Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1298 – 1306
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doi:10.3906/mat-1706-68

On some multivariate LCM and GCD sums

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Received: 22.06.2017 • **Accepted/Published Online:** 07.11.2017 • **Final Version:** 08.05.2018

Abstract: In this paper we obtain an asymptotic formula with a power saving error term for the summation function of a family of generalized least common multiple and greatest common divisor functions of several integer variables.

Key words: Arithmetic functions of several variables, multiplicative functions, least common multiple, greatest common divisor, asymptotic formula

1. Introduction

Let $[n_1,\dots,n_k]$ denote the least common multiple (LCM) and $(n_1,\dots,n_k)$ denote the greatest common divisor (GCD) of positive integers $n_1,\dots,n_k$. Although looking simple, their statistical behavior is nontrivial; see, for example, the recent study [1] of the least common multiple function from the probabilistic point of view. A related and natural question would be to study asymptotic formulas for mean values of the GCD and LCM functions of several integer variables. For example, Diaconis and Erdős in [2] obtained the following asymptotic formulas in the case of $k=2$ variables:

$$\sum_{m,n\le x}(m,n)=\frac{x^2}{\zeta(2)}\left(\log x+2\gamma-\frac{1}{2}-\frac{\zeta(2)}{2}-\frac{\zeta'(2)}{\zeta(2)}\right)+O(x^{3/2}\log x)$$

and

$$\sum_{m,n\le x}[m,n]^a=\frac{\zeta(a+2)}{(a+1)^2\zeta(2)}\,x^{2a+2}+O(x^{2a+1}\log x),$$

where a is any positive real number, γ is Euler's constant, and $\zeta(s)$ is the Riemann zeta function. In [4] the authors considered also the problem of establishing an asymptotic formula for the summation function of the quotient $\frac{[m,n]}{(m,n)}$ of the least common multiple and the greatest common divisor of integers m and n and obtained the formula

$$\sum_{m,n\le x}\frac{[m,n]}{(m,n)}=\frac{\pi^2}{60}\,x^4+O(x^3\log x).$$

For some interesting properties of the arithmetic function $\frac{[m,n]}{(m,n)}$ we refer the reader to the recent paper [3], and a more extensive bibliography of the related results in this area is presented in the introductory section of [4].

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2010 *AMS Mathematics Subject Classification:* 11A25, 11N37, 11N60, 11A05

Moreover, Hilberdink and Tóth in [4] derived more general asymptotic formulas, concerning the summation over $k \geq 3$ arguments: for any real $a > 0$ and for any $\epsilon > 0$ they obtained

$$\sum_{n_1,\ldots,n_k \leq x} [n_1,\ldots,n_k]^a = C_{a,k}\, x^{k(a+1)} + O_\epsilon\left(x^{k(a+1)-\frac{1}{2}+\epsilon}\right)$$

and

$$\sum_{n_1,\ldots,n_k \leq x} \left(\frac{[n_1,\ldots,n_k]}{(n_1,\ldots,n_k)}\right)^a = D_{a,k}\, x^{k(a+1)} + O_\epsilon\left(x^{k(a+1)-\frac{1}{2}+\epsilon}\right)$$

for some positive constants $C_{a,k}$ and $D_{a,k}$.

In this paper we will generalize these results (in the case of integer a) further and consider the arithmetic function of $k+\ell$ variables:

$$f(n_1,\ldots,n_{k+\ell}) := \left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c}, \frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right].$$

This function satisfies

$$f(m_1n_1,\ldots,m_{k+\ell}n_{k+\ell}) = f(m_1,\ldots,m_{k+\ell})f(n_1,\ldots,n_{k+\ell}),$$

for any $m_1,\ldots,m_{k+\ell},n_1,\ldots,n_{k+\ell} \in \mathbb{N}$ such that $(m_1\ldots m_{k+\ell}, n_1\ldots n_{k+\ell}) = 1$, i.e. f is an example of a multiplicative arithmetic function of several variables. Using the methods from [4], we will prove the following theorem:

