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Fréchet–Hilbert spaces and the property SCBS

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Abstract: In this note, we obtain that all separable Fréchet–Hilbert spaces have the property of smallness up to a complemented Banach subspace (SCBS). Djakov, Terzioğlu, Yurdakul, and Zahariuta proved that a bounded perturbation of an automorphism on Fréchet spaces with the SCBS property is stable up to a complemented Banach subspace. Considering Fréchet–Hilbert spaces we show that the bounded perturbation of an automorphism on a separable Fréchet–Hilbert space still takes place up to a complemented Hilbert subspace. Moreover, the strong dual of a real Fréchet–Hilbert space has the SCBS property.

Key words: Locally convex spaces, Fréchet–Hilbert spaces, the SCBS property

1. Introduction

We denote by $\mathcal{U}(X)$ the collection of all closed, absolutely convex neighborhoods of the origin, and $\mathcal{B}(X)$ the collection of all closed, absolutely convex, and bounded subsets of a locally convex space X. We write $(X,Y) \in \mathcal{BF}$ when every continuous linear operator from X into X that factors over Y is bounded.

A locally convex space X satisfies the property of smallness up to a complemented Banach subspace (SCBS) if for all $A \in \mathcal{B}(X)$, for all $U \in \mathcal{U}(X)$ and for all $\epsilon > 0$, there are complementary subspaces B and C of X such that B is a Banach space and $A \subset B + \epsilon U \cap C$. In [4], it was obtained that a bounded perturbation of an automorphism on a Fréchet space X with the property SCBS is stable up to some complemented Banach subspace and all Banach valued ℓ^p-Köthe spaces have this property. It was essential there to get the generalized Douady lemma in [9].

By the Fredholm operator theory, isomorphisms of the spaces $X_1 \times Y_1 \cong X_2 \times Y_2$ such that any continuous linear operator from X_1 to Y_2 and from X_2 to Y_1 is compact implies an isomorphism of Cartesian factors up to some finite dimensional subspace. This approach was generalized by using boundedness instead of compactness and obtained the isomorphism up to some complemented Banach subspace [4].

In particular, more general forms of Köthe, say ℓ-Köthe spaces, are considered in [1] and it was proved that all ℓ-Köthe spaces satisfy the SCBS property. Moreover, this property does not pass to subspaces or quotients.

In this work, we consider the Fréchet–Hilbert spaces (hilbertizable Fréchet spaces) and prove that all separable Fréchet–Hilbert spaces have the SCBS property. In this case, we show that a bounded perturbation

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of an automorphism on a separable Fréchet–Hilbert space X will be stable up to some complemented Hilbert subspace. In addition, the strong dual of real Fréchet–Hilbert spaces also satisfies the SCBS property.

Since normable Fréchet–Hilbert spaces are exactly Hilbert spaces, we consider only nonnormable Fréchet–Hilbert spaces.

2. Preliminaries

A Fréchet space X is called a Fréchet–Hilbert space if its topology is generated by a fundamental system of Hilbert seminorms $p_n(x) = \langle x, x\rangle_n^{1/2}$, $(n \in \mathbb{N})$, and X is complete in each seminorm p_n.

If X is the strict inductive limit of the sequence X_n of locally convex spaces, we write $X = s - \varinjlim X_n$. If X_n are Banach spaces (resp. Hilbert spaces) then X is strict $LB-$spaces (resp. strict $LH-$spaces).

Recall that a Fréchet space X is called a quojection if, for every continuous seminorm p on X, the space $X/Kerp$ is Banach when endowed with the quotient topology. It is easily seen that X is a quojection if it is isomorphic to the projective limit of a sequence of Banach spaces with respect to surjective mappings [7].

