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Diagonal lift in the semi-cotangent bundle and its applications

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Abstract: The present paper is devoted to some results concerning the diagonal lift of tensor fields of type (1,1) from manifold M to its semi-cotangent bundle t*M. In this context, cross-sections in the semi-cotangent (pull-back) bundle t*M of cotangent bundle T*M by using projection (submersion) of the tangent bundle TM can be also defined.

Key words: Vector field, complete lift, diagonal lift, pull-back bundle, cross-section, semi-cotangent bundle

1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞, and let $(T(M_n),\pi_1,M_n)$ be a tangent bundle over M_n. We use the notation $(x^i)=(x^{\overline{\alpha}},x^\alpha)$, where the indices $i,j,...$ run from 1 to $2n$, the indices $\overline{\alpha},\overline{\beta},...$ from 1 to n, and the indices $\alpha,\beta,...$ from $n+1$ to $2n$, while x^α are coordinates in M_n and $x^{\overline{\alpha}}=y^\alpha$ are fiber coordinates of the tangent bundle $T(M_n)$ (for definition of the pull-back bundle, see, for example, [1],[3],[4],[5],[6]).

Now let $(T^*(M_n),\widetilde{\pi},M_n)$ be a cotangent bundle with base space M_n and let $T(M_n)$ be a tangent bundle determined by a natural projection (submersion) $\pi_1:T(M_n)\rightarrow M_n$. The semi-cotangent ([8],[9]) bundle (induced or pull-back bundle) of the cotangent bundle $(T^*(M_n),\widetilde{\pi},M_n)$ is the bundle $(t^*(M_n),\pi_2,T(M_n))$ over tangent bundle $T(M_n)$ with a total space

$$\begin{aligned} t^*(M_n) &= \left\{((x^{\overline{\alpha}},x^\alpha),x^{\overline{\overline{\alpha}}})\in T(M_n)\times T_x^*(M_n):\pi_1\left(x^{\overline{\alpha}},x^\alpha\right)=\widetilde{\pi}\left(x^\alpha,x^{\overline{\overline{\alpha}}}\right)=(x^\alpha)\right\}\\ &\subset T(M_n)\times T_x^*(M_n)\end{aligned}$$

and with the projection map $\pi_2:t^*(M_n)\rightarrow T(M_n)$ defined by $\pi_2(x^{\overline{\alpha}},x^\alpha,x^{\overline{\overline{\alpha}}})=\left(x^{\overline{\alpha}},x^\alpha\right)$, where $T_x^*(M_n)$ $\left(x=\pi_1\left(\widetilde{x}\right),\widetilde{x}=\left(x^{\overline{\alpha}},x^\alpha\right)\in T(M_n)\right)$ is the cotangent space at a point x of M_n, where $x^{\overline{\overline{\alpha}}}=p_\alpha$ $(\overline{\overline{\alpha}},\overline{\overline{\beta}},...=2n+1,...,3n)$ are fiber coordinates of the cotangent bundle $T^*(M_n)$. If $(x^{i'})=(x^{\overline{\alpha}'},x^{\alpha'},x^{\overline{\overline{\alpha}}'})$ is another system of local adapted coordinates in the semi-cotangent bundle $t^*(M_n)$, then we have

$$\left\{\begin{array}{l} x^{\overline{\alpha}'}=\frac{\partial x^{\alpha'}}{\partial x^\beta}y^\beta,\\ x^{\alpha'}=x^{\alpha'}\left(x^\beta\right),\\ x^{\overline{\overline{\alpha}}'}=\frac{\partial x^\beta}{\partial x^{\alpha'}}p_\beta.\end{array}\right. \tag{1.1}$$

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The Jacobian of (1.1) has components

$$\overline{A} = (A_J^{I'}) = \begin{pmatrix} A_\beta^{\alpha'} & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & p_\sigma A_\beta^{\beta'} A_{\beta'\alpha'}^{\sigma} & A_{\alpha'}^{\beta} \end{pmatrix}, \tag{1.2}$$

where

$$A_{\beta\varepsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon}, \quad A_{\beta'\alpha'}^{\alpha} = \frac{\partial^2 x^{\alpha}}{\partial x^{\beta'} \partial x^{\alpha'}}.$$

We denote by $\Im_q^p(T(M_n))$ and $\Im_q^p(M_n)$ the modules over $F(T(M_n))$ and $F(M_n)$ of all tensor fields of type (p,q) on $T(M_n)$ and M_n, respectively, where $F(T(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞-functions on $T(M_n)$ and M_n, respectively.