Theorem 1.1 *Let $k \geq 2$, $\ell \geq 1$, $a \geq c \geq 1$, and $b \geq d \geq 0$ be fixed integers. Then for every $\epsilon > 0$ we have*

$$\sum_{n_1,\ldots,n_{k+\ell} \leq x} \left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c}, \frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right] = \frac{C_{k,a,c;\ell,b,d}}{(a+1)^k(b+1)^\ell}\quad x^{k(a+1)+\ell(b+1)} + O_\epsilon\left(x^{k(a+1)+\ell(b+1)-\frac{1}{2}+\epsilon}\right) \tag{1.1}$$

and

$$\sum_{n_1,\ldots,n_{k+\ell} \leq x} \frac{\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c}, \frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]}{(n_1\ldots n_k)^a(n_{k+1}\ldots n_{k+\ell})^b} = C_{k,a,c;\ell,b,d}\, x^{k+\ell} + O_\epsilon\left(x^{k+\ell-\frac{1}{2}+\epsilon}\right), \tag{1.2}$$

where the constant $C_{k,a,c;\ell,b,d}$ is given by the Euler product

$$\prod_p \left(1-\frac{1}{p}\right)^{k+\ell} \sum_{\nu_1,\ldots,\nu_{k+\ell}=0}^{\infty} \frac{p^{\max\{(a\max - c\min)\{\nu_1,\ldots,\nu_k\},(b\max - d\min)\{\nu_{k+1},\ldots,\nu_{k+\ell}\}\}}}{p^{(a+1)(\nu_1+\cdots+\nu_k)+(b+1)(\nu_{k+1}+\cdots+\nu_{k+\ell})}}.$$

Here and throughout the paper we will use the following notation:

$$(a\max - c\min)\{\nu_1,\ldots,\nu_k\} := a\cdot\max\{\nu_1,\ldots,\nu_k\} - c\cdot\min\{\nu_1,\ldots,\nu_k\}.$$

As an illustration, we obtain the following corollaries:

Corollary 1.2 *For every $\epsilon > 0$ we have*

$$\sum_{n_1,n_2,n_3\le x}\left[\frac{[n_1,n_2]}{(n_1,n_2)},n_3\right]=\frac{C_{2,1,1;1,1,0}}{8}\,x^6+O_\epsilon\left(x^{\frac{11}{2}+\epsilon}\right)$$

and

$$\sum_{n_1,n_2,n_3\le x}\frac{\left[\frac{[n_1,n_2]}{(n_1,n_2)},n_3\right]}{n_1n_2n_3}=C_{2,1,1;1,1,0}\,x^3+O_\epsilon\left(x^{\frac{5}{2}+\epsilon}\right)$$

where

$$C_{2,1,1;1,1,0}=\zeta(3)\zeta(4)\prod_p\left(1-\frac{3}{p^2}+\frac{3}{p^3}-\frac{2}{p^4}+\frac{1}{p^5}\right). \tag{1.3}$$

Corollary 1.3 *For every $\epsilon > 0$ we have*

$$\sum_{n_1,n_2,n_3\le x}\left[\frac{[n_1,n_2]^3}{(n_1,n_2)},n_3^2\right]=\frac{C_{2,3,1;1,2,0}}{48}\,x^{11}+O_\epsilon\left(x^{\frac{21}{2}+\epsilon}\right)$$

and

$$\sum_{n_1n_2,n_3\le x}\frac{\left[\frac{[n_1,n_2]^3}{(n_1,n_2)},n_3^2\right]}{n_1^3n_2^3n_3^2}=C_{2,3,1;1,2,0}\,x^3+O_\epsilon\left(x^{\frac{5}{2}+\epsilon}\right),$$

where

$$C_{2,3,1;1,2,0}=\zeta(3)\zeta(6)\zeta(9)\zeta(11)\prod_p\left(1-\frac{3}{p^2}+\frac{1}{p^3}+\frac{2}{p^4}-\frac{1}{p^5}+\frac{2}{p^6}-\frac{7}{p^7}+\frac{10}{p^8}-\frac{9}{p^9}\right.$$

$$\left.+\frac{5}{p^{10}}-\frac{1}{p^{11}}-\frac{1}{p^{12}}+\frac{5}{p^{13}}-\frac{9}{p^{14}}+\frac{10}{p^{15}}-\frac{7}{p^{16}}+\frac{2}{p^{17}}-\frac{1}{p^{18}}+\frac{2}{p^{19}}+\frac{1}{p^{20}}-\frac{3}{p^{21}}+\frac{1}{p^{23}}\right). \tag{1.4}$$

By the method of the proof in Theorem 1.1 we obtained the relative error of size $O(x^{-1/2+\epsilon})$. It remains an interesting open question to determine the best possible exponent in the error term.