Following Nachbin [8], a locally convex space X is said to have the openness condition if

$$\forall U \in \mathcal{U}(X) \quad \exists V \in \mathcal{U}(X) \quad \forall W \in \mathcal{U}(X) \quad \exists c > 0 : V \subset cW + Kerp_U,$$

that is, $X/Kerp_U$ with the quotient topology is normable for every $U \in \mathcal{U}(X)$. Fréchet spaces with the openness condition are exactly quojections [2]. On the other hand, quojections satisfy the openness condition. For a quojection X the strong dual X' is a strict $(LB)-$space (see [10]).

3. Fréchet–Hilbert spaces

In this section, we consider the separable Fréchet–Hilbert spaces and obtain some applications. First, we prove the following.

Proposition 1 *Every separable Fréchet–Hilbert space X has the SCBS property.*

Proof Every separable Fréchet–Hilbert space is isomorphic to the space $\omega = \mathbb{R}^{\mathbb{N}}$ of all scalar sequences, or to the space $\ell^2 \times \omega$, or to the space $(\ell^2)^{\mathbb{N}}$, in particular the spaces ℓ^2_{loc} of locally square summable double sequences and the space $\mathcal{L}^2_{loc}(\mathbb{R})$ isomorphic to the space $(\ell^2)^{\mathbb{N}}$ [10]. We see that these countable products of Banach-spaces can be understood as Banach valued Köthe spaces, that is, the kth row of the Köthe matrix will be $e_1 + e_2 + \cdots + e_k$, and it is known that all Banach valued Köthe spaces have the SCBS property [4]. □

Now we have the following result if we follow the steps of the proof in [4].

Theorem 1 *If X is a separable Fréchet–Hilbert space and $T : X \longrightarrow X$ is a bounded (resp.compact) operator, then there exist complementary subspaces H and C such that:*
(i) H is Hilbert (resp. finite-dimensional) space; and
(ii) the operator $1_C - \pi_C T i_C$ is an automorphism on C, where π_C and i_C are the canonical projection onto C and embedding into X, respectively.

Proof Let $T : X \longrightarrow X$ be a linear bounded operator. Then $\exists V \in \mathcal{U}(X)$ such that $T(V)$ is bounded in X, i.e.

$$\forall U \in \mathcal{U}(X) \quad \exists M_U > 0 : p_U(Tx) \leq M_U p_V(x)$$

Since every separable Fréchet–Hilbert space has the SCBS property, there exist complementary subspaces H and C of X such that H is a Hilbert (resp. finite dimensional) space and

$$T(V) \subset H + \frac{1}{2}V \cap C.$$

Let $T_1 = \pi_C T i_C : C \longrightarrow C$. Then we obtain

$$p_V(T_1 x) \leq \frac{1}{2} p_V(x), \quad \forall x \in C.$$

We show that $1_C - T_1$ is an automorphism on C. Consider the series

$$Sx = x + T_1 x + T_1^2 x + \cdots + T_1^m x + \cdots, \quad \forall x \in C \tag{3.1}$$

It is convergent in C because $\forall U \in \mathcal{U}(X)$, $m = 1, 2, \ldots$, and we obtain

$$p_U(T_1^m x) \leq M_U p_V(T_1^{m-1} x) \leq M_U \left(\frac{1}{2}\right) p_V(T_1^{m-2} x) \leq M_U \left(\frac{1}{2}\right)^{m-1} p_V(x)$$

and so by Banach–Steinhaus theorem, (3.1) defines a continuous linear operator $S : C \longrightarrow C$. Since $(1_C - T_1)Sx = S(1_C - T_1)x = x$, the operator S is the inverse to the operator $1_C - T_1$. □

Now, by the theorem above, we obtain a modification of the generalized Douady lemma in [9], in exactly the same way as in [4].

Theorem 2 *Suppose X_1 is a separable Fréchet–Hilbert space and X_2, Y_1, Y_2 are topological vector spaces. If $X_1 \times Y_1 \cong X_2 \times Y_2$ and $(X_1, Y_2) \in \mathcal{BF}$ then there exist complementary subspaces H_1, C_1 in X_1 and H_2, C_2 in X_2 such that H_1 is a Hilbert space, $C_1 \cong C_2$, and $H_1 \times Y_1 \cong H_2 \times Y_2$.*

If in addition $(X_2, Y_1) \in \mathcal{BF}$, then H_2 is a Hilbert space.