Let θ be a covector field on $T(M_n)$. Then the transformation $p \to \theta_p$, θ_p being the value of θ at $p \in T(M_n)$, determines a cross-section β_θ of a semi-cotangent bundle. Thus, if $\sigma : M_n \to T^*(M_n)$ is a cross-section of $(T^*(M_n), \widetilde{\pi}, M_n)$, such that $\widetilde{\pi} \circ \sigma = I_{(M_n)}$, an associated cross-section $\beta_\theta : T(M_n) \to t^*(M_n)$ of semi-cotangent (pull-back) bundle $(t^*(M_n), \pi_2, T(M_n))$ of cotangent bundle by using projection (submersion) of the tangent bundle $T(M_n)$ defined by [[2], p. 217–218], [[7], p. 301]:

$$\beta_\theta\left(x^{\overline{\alpha}}, x^\alpha\right) = \left(x^{\overline{\alpha}}, x^\alpha, \sigma \circ \pi_1\left(x^{\overline{\alpha}}, x^\alpha\right)\right) = \left(x^{\overline{\alpha}}, x^\alpha, \sigma\left(x^\alpha\right)\right) = \left(x^{\overline{\alpha}}, x^\alpha, \theta_\alpha\left(x^\beta\right)\right).$$

If the covector field θ has the local components $\theta_\alpha\left(x^\beta\right)$, the cross-section $\beta_\theta\left(T(M_n)\right)$ of $t^*(M_n)$ is locally expressed by

$$x^{\overline{\alpha}} = y^\alpha = V^\alpha\left(x^\beta\right), \quad x^\alpha = x^\alpha, \quad x^{\overline{\overline{\alpha}}} = p_\alpha = \theta_\alpha\left(x^\beta\right) \tag{1.3}$$

with respect to the coordinates $x^A = (x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$. $x^{\overline{\alpha}} = y^\alpha$ are considered as parameters. Taking the derivative of (1.3) with respect to $x^{\overline{\alpha}} = y^\alpha$, we have vector fields $B_{(\overline{\beta})}$ $(\overline{\beta} = 1, ..., n)$ with components

$$B_{(\overline{\beta})} = \frac{\partial x^A}{\partial x^{\overline{\beta}}} = \partial_{\overline{\beta}} x^A = \begin{pmatrix} \partial_{\overline{\beta}} V^\alpha \\ \partial_{\overline{\beta}} x^\alpha \\ \partial_{\overline{\beta}} \theta_\alpha \end{pmatrix},$$

which are tangent to the cross-section $\beta_\theta\left(T(M_n)\right)$.

Thus, $B_{(\overline{\beta})}$ have components

$$B_{(\overline{\beta})} : \left(B_{(\overline{\beta})}^A\right) = \begin{pmatrix} \delta_{\overline{\beta}}^{\alpha} \\ 0 \\ 0 \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$, where

$$\delta_{\overline{\beta}}^{\alpha} = A_{\overline{\beta}}^{\alpha} = \frac{\partial x^\alpha}{\partial x^{\overline{\beta}}}.$$

Let $X \in \Im_0^1(T(M_n))$, i.e. $X = X^\alpha \partial_\alpha$. We denote by BX the vector field with local components

$$BX : \left(B^A_{(\overline{\beta})} X^{\overline{\beta}}\right) = \begin{pmatrix} \delta^{\alpha}_{\overline{\beta}} X^{\overline{\beta}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^{\alpha}_{\overline{\beta}} X^{\overline{\beta}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X^\alpha \\ 0 \\ 0 \end{pmatrix} \tag{1.4}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$, which is defined globally along $\beta_\theta(T(M_n))$. Then a mapping

$$B : \Im_0^1(T(M_n)) \to \Im_0^1(\beta_\theta(T(M_n)))$$

is defined by (1.4). The mapping B is the differential of $\beta_\theta : T(M_n) \to t^*(M_n)$ and so an isomorphism of $\Im_0^1(T(M_n))$ onto $\Im_0^1(\beta_\theta(T(M_n)))$.

