(i) If $\mid I \mid = 3$, $\Gamma_I(X)$ is planar, then $\mid X \mid \leq 5$.

(ii) If $\mid I \mid = 2$, $\Gamma_I(X)$ is outerplanar, then $\mid X \mid \leq 4$.

(iii) If $\mid I \mid = 7$, $\Gamma_I(X)$ is toroidal, then $\mid X \mid \leq 9$.

Proof

(i) Suppose that $\Gamma_I(X)$ is planar. Assume on the contrary that $\mid X \mid \geq 6$. Now put $V_1 := I$ and $V_2 := \{x_1, x_2, x_3\} \subseteq X - I$. Since for all $a \in I, i = 1, 2, 3$, $a * x_i \leq a$, I is an ideal, then $a * x_i \in I$. Thus, one can find a copy of $k_{\{3,3\}}$ in $\Gamma_I(X)$. Therefore, by Kuratowski's theorem, $\Gamma_I(X)$ is not planar, which is impossible. Hence, $\mid X \mid \leq 5$.

(ii) Suppose that $\Gamma_I(X)$ is outerplanar. Assume on the contrary that $\mid X \mid \geq 5$. Now put $V_1 := I$ and $V_2 := \{x_1, x_2\} \subseteq X - I$. Since for all $a \in I, i = 1, 2$, $a * x_i \leq a$, I is ideal, then $a * x_i \in I$. Thus, one can find a copy of $K_{\{2,3\}}$ in $\Gamma_I(X)$. Therefore, by Definition 2.5, $\Gamma_I(X)$ is not outerplanar, which is impossible. Therefore, $\mid X \mid \leq 4$.

(iii) Suppose that $\Gamma_I(X)$ is toroidal. Assume on the contrary that $\mid X \mid \geq 10$. Now put $V_1 := I$ and $V_2 := \{x_1, x_2, x_3\} \subseteq X - I$. Since for all $a \in I, i = 1, 2, 3$, $a * x_i \leq a$, I is ideal, then $a * x_i \in I$. Thus, one can find a copy of $K_{\{3,7\}}$ in $\Gamma_I(X)$. Therefore, by Theorem 2.8, $\Gamma_I(X)$ is not toroidal, which is impossible. Therefore, $\mid X \mid \leq 9$.

□

Theorem 5.23 *Let ideal I satisfy in the conditions of Theorem 3.6. Then the following statements hold:*

(i) $\Gamma^I(X)$ is a complete graph on $X - I$.

(ii) $\Gamma^I(X)$ is an empty graph on I.

(iii) $\alpha(\Gamma^I(X)) = \mid I \mid$.

(iv) $\Gamma^I(X)$ is a planar graph if and only if $\mid X - I \mid \leq 4$.

(v) $\Gamma^I(X)$ is an outerplanar graph if and only if $\mid X - I \mid \leq 3$.

(vi) $\Gamma^I(X)$ is a toroidal graph if and only if $\mid X - I \mid \leq 7$.

Proof Straightforward by Definition 5.1 of graph $\Gamma^I(X)$, Theorems 2.8 and 3.6. □

Theorem 5.24 *Let I be an ideal of X. Then $\Gamma^I(X)$ is chordal.*

Proof Let $x_0, x_1, \ldots, x_n$ be a cycle of length $n \geq 4$. Therefore, we have $x_i * x_{i+1} \in I, x_{i+1} * x_{i+2} \in I$ for all $i = 0, \ldots, n-2$. By the transitive property of * in Theorem 3.7, $x_i * x_{i+2} \in I$ for all $i = 0, \ldots, n-2$. Hence, $\Gamma^I(X)$ is chordal, completing the proof. □

Note that the following example says that $\Gamma_I(X)$ is not chordal, whereas Theorem 5.24 shows that $\Gamma^I(X)$ is chordal.