2. Proof of Theorem 1.1

To prove this theorem we need the following lemma:

Lemma 2.1 *For integers $k \ge 2$, $\ell \ge 1$, $a \ge c \ge 1$, and $b \ge d \ge 0$ we have*

$$L(z_1,\ldots,z_{k+\ell}):=\sum_{n_1,\ldots,n_{k+\ell}=1}^{\infty}\frac{\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c},\frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]}{n_1^{z_1}\ldots n_k^{z_k}n_{k+1}^{z_{k+1}}\ldots n_{k+\ell}^{z_{k+\ell}}}$$

$$=\zeta(z_1-a)\ldots\zeta(z_k-a)\zeta(z_{k+1}-b)\ldots\zeta(z_{k+\ell}-b)\,H(z_1,\ldots,z_{k+\ell}), \tag{2.1}$$

where the multiple Dirichlet series $H(z_1,\ldots,z_{k+\ell})$ is absolutely convergent for

$$\Re z_1,\ldots,\Re z_k > a+\frac{1}{2} \qquad \text{and} \qquad \Re z_{k+1},\ldots,\Re z_{k+\ell} > b+\frac{1}{2}. \tag{2.2}$$

Proof Since the function

$$(n_1,\dots,n_{k+\ell}) \longmapsto \left[\frac{[n_1,\dots,n_k]^a}{(n_1,\dots,n_k)^c},\ \frac{[n_{k+1},\dots,n_{k+\ell}]^b}{(n_{k+1},\dots,n_{k+\ell})^d}\right]$$

is a multiplicative function of $k+\ell$ variables, the multiple Dirichlet series $L(z_1,\dots,z_{k+\ell})$ has the following Euler product expansion:

$$L(z_1,\dots,z_{k+\ell}) = \prod_p \sum_{\nu_1,\dots,\nu_{k+\ell}=0}^{\infty} \frac{p^{\max\{(a\max - c\min)\{\nu_1,\dots,\nu_k\},(b\max - d\min)\{\nu_{k+1},\dots,\nu_{k+\ell}\}\}}}{p^{\nu_1 z_1+\dots+\nu_k z_k+\nu_{k+1}z_{k+1}+\dots+\nu_{k+\ell}z_{k+\ell}}}.$$

In each Euler's local factor, we single out the contribution of the terms for which $\nu_1+\dots+\nu_{k+\ell}\le 1$:

$$\begin{aligned} L(z_1,\dots,z_{k+\ell}) =& \prod_p \Bigg(1+\frac{p^a}{p^{z_1}}+\dots+\frac{p^a}{p^{z_k}}+\frac{p^b}{p^{z_{k+1}}}+\dots+\frac{p^b}{p^{z_{k+\ell}}}+ \\ &+\sum_{\nu_1+\dots+\nu_{k+\ell}\ge 2} \frac{p^{\max\{(a\max - c\min)\{\nu_1,\dots,\nu_k\},(b\max - d\min)\{\nu_{k+1},\dots,\nu_{k+\ell}\}\}}}{p^{\nu_1 z_1+\dots+\nu_{k+\ell}z_{k+\ell}}}\Bigg). \qquad (2.3)\end{aligned}$$

Now, for $(z_1,\dots,z_{k+\ell})$ in the region $\Re z_1,\dots,\Re z_k \ge \delta_1 > a$, $\Re z_{k+1},\dots,\Re z_{k+\ell} \ge \delta_2 > b$ (for some fixed $\delta_1 > a$, $\delta_2 > b$), we have that

$$\begin{aligned}\left|\frac{p^{\max\{(a\max - c\min)\{\nu_1,\dots,\nu_k\},(b\max - d\min)\{\nu_{k+1},\dots,\nu_{k+\ell}\}\}}}{p^{\nu_1 z_1+\dots+\nu_{k+\ell}z_{k+\ell}}}\right| &\le \frac{p^{a(\nu_1+\dots+\nu_k)+b(\nu_{k+1}+\dots+\nu_{k+\ell})}}{p^{\delta_1(\nu_1+\dots+\nu_k)+\delta_2(\nu_{k+1}+\dots+\nu_{k+\ell})}} \\ &= \frac{1}{p^{(\delta_1-a)(\nu_1+\dots+\nu_k)+(\delta_2-b)(\nu_{k+1}+\dots+\nu_{k+\ell})}}.\end{aligned}$$