Since a cross-section is locally expressed by $x^{\overline{\alpha}} = y^\alpha = const.$, $x^{\overline{\overline{\alpha}}} = p_\alpha = const.$, $x^\alpha = x^\alpha$, x^α being considered as parameters. Taking the derivative of (1.3) with respect to x^α, we have vector fields $C_{(\beta)}$ $(\beta = n+1, ..., 2n)$ with components

$$C_{(\beta)} = \frac{\partial x^A}{\partial x^\beta} = \partial_\beta x^A = \begin{pmatrix} \partial_\beta V^\alpha \\ \partial_\beta x^\alpha \\ \partial_\beta \theta_\alpha \end{pmatrix},$$

which are tangent to the cross-section $\beta_\theta(T(M_n))$.

Thus, $C_{(\beta)}$ have components

$$C_{(\beta)} : \left(C^A_{(\beta)}\right) = \begin{pmatrix} \partial_\beta V^\alpha \\ \delta^\alpha_\beta \\ \partial_\beta \theta_\alpha \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$, where

$$\delta^\alpha_\beta = A^\alpha_\beta = \frac{\partial x^\alpha}{\partial x^\beta}.$$

Let $X \in \Im_0^1(T(M_n))$. Then we denote by CX the vector field with local components

$$CX : \left(C^A_{(\beta)} X^\beta\right) = \begin{pmatrix} X^\beta \partial_\beta V^\alpha \\ X^\alpha \\ X^\beta \partial_\beta \theta_\alpha \end{pmatrix} \tag{1.5}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$, which is defined globally along $\beta_\theta(T(M_n))$. Then a mapping

$$C : \Im_0^1(T(M_n)) \to \Im_0^1(\beta_\theta(T(M_n)))$$

is defined by (1.5). The mapping C is the differential of $\beta_\theta : T(M_n) \to t^*(M_n)$ and so an isomorphism of $\Im_0^1(T(M_n))$ onto $\Im_0^1(\beta_\theta(T(M_n)))$.

Now, considering $\omega \in \Im_1^0(M_n)$ and vector field $X \in \Im_0^1(T(M_n))$, then ${}^{vv}\omega$ (vertical lift), ${}^{cc}X$ (complete lift), and ${}^{HH}X$ (horizontal lift) have, respectively, components on the semi-cotangent bundle $t^*(M_n)$ [8]:

$${}^{vv}\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix}, \quad {}^{cc}X = \begin{pmatrix} y^\varepsilon \partial_\varepsilon X^\alpha \\ X^\alpha \\ -p_\sigma(\partial_\alpha X^\sigma) \end{pmatrix}, \quad {}^{HH}X = \begin{pmatrix} -\Gamma^\alpha_\beta X^\beta \\ X^\alpha \\ X^\beta \Gamma_{\beta\alpha} \end{pmatrix} \tag{1.6}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$, where

$$\Gamma^{\alpha}_{\beta} = V^{\varepsilon}\Gamma^{\alpha}_{\varepsilon\,\beta}, \quad \Gamma_{\beta\alpha} = \theta_{\varepsilon}\Gamma^{\varepsilon}_{\beta\alpha}.$$

On the other hand, the fiber is locally represented by

$$x^{\overline{\alpha}} = y^{\alpha} = const., \quad x^{\alpha} = const., \quad x^{\overline{\alpha}} = p_{\alpha} = p_{\alpha},$$

p_{α} being considered as parameters. Thus, on differentiating with respect to p_{α}, we easily see that the vector fields $E_{\left(\overline{\overline{\beta}}\right)} =^{vv} \left(dx^{\beta}\right)$ $(\overline{\overline{\beta}} = 2n+1, ..., 3n)$ with components

$$E_{\left(\overline{\overline{\beta}}\right)} : \left(E^{A}_{\left(\overline{\overline{\beta}}\right)}\right) = \partial_{\left(\overline{\overline{\beta}}\right)} x^{A} = \begin{pmatrix} \partial_{\overline{\overline{\beta}}} y^{\alpha} \\ \partial_{\overline{\overline{\beta}}} x^{\alpha} \\ \partial_{\overline{\overline{\beta}}} p_{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta^{\beta}_{\alpha} \end{pmatrix}$$

are tangent to the fiber, where

$$\delta^{\beta}_{\alpha} = A^{\beta}_{\alpha} = \frac{\partial x^{\beta}}{\partial x^{\alpha}}.$$