Example 5.25 *Let X be a BCK-algebra defined in Example 3.2, and let $I = \{0\}$ be an ideal of X. Consider $x_1, x_2, x_3, x_4 \in X$ in such a way that $x_1 \le x_2, x_1 \le x_3, x_4 \le x_2, x_4 \le x_3$. Therefore, $\Gamma_I(X)$ has a cycle isomorphic to C_4 on vertex set $\{x_1, x_2, x_3, x_4\}$. Since $x_1 \le x_2, x_1 \le x_3, x_4 \le x_2, x_4 \le x_3$, we have $x_1 * x_2 = x_1 * x_3 = x_4 * x_2 = x_4 * x_3 = 0 \in I$ by the definition of operation *. Thus, by Definition 5.1 of graph $\Gamma_I(X)$, $x_1x_2, x_1x_3, x_4x_2, x_4x_3 \in E(\Gamma_I(X))$. On the other hand, $x_2 * x_3 = x_2 \notin I, x_3 * x_2 = x_3 \notin I, x_4 * x_1 = x_4 \notin I, x_1 * x_4 = x_1 \notin I$ since x_2 to x_3, x_1 to x_4 are not comparable. Therefore, $x_1x_4, x_2x_3 \notin E(\Gamma_I(X))$, and thus graph $\Gamma_I(X)$ is not chordal, completing the proof.*

Remark 5.26 *Let I be an ideal of X, $A = \{x, y \in X; x * y \notin I, y * x \notin I\}$. Then we have the following statements:*

(i) *$\Gamma_I(X - A)$ is planar if and only if $\Gamma_I(X)$ is planar.*

(ii) *$\Gamma_I(X - A)$ is outerplanar if and only if $\Gamma_I(X)$ is outerplanar.*

The following theorem gives us a characterization of graph $\Gamma^I(X)$.

Theorem 5.27 *Let I be an ideal of X. Then the following statements hold:*

(i) *$\Gamma^I([x]_I)$ is a complete graph, for any $x \in X$,*

(ii) *$\Gamma^I(X) = \bigcup_{x \in X} \Gamma^I([x]_I)$,*

(iii) *$\Gamma^I(X)$ is a graph with $\mid X/I \mid$ components,*

(iv) *$\Gamma^I(X)$ is a planar graph if and only if $\mid [x]_I \mid \le 4$, for all $x \in X$,*

(v) *$\Gamma^I(X)$ is an outerplanar graph if and only if $\mid [x]_I \mid \le 3$, for all $x \in X$,*

(vi) *$\Gamma^I(X)$ is a toroidal graph if and only if $\mid [x]_I \mid \le 7$, for all $x \in X$,*

(vii) *$\omega(\Gamma^I(X)) = \max\{\mid [x]_I \mid ; x \in X\}$.*

Proof

(i) Letting $u, v \in [x]_I$, then by Definition 3.8 of $\equiv_I$, $u * x \in I, x * u \in I, v * x \in I, x * v \in I$ so $u * v \in I, v * u \in I$ since Theorem 3.7 says that operation * has a transitive property. Thus, by Definition 5.1 of graph $\Gamma^I(X), uv \in E(\Gamma^I([x]_I)$, and then $\Gamma^I([x]_I)$ is a complete graph.

(ii) Since $X = \bigcup_{x \in X}[x]_I$, then $V(\Gamma^I(X)) = V(\bigcup_{x \in X} \Gamma^I([x]_I))$. Clearly, $E(\bigcup_{x \in X} \Gamma^I([x]_I)) \subseteq E(\Gamma^I(X))$. Now let $xy \in E(\Gamma^I(X))$. Then $x * y \in I, y * x \in I$, and so $xy \in \Gamma^I([x]_I)$. Hence, $E(\bigcup_{x \in X} \Gamma^I([x]_I)) = E(\Gamma^I(X))$.