Therefore, since $N_k(n) := \#\{(\nu_1,\dots,\nu_k) \mid \nu_1+\dots+\nu_k = n\} = \binom{n+k-1}{k-1}$, the sum over $\nu_1+\dots+\nu_{k+\ell}\ge 2$ in (2.3) is bounded by

$$\sum_{m+n\ge 2} \frac{N_k(m)N_\ell(n)}{p^{(\delta_1-a)m+(\delta_2-b)n}} = O\left(\frac{1}{p^{2(\delta_1-a)}}+\frac{1}{p^{2(\delta_2-b)}}\right).$$

Hence, for $\Re z_j > \max\{a+1,\delta_1\}$ for all $1\le j\le k$ and $\Re z_j > \max\{b+1,\delta_2\}$ for all $k+1\le j\le k+\ell$ we have

$$\begin{aligned} H(z_1,\dots,z_{k+\ell}) :=& L(z_1,\dots,z_{k+\ell})\zeta^{-1}(z_1-a)\dots\zeta^{-1}(z_k-a)\zeta^{-1}(z_{k+1}-b)\dots\zeta^{-1}(z_{k+\ell}-b) \\ =& \prod_p \left(1-\frac{1}{p^{z_1-a}}\right)\dots\left(1-\frac{1}{p^{z_k-a}}\right)\left(1-\frac{1}{p^{z_{k+1}-b}}\right)\dots\left(1-\frac{1}{p^{z_{k+\ell}-b}}\right) \\ &\times\left(1+\frac{1}{p^{z_1-a}}+\dots+\frac{1}{p^{z_k-a}}+\frac{1}{p^{z_{k+1}-b}}+\dots+\frac{1}{p^{z_{k+\ell}-b}}+O\left(\frac{1}{p^{2(\delta_1-a)}}+\frac{1}{p^{2(\delta_2-b)}}\right)\right) \\ =& \prod_p \left(1+O\left(\frac{1}{p^{2(\delta_1-a)}}+\frac{1}{p^{2(\delta_2-b)}}\right)\right), \qquad (2.4)\end{aligned}$$

since the terms $\pm\frac{1}{p^{z_j-a}}$ and $\pm\frac{1}{p^{z_j-b}}$ cancel out.

The Euler product in (2.4) converges absolutely for $\delta_1 > a+\frac{1}{2}$ and $\delta_2 > b+\frac{1}{2}$. Thus, the identity (2.1) holds in the product of half-planes (2.2). □

Proof (*of Theorem 1.1*). Let us define the multiplicative function $h(n_1,\dots,n_{k+\ell})$ as coefficients of the multiple Dirichlet series expansion of the function $H(z_1,\dots,z_{k+\ell})$ from Lemma 2.1:

$$H(z_1,\dots,z_{k+\ell}) = \sum_{n_1,\dots,n_{k+\ell}=1}^{\infty} \frac{h(n_1,\dots,n_{k+\ell})}{n_1^{z_1}\dots n_{k+\ell}^{z_{k+\ell}}}.$$

From the identity (2.1) we obtain the following convolution identity between the corresponding arithmetic functions:

$$\left[\frac{[n_1,\dots,n_k]^a}{(n_1,\dots,n_k)^c}, \frac{[n_{k+1},\dots,n_{k+\ell}]^b}{(n_{k+1},\dots,n_{k+\ell})^d}\right] = \sum_{j_1d_1=n_1,\dots,j_{k+\ell}d_{k+\ell}=n_{k+\ell}} j_1^a\dots j_k^a\, j_{k+1}^b\dots j_{k+\ell}^b\, h(d_1,\dots,d_{k+\ell}). \tag{2.5}$$