A locally convex space X is said to satisfy the vanishing sequence property (vsp) if given any strongly convergent sequence (x_n) there is $m \in \mathbb{N}$ such that $x_n = 0$ if $n > m$. Furthermore, a Fréchet space satisfies the vsp if and only if it has a continuous norm (see [3]). Using this fact, we obtain the following application.

Proposition 2 *Suppose X_1, X_2 are separable Fréchet–Hilbert spaces and Y_1, Y_2 are Fréchet spaces admitting continuous norms. If $X_1 \times Y_1 \cong X_2 \times Y_2$, then there exist complementary subspaces H_1, C_1 in X_1 and H_2, C_2 in X_2 such that H_1 and H_2 are Hilbert spaces, $C_1 \cong C_2$, and $H_1 \times Y_1 \cong H_2 \times Y_2$.*

Proof Since Fréchet–Hilbert spaces are reflexive and separable ones are trivial quojections (see [10]), from Corollary 10 in [3], we obtain that $(X_1, Y_2) \in \mathcal{B}$, $(X_2, Y_1) \in \mathcal{B}$. Since X_1 and X_2 satisfy the SCBS propery, by Theorem 2, there exist complementary subspaces H_1, C_1 in X_1 and H_2, C_2 in X_2 such that H_1 and H_2 are Hilbert spaces, $C_1 \cong C_2$, and $H_1 \times Y_1 \cong H_2 \times Y_2$. In this application, we used the relation $\mathcal{B}$ instead of the weaker relation $\mathcal{BF}$. □

4. Strong dual of Fréchet–Hilbert spaces

It is known that, if X is a strict inductive limit of Banach spaces and $A \in \mathcal{B}(X)$, then A is contained and bounded in some Banach space B from the inductive limit system. Since the limit is strict then B is a subspace of X (see 2.5.13 in [5]).

It is not hard to verify that a Fréchet space X is complete in a seminorm p_n if and only if the quotient space $X/Kerp_n$ is a Banach space in the associated norm $\tilde{p}_n$. Therefore, in the case of a Fréchet–Hilbert space X, the quotient spaces $X/Kerp_n$ are Hilbert spaces in the norms $\tilde{p}_n$.

Theorem 3 *If X is a real Fréchet–Hilbert space, the strong dual of X has the $SCBS$.*

Proof Let X be a real Fréchet–Hilbert space. Then its strong dual $X' = s - \varinjlim H_n$ is a strict (LH)-space where each Hilbert subspace H_n has a topological complement in X' [10]. If $A \in \mathcal{B}(X')$, then A is contained and bounded in some Hilbert space H from the inductive system, which is complemented. That is X' has the $SCBS$ property. Here the smallness is trivial since the Hilbert subspace chosen is independent of ϵ and U. □

Remark 1 *The space of all sequences $\omega = \prod_{k=1}^{\infty} X_k$, where $X_k = \mathbb{R}$ is the real line for all k, is a Fréchet–Hilbert space [10]. Strong dual X' is the direct sum of Banach spaces dual to X_k, i.e. $X' = \sum_{k=1}^{\infty} {X_k}'$. Locally convex direct sum of Banach spaces has the SCBS property trivially, since every bounded subset of it is contained and bounded in a finite direct sum of its components (which is Banach) from the direct system, which is complemented [6].*

Corollary 1 *If X is a real Fréchet–Hilbert space, then X has the openness condition.*

Proof Let X be a real Fréchet–Hilbert space. Then it is the strict projective limit of a sequence of complemented Hilbert subspaces of it [10]. Therefore, being a quojection, it has the openness condition. □

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