(iii) We want to show that there is not any path between elements of $[x]_I$ and $[y]_I$ for all distinct elements $x, y \in X$. Let x, y be distinct elements of X, $a \in [x]_I$ and $b \in [y]_I$. If there is a path $a, a_1, a_2, \ldots, a_n, b$ that links a to b, then $a * a_1 \in I, a_1 * a \in I$ and so by Definition 3.8 of $\equiv_I$ we have $a_1 \in [a]_I = [x]_I$. In a similar way, we have $a_2, \ldots, a_n, b \in [x]_I$ so $b \in [x]_I \bigcap [y]_I$, which is contrary to $[x]_I \bigcap [y]_I = \emptyset$.

(iv) We know that $\Gamma^I([x]_I)$ is a complete graph by (i), if $\mid [x]_I \mid > 4$, and then $\Gamma^I(X)$ has induced subgraph isomorphic to K_5, so by Kuratowski's theorem $\Gamma^I([x]_I)$ is not planar.

(v) We know that $\Gamma^I([x]_I)$ is a complete graph by (i), if $\mid [x]_I \mid > 3$, and then $\Gamma^I(X)$ has induced subgraph isomorphic to K_4, so by Definition 2.5 $\Gamma^I([x]_I)$ is not outerplanar.

(vi) We know that $\Gamma^I([x]_I)$ is a complete graph by (i), if $\mid [x]_I \mid > 7$, and then $\Gamma^I(X)$ has induced subgraph isomorphic to K_8, so by Theorem 2.8 $\Gamma^I([x]_I)$ is not toroidal.

(vii) We know by (i) and (ii) that $\Gamma^I([x]_I)$ is a complete graph, $\Gamma^I(X) = \bigcup \Gamma^I([x]_I)$, since $\omega(\Gamma^I(X))$ is length of the greatest induced complete subgraph in $\Gamma^I(X)$, and we have $\omega(\Gamma^I(X)) = \max\{\mid [x]_I \mid; x \in X\}$.

□

Theorem 5.28 *Let I be an ideal of X. If $t = \mid X/I \mid, X = \bigcup_{i=1,\dots,t}[x_i]_I$. Then $\alpha(\Gamma^I(X)) \geq t$.*

Proof Let $z_1 \in [x_i]_I, z_2 \in [x_j]_I, i, j = 1, \dots, t$. By proof of Theorem 5.27, we have $z_1 z_2 \notin E(\Gamma^I(X))$. Then by Definition 2.1 of an independent set that is the maximum size of the vertex set in such a way that they do not connect to each other, Theorem 5.27 omit (i), we have $\alpha(\Gamma^I(X)) \geq t$. □

Theorem 5.29 *Let I be an ideal of X. Then $\Gamma^I([x]_I)$ is an Euler graph if and only if $\mid [x]_I \mid$ is odd.*

Proof According to Theorem 5.27(i), we know that $\Gamma^I([x]_I)$ is a complete graph. If $\mid [x]_I \mid$ is odd, then the degree of every vertex of $\Gamma^I([x]_I)$ is even, so based on Euler's theorem, which says that a connected graph is an Euler graph if and only if the degree of every vertex is even, we gain that $\Gamma^I([x]_I)$ is an Eulerian graph. □

Proposition 5.30 *Let $(X_1, *_1, 0)$, $(X_2, *_2, 0)$ be two BCK-algebras in such a way that $X_1 \bigcap X_2 = \{0\}$, $I_1 \lhd X_1, I_2 \lhd X_2$, $I = I_1 \bigcup I_2$, and $S = \{xy \mid (x \in X_1, x \in I_1)$ or $(y \in X_2, y \in I_2)\}$. Then $E(\Gamma_I(X_1 \bigcup X_2)) = E(\Gamma_{I_1}(X_1) \bigcup \Gamma_{I_2}(X_2)) \bigcup S$.*

Proof Let $x, y \in X_1 \bigcup X_2, xy \in E(\Gamma_I(X_1 \bigcup X_2))$. Therefore, we have the following cases:

(i) If $x, y \in X_1$, we have $x * y = x *_1 y$. Therefore, $x * y \in I$ implies $x *_1 y \in I_1$, by Proposition 3.3, since $X_1 \bigcap X_2 = \{0\}$. Then $xy \in E(\Gamma_{I_1}(X_1))$.