By using this identity we get

$$\begin{aligned}
&\sum_{n_1,\dots,n_{k+\ell}\le x}\left[\frac{[n_1,\dots,n_k]^a}{(n_1,\dots,n_k)^c}, \frac{[n_{k+1},\dots,n_{k+\ell}]^b}{(n_{k+1},\dots,n_{k+\ell})^d}\right]\\
&= \sum_{j_1d_1\le x,\dots,j_{k+\ell}d_{k+\ell}\le x} j_1^a\dots j_k^a\, j_{k+1}^b\dots j_{k+\ell}^b\, h(d_1,\dots,d_{k+\ell})\\
&= \sum_{d_1,\dots,d_{k+\ell}\le x} h(d_1,\dots,d_{k+\ell}) \sum_{j_1\le \frac{x}{d_1}} j_1^a \dots \sum_{j_k\le \frac{x}{d_k}} j_k^a \sum_{j_{k+1}\le \frac{x}{d_{k+1}}} j_{k+1}^b \dots \sum_{j_{k+\ell}\le \frac{x}{d_{k+\ell}}} j_{k+\ell}^b\\
&= \sum_{d_1,\dots,d_{k+\ell}\le x} h(d_1,\dots,d_{k+\ell}) \left(\frac{x^{a+1}}{(a+1)d_1^{a+1}} + O\left(\frac{x^a}{d_1^a}\right)\right)\dots\left(\frac{x^{b+1}}{(b+1)d_{k+\ell}^{b+1}} + O\left(\frac{x^b}{d_{k+\ell}^b}\right)\right)\\
&= \frac{x^{k(a+1)+\ell(b+1)}}{(a+1)^k(b+1)^\ell} \sum_{d_1,\dots,d_{k+\ell}\le x} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}} + R(x),
\end{aligned} \tag{2.6}$$

where the remainder term $R(x)$ is bounded by

$$R(x) \ll \sum_{\substack{u_1,\dots,u_k\in\{a,a+1\},\\ v_1,\dots,v_\ell\in\{b,b+1\},\\ (u_1,\dots,u_k,v_1,\dots,v_\ell)\neq\\ (a+1,\dots,a+1,b+1,\dots,b+1)}} x^{u_1+\dots+u_k+v_1+\dots+v_\ell} \sum_{d_1,\dots,d_{k+\ell}\le x} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^{u_1}\dots d_k^{u_k} d_{k+1}^{v_1}\dots d_{k+\ell}^{v_\ell}}. \tag{2.7}$$

Here the first summation is over $2^{k+\ell}-1$ $(k+\ell)$-tuples $(u_1,\dots,u_k,v_1,\dots,v_\ell)$ in which at least one u_i is a or at least one v_j is b. Let $(u_1,\dots,u_k,v_1,\dots,v_\ell)$ be one such fixed $(k+\ell)$-tuple with $s\ge 1$ u_i-coordinates equal to a, for example $(u_1,\dots,u_s,u_{s+1},\dots,u_k,v_1,\dots,v_\ell) = (a,\dots,a,a+1,\dots,a+1,b+1,\dots,b+1)$. The corresponding contribution on the right-hand side of (2.7) is bounded by

$$\ll x^{sa+(k-s)(a+1)+\ell(b+1)} \sum_{d_1,\dots,d_{k+\ell}\le x} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^a\dots d_s^a d_{s+1}^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}}$$

$$= x^{sa+(k-s)(a+1)+\ell(b+1)} \sum_{d_1,\dots,d_{k+\ell}\le x} \frac{|h(d_1,\dots,d_{k+\ell})| d_1^{\frac{1}{2}+\epsilon}\dots d_s^{\frac{1}{2}+\epsilon}}{d_1^{a+\frac{1}{2}+\epsilon}\dots d_s^{a+\frac{1}{2}+\epsilon} d_{s+1}^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}}$$

$$\le x^{sa+(k-s)(a+1)+\ell(b+1)+s(\frac{1}{2}+\epsilon)} \sum_{d_1,\dots,d_{k+\ell}=1}^{\infty} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^{a+\frac{1}{2}+\epsilon}\dots d_s^{a+\frac{1}{2}+\epsilon} d_{s+1}^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}},$$

for any $\epsilon > 0$. Here, the exponents $(a+\frac{1}{2}+\epsilon,\dots,a+\frac{1}{2}+\epsilon,a+1,\dots,a+1,b+1,\dots,\ b+1)$ belong to the region of absolute convergence (2.2) and hence, by Lemma 2.1, the last multiple Dirichlet series converges to a constant and we obtain the bound

$$\ll x^{k(a+1)+\ell(b+1)-\frac{s}{2}+s\epsilon}.$$

This is maximal for $s = 1$. Similarly we bound the contributions in (2.7) corresponding to all other $(k+\ell)$-tuples $(u_1,\dots,u_k,v_1,\dots,v_\ell)$. Therefore, we get

$$R(x) \ll x^{k(a+1)+\ell(b+1)-\frac{1}{2}+\epsilon}. \tag{2.8}$$

Next we write the sum in the main term in (2.6) as follows:

$$\sum_{d_1,\dots,d_{k+\ell}\le x} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}} = \sum_{d_1,\dots,d_{k+\ell}=1}^{\infty} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}}$$
$$- \sum_{\emptyset\ne I\subseteq\{1,\dots,k+\ell\}} \sum_{\substack{d_i>x, i\in I\\ d_j\le x, j\notin I}} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}}. \tag{2.9}$$