(ii) If $x, y \in X_2$, in a similar way, we can prove that $xy \in E(\Gamma_{I_2}(X_2))$.

(iii) If $x \in X_1, y \in X_2$, by Proposition 3.3, we have $x * y = x$. Hence, $x * y \in I$ implies $x * y \in I_1$, since $X_1 \bigcap X_2 = \{0\}$. Therefore, $xy \in E(\Gamma_{I_1}(X_1))$.

(iv) If $x \in X_2, y \in X_1$. Similarly, we can prove that $xy \in E(\Gamma_{I_2}(X_2))$.

Clearly, $E(\Gamma_{I_1}(X_1) \bigcup \Gamma_{I_2}(X_2)) \bigcup S \subseteq E(\Gamma_I(X_1 \bigcup X_2))$, completing the proof. □

Corollary 5.31 *Let $(X_1, *_1, 0), (X_2, *_2, 0)$ be two BCK-algebras, such that $X_1 \bigcap X_2 = \{0\}$, and let $0 = I_1 \lhd X_1, 0 = I_2 \lhd X_2$. Then $\Gamma_I(X_1 \bigcup X_2) = \Gamma_{I_1}(X_1) \bigcup \Gamma_{I_2}(X_2)$.*

$*_1$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	a
c	c	c	c	0

Table 8.

$*_2$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Table 9.

Example 5.32 *Let $X_1 = \{0, a, b, c\}$ and $X_2 = \{0, 1, 2\}$. Define the binary operations $*_1$ and $*_2$ on X_1 and X_2, respectively, by Tables 8 and 9:*

Therefore, $(X_1, *_1, 0)$ and $(X_2, *_2, 0)$ are BCK-algebras and $I_1 = \{0, a, b\}, I_2 = \{0\}$ are two ideals of X_1, X_2, respectively. Also, $\Gamma_{I_1}(X_1), \Gamma_{I_2}(X_2)$ and $\Gamma_{I_1}(X_1) \bigcup \Gamma_{I_2}(X_2)$ are given by Figures 13, 14, and 15.

Let $X = X_1 \bigcup X_2, I = I_1 \bigcup I_2$. If S is the set that was defined in Proposition 5.30, then $S = \{0a, 0b, 0c, 01, 02, , a1, a2, b1, b2\}$. Therefore, $E(\Gamma_I(X)) = E(\Gamma_{I_1}(X_1) \bigcup \Gamma_{I_2}(X_2)) \bigcup S$ and so $\Gamma_I(X)$ is given in Figure 16.

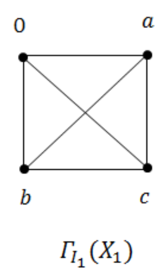

Figure 13.

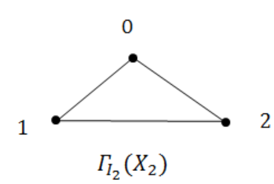

Figure 14.

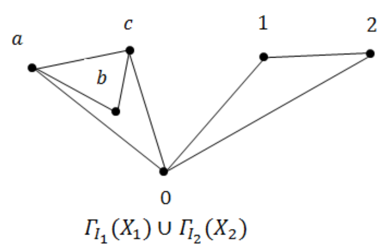

Figure 15.

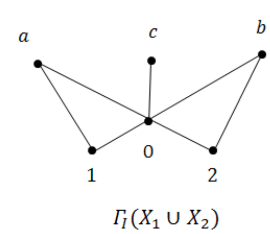

Figure 16.

Theorem 5.33 *Let X be finite and I be an ideal of X. Then we have:*

$\chi(\Gamma^I(X), \lambda) = \prod_{a_t \in X} \chi(\Gamma^I([a_t]_I), \lambda) = \prod_{t=1,\ldots,m} (\lambda - r_t + 1)(\lambda + 1)^{r_t}$, *where* $r_t =\mid [a_t]_I \mid_I$

Proof Let $m \in N, X/I = \{[a_1]_I, \dots, [a_m]_I\}, [a_t]_I = \{x_{1t}, \dots, x_{r_t t}\}$, and $A_t = [b_{i,j}]$ be the adjacency matrix of $\Gamma^I([a_t]_I)$, for all $t \in \{1, 2, \dots, m\}$. Then $X = \{x_{1,1}, x_{2,1}, \dots, x_{r_1,1}, x_{1,2}, x_{2,2}, \dots, x_{r_2,2}, \dots, x_{1,m}, x_{2,m}, \dots, x_{r_m,m}\}$. Since $[a_i]_I \bigcap [a_j]_I = \emptyset$, for all distinct $i, j \in \{1, 2, \dots, m\}$, then by Theorem 5.27 (ii), the adjacent matrix of $\Gamma^I(X)$ is of the form

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_m \end{bmatrix}$$

where A_t is isomorphic to the adjacent matrix of a complete graph K_{r_t} on r_t vertices, for all $t \in \{1, 2, \dots, m\}$. By the properties of the determinate, we have,

$$\chi(\Gamma^I(X), \lambda) = det(\lambda I - A) = det(\lambda I_1 - A_1) \times \det(\lambda I_2 - A_2) \times \dots \times \det(\lambda I_m - A_m) = \prod \chi(\Gamma^I([a_t]_I), \lambda),$$

where I_t is an $r_t \times r_t$ identity matrix, for all $t \in \{1, 2, \dots, m\}$. On the other hand, by Theorems 2.10 and 5.27, part (i), we have $\chi(\Gamma^I(X), \lambda) = \prod_{t=1,\dots,m}(\lambda - r_t + 1)(\lambda + 1)^{r_t}$. □

Corollary 5.34 *Let X be finite and t be the number of elements $a \in X$ such that $\mid [a]_I \mid= 1$.*

(i) $\chi(\Gamma^I(X), \lambda) = \lambda^t \times f(\lambda)$, for some polynomial $f(\lambda)$.

(ii) $I = \{0\}$ if and only if $\chi(\Gamma^I(X), \lambda) = \lambda^n$, for some $n \in N$.

Proof

(i) Let $\mid X \mid= n$ and $\{a_1, \dots, a_t\}$ be the set of all elements of X such that $\mid [a_i]_I \mid= 1$, for all $i \in \{1, 2, \dots, t\}$. Therefore, by using the proof of Theorem 5.27 (iii), the adjacent matrix of $\Gamma^I(X)$ is of the form

$$\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}_{n \times n}$$

where B is an $(n - t \times n - t)$ matrix. Hence, by properties of the determinant, we have $\chi(\Gamma^I(X), \lambda) = \det(\lambda I_t) \times \det(\lambda I_{n-t} - B) = \lambda^t \times \det(\lambda I_{n-t} - B)$. Let $f(\lambda) = \det(\lambda I_{n-t} - B)$, and then $\chi(\Gamma^I(X), \lambda) = \lambda^t \times f(\lambda)$.

(ii) Since $I = \{0\}$, then $\mid [a]_I \mid= 1$, for all $a \in X$. Therefore, by (i), $\chi(\Gamma^I(X), \lambda) = \lambda^n$, where $\mid X \mid= n$. Conversely, let $\chi(\Gamma^I(X), \lambda) = \lambda^n$, for some $n \in N$. Then $\Gamma^I(X)$ is an empty graph. Therefore, by Theorem 5.20, $I = \{0\}$.

□

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