The complete multiple Dirichlet series in (2.9) converges by Lemma 2.1 and is equal to $H(a+1,\dots,a+1,b+1,\dots,b+1)$. For one fixed subset I, say $I = \{1,2,\dots,s,\ k+1,k+2,\dots,k+t\}$, $1\le s\le k$, $1\le t\le \ell$, the corresponding contribution in (2.9) is bounded by

$$\sum_{\substack{d_1,\dots,d_s>x\\ d_{s+1},\dots,d_k\le x\\ d_{k+1},\dots,d_{k+t}>x\\ d_{k+t+1},\dots,d_{k+\ell}\le x}} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}}$$

$$= \sum_{\substack{d_1,\dots,d_s>x\\ d_{s+1},\dots,d_k\le x\\ d_{k+1},\dots,d_{k+t}>x\\ d_{k+t+1},\dots,d_{k+\ell}\le x}} \frac{|h(d_1,\dots,d_{k+\ell})| d_1^{-\frac{1}{2}+\epsilon}\dots d_s^{-\frac{1}{2}+\epsilon} d_{k+1}^{-\frac{1}{2}+\epsilon}\dots d_{k+t}^{-\frac{1}{2}+\epsilon}}{d_1^{a+\frac{1}{2}+\epsilon}\dots d_s^{a+\frac{1}{2}+\epsilon} d_{s+1}^{a+1}\dots d_k^{a+1} d_{k+1}^{b+\frac{1}{2}+\epsilon}\dots d_{k+t}^{b+\frac{1}{2}+\epsilon} d_{k+t+1}^{b+1}\dots d_{k+\ell}^{b+1}}$$

$$\le x^{(s+t)(-\frac{1}{2}+\epsilon)} \sum_{d_1,\dots,d_{k+\ell}=1}^{\infty} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^{a+\frac{1}{2}+\epsilon}\dots d_s^{a+\frac{1}{2}+\epsilon} d_{s+1}^{a+1}\dots d_k^{a+1} d_{k+1}^{b+\frac{1}{2}+\epsilon}\dots d_{k+t}^{b+\frac{1}{2}+\epsilon} d_{k+t+1}^{b+1}\dots d_{k+\ell}^{b+1}}.$$

Here, the multiple Dirichlet series converges to a constant by Lemma 2.1 since

$$\left(a+\frac{1}{2}+\epsilon,\dots,a+\frac{1}{2}+\epsilon,a+1,\dots a+1,b+\frac{1}{2}+\epsilon,\dots,b+\frac{1}{2}+\epsilon,b+1,\dots,b+1\right)$$

belongs to the region (2.2). For all $I \neq \emptyset$ we have that $s+t \geq 1$ and hence the total error introduced by completing the series in (2.6) is again

$$O\left(x^{k(a+1)+\ell(b+1)-\frac{1}{2}+\epsilon}\right)$$

and matches the bound for the remainder (2.8).

This proves the asymptotic formula (1.1), with the constant $C_{k,a,c;\ell,b,d} = H(a+1,\dots,a+1,b+1,\dots,b+1)$, which can be explicitly calculated from Lemma 2.1:

$$C_{k,a,c;\ell,b,d} = \prod_p \left(1-\frac{1}{p}\right)^{k+\ell} \times \sum_{\nu_1,\dots,\nu_{k+\ell}=0}^{\infty} \frac{p^{\max\{(a\max - c\min)\{\nu_1,\dots,\nu_k\},\,(b\max - d\min)\{\nu_{k+1},\dots,\nu_{k+\ell}\}\}}}{p^{(a+1)(\nu_1+\dots+\nu_k)+(b+1)(\nu_{k+1}+\dots+\nu_{k+\ell})}}.$$

Proof of (1.2): From Lemma 2.1, i.e. from the convolution identity (2.5), we obtain

$$\frac{\left[\frac{[n_1,\dots,n_k]^a}{(n_1,\dots,n_k)^c}, \frac{[n_{k+1},\dots,n_{k+\ell}]^b}{(n_{k+1},\dots,n_{k+\ell})^d}\right]}{(n_1\dots n_k)^a (n_{k+1}\dots n_{k+\ell})^b} = \sum_{j_1 d_1 = n_1,\dots,j_{k+\ell} d_{k+\ell} = n_{k+\ell}} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^a \dots d_k^a d_{k+1}^b \dots d_{k+\ell}^b}.$$

Replacing this to the left-hand side of (1.2) we get

$$\begin{aligned}
\sum_{n_1,\dots,n_{k+\ell}\leq x} \frac{\left[\frac{[n_1,\dots,n_k]^a}{(n_1,\dots,n_k)^c}, \frac{[n_{k+1},\dots,n_{k+\ell}]^b}{(n_{k+1},\dots,n_{k+\ell})^d}\right]}{(n_1\dots n_k)^a (n_{k+1}\dots n_{k+\ell})^b} &= \sum_{d_1,\dots,d_{k+\ell}\leq x} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^a \dots d_k^a d_{k+1}^b \dots d_{k+\ell}^b} \sum_{j_1 \leq \frac{x}{d_1}} 1 \cdots \sum_{j_{k+\ell}\leq \frac{x}{d_{k+\ell}}} 1 \\
&= \sum_{d_1,\dots,d_{k+\ell}\leq x} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^a \dots d_k^a d_{k+1}^b \dots d_{k+\ell}^b} \left(\frac{x}{d_1} + O(1)\right) \dots \left(\frac{x}{d_{k+\ell}} + O(1)\right) \\
&= x^{k+\ell} \sum_{d_1,\dots,d_{k+\ell}\leq x} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^{a+1} \dots d_k^{a+1} d_{k+1}^{b+1} \dots d_{k+\ell}^{b+1}} + R_1(x), \qquad (2.10)
\end{aligned}$$

where the remainder $R_1(x)$ is bounded by

$$R_1(x) \ll \sum_{\substack{u_1,\dots,u_{k+\ell}\in\{0,1\} \\ (u_1,\dots,u_{k+\ell})\neq(1,1,\dots,1)}} x^{u_1+\dots+u_{k+\ell}} \sum_{d_1,\dots,d_{k+\ell}\leq x} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^{u_1}\dots d_{k+\ell}^{u_{k+\ell}}}.$$

By the same arguments leading to the bound (2.8), here we obtain

$$R_1(x) \ll x^{k+\ell-\frac{1}{2}+\epsilon}.$$

Similarly, we can complete the multiple Dirichlet series in the main term in (2.10) with the cost of the error term $O\left(x^{k+\ell-\frac{1}{2}+\epsilon}\right)$, which proves (1.2). □

3. Proofs of corollaries

Proof (*of Corollary 1.2*). By Theorem 1.1, we have that

$$C_{2,1,1;1,1,0} = \prod_p \left(1 - \frac{1}{p}\right)^3 \sum_{a,b,c \geq 0} \frac{p^{\max\{\max\{a,b\} - \min\{a,b\},\, c\}}}{p^{2(a+b+c)}}$$

$$= \prod_p \left(1 - \frac{1}{p}\right)^3 (S_1 + 2S_2 + 2S_3),$$

where

$$S_1 = \sum_{\substack{a=b\geq 0\\ c\geq 0}} \frac{1}{p^{4a+c}} = \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{1-\frac{1}{p^4}},$$

$$S_2 = \sum_{\substack{a>b\geq 0\\ 0\leq c<a-b}} \frac{1}{p^{a+3b+2c}} = \frac{1}{1-\frac{1}{p^2}} \left(\sum_{a>b\geq 0} \frac{1}{p^{a+3b}} - \sum_{a>b\geq 0} \frac{1}{p^{3a+b}} \right)$$

$$= \frac{1}{1-\frac{1}{p^2}} \left(\frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{b\geq 0} \frac{1}{p^{4b}} - \frac{1}{p^3\left(1-\frac{1}{p^3}\right)} \sum_{b\geq 0} \frac{1}{p^{4b}} \right)$$

$$= \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{1-\frac{1}{p^4}} \cdot \frac{1}{p\left(1-\frac{1}{p^3}\right)}$$

and

$$S_3 = \sum_{\substack{a>b\geq 0\\ c\geq a-b}} \frac{1}{p^{2a+2b+c}} = \frac{1}{1-\frac{1}{p}} \sum_{a>b\geq 0} \frac{1}{p^{3a+b}} = \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{1-\frac{1}{p^4}} \cdot \frac{1}{p^3\left(1-\frac{1}{p^3}\right)}.$$

Then we obtain

$$S_1 + 2S_2 + 2S_3 = \frac{1+\frac{2}{p}+\frac{1}{p^3}}{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^3}\right)\left(1-\frac{1}{p^4}\right)}$$

and hence

$$C_{2,1,1;1,1,0} = \prod_p \frac{\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}+\frac{1}{p^3}\right)}{\left(1-\frac{1}{p^3}\right)\left(1-\frac{1}{p^4}\right)},$$

which gives (1.3). □

Proof (*of Corollary 1.3*). By Theorem 1.1, we have that

$$C_{2,3,1;1,2,0} = \prod_p \left(1 - \frac{1}{p}\right)^3 \sum_{a,b,c \geq 0} \frac{p^{\max\{3\max\{a,b\} - \min\{a,b\},\, 2c\}}}{p^{4(a+b)+3c}}$$

$$= \prod_p \left(1 - \frac{1}{p}\right)^3 (S_1 + S_2 + 2S_3 + 2S_4),$$

where

$$S_1 = \sum_{0\le a=b<c} \frac{1}{p^{8a+c}} = \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{a\ge 0} \frac{1}{p^{9a}} = \frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^9}\right)},$$

$$S_2 = \sum_{0\le c\le a=b} \frac{1}{p^{6a+3c}} = \frac{1}{\left(1-\frac{1}{p^6}\right)\left(1-\frac{1}{p^9}\right)},$$

$$\begin{aligned} S_3 &= \sum_{\substack{a>b\ge 0\\ 3a-b<2c}} \frac{1}{p^{4a+4b+c}} \\ &= \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{\substack{a>b\ge 0\\ a\equiv b \pmod 2}} \frac{1}{p^{\frac{11}{2}a+\frac{7}{2}b}} + \frac{1}{p^{\frac{1}{2}}\left(1-\frac{1}{p}\right)} \sum_{\substack{a>b\ge 0\\ a\not\equiv b \pmod 2}} \frac{1}{p^{\frac{11}{2}a+\frac{7}{2}b}} \\ &= \frac{1}{p^{12}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{11}}\right)} \sum_{b\ge 0} \frac{1}{p^{9b}} + \frac{1}{p^{6}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{11}}\right)} \sum_{b\ge 0} \frac{1}{p^{9b}} \\ &= \frac{1}{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^9}\right)\left(1-\frac{1}{p^{11}}\right)} \left(\frac{1}{p^6}+\frac{1}{p^{12}}\right) \end{aligned}$$

and

$$\begin{aligned} S_4 &= \sum_{\substack{a>b\ge 0\\ 3a-b\ge 2c\ge 0}} \frac{1}{p^{a+5b+3c}} \\ &= \frac{1}{1-\frac{1}{p^3}} \left(\sum_{\substack{a>b\ge 0\\ a\equiv b \pmod 2}} \frac{1}{p^{a+5b}} - \frac{1}{p^3} \sum_{\substack{a>b\ge 0\\ a\equiv b \pmod 2}} \frac{1}{p^{\frac{11}{2}a+\frac{7}{2}b}} + \sum_{\substack{a>b\ge 0\\ a\not\equiv b \pmod 2}} \frac{1}{p^{a+5b}} - \frac{1}{p^{\frac{3}{2}}} \sum_{\substack{a>b\ge 0\\ a\not\equiv b \pmod 2}} \frac{1}{p^{\frac{11}{2}a+\frac{7}{2}b}} \right) \\ &= \frac{1}{1-\frac{1}{p^3}} \left(\frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^6}\right)} - \frac{1}{\left(1-\frac{1}{p^9}\right)\left(1-\frac{1}{p^{11}}\right)} \left(\frac{1}{p^7}+\frac{1}{p^{14}}\right) \right). \end{aligned}$$

Collecting everything, we obtain (1.4). □

Acknowledgment

The author would like to express her sincere thanks to the anonymous referees for a careful reading and valuable comments.

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