Let ω be a 1-form with local components ω_{α} on M_n, so that ω is a 1-form with local expression $\omega = \omega_{\alpha} dx^{\alpha}$. We denote by $E\omega$ the vector field with local components

$$E\omega : \left(E^{A}_{\left(\overline{\overline{\beta}}\right)} \omega_{\beta}\right) = \begin{pmatrix} 0 \\ 0 \\ \omega_{\alpha} \end{pmatrix}, \tag{1.7}$$

which is tangent to the fiber. Then a mapping

$$E : \Im^{0}_{1}(M_n) \to \Im^{1}_{0}(t^{*}(M_n))$$

is defined by (1.7) and so an isomorphism of $\Im^{0}_{1}(M_n)$ in to $\Im^{1}_{0}(t^{*}(M_n))$.

From (1.4), (1.5), and (1.7), we obtain:

Theorem 1 *Let X and Y be vector fields on $T(M_n)$. For the Lie product, we have*

(i) $[CX, CY] = C[X, Y]$,

(ii) $[BX, BY] = 0$,

(iii) $[E\psi, E\omega] = 0$,

for any $\psi, \omega \in \Im^{0}_{1}(M_n)$.

Proof

(i) If X and Y are vector fields on $T(M_n)$ and $\begin{pmatrix} [CX,CY]^{\overline{\beta}} \\ [CX,CY]^{\beta} \\ [CX,CY]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[CX,CY]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t^*(M_n)$, then we have

$$[CX,CY]^J = (CX)^I\,\partial_I(CY)^J - (CY)^I\partial_I(CX)^J.$$

First, if $J = \overline{\beta}$, we have

$$\begin{aligned}
[CX,CY]^{\overline{\beta}} &= (CX)^I\,\partial_I(CY)^{\overline{\beta}} - (CY)^I\partial_I(CX)^{\overline{\beta}} \\
&= (CX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(CY)^{\overline{\beta}} + (CX)^{\alpha}\,\partial_{\alpha}(CY)^{\overline{\beta}} + (CX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(CY)^{\overline{\beta}} \\
&\quad -(CY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(CX)^{\overline{\beta}} - (CY)^{\alpha}\partial_{\alpha}(CX)^{\overline{\beta}} - (CY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(CX)^{\overline{\beta}} \\
&= X^{\beta}\partial_{\beta}V^{\alpha}\partial_{\overline{\alpha}}Y^{\gamma}\partial_{\gamma}V^{\beta} + X^{\alpha}\partial_{\alpha}Y^{\gamma}\partial_{\gamma}V^{\beta} \\
&\quad -Y^{\beta}\partial_{\beta}V^{\alpha}\partial_{\overline{\alpha}}X^{\gamma}\partial_{\gamma}V^{\beta} - Y^{\alpha}\partial_{\alpha}X^{\gamma}\partial_{\gamma}V^{\beta} \\
&= (X^{\alpha}\partial_{\alpha}Y^{\gamma} - Y^{\alpha}\partial_{\alpha}X^{\gamma})\,\partial_{\gamma}V^{\beta} \\
&= [X,Y]^{\gamma}\,\partial_{\gamma}V^{\beta}
\end{aligned}$$

by virtue of (1.5). Second, if $J = \beta$, we have

$$\begin{aligned}
[CX,CY]^{\beta} &= (CX)^I\,\partial_I(CY)^{\beta} - (CY)^I\partial_I(CX)^{\beta} \\
&= (CX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(CY)^{\beta} + (CX)^{\alpha}\,\partial_{\alpha}(CY)^{\beta} + (CX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(CY)^{\beta} \\
&\quad -(CY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(CX)^{\beta} - (CY)^{\alpha}\partial_{\alpha}(CX)^{\beta} - (CY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(CX)^{\beta} \\
&= X^{\overline{\alpha}}\partial_{\overline{\alpha}}Y^{\beta} + X^{\alpha}\partial_{\alpha}Y^{\beta} + X^{\beta}\partial_{\beta}\theta_{\alpha}\partial_{\overline{\overline{\alpha}}}Y^{\beta} \\
&\quad -Y^{\overline{\alpha}}\partial_{\overline{\alpha}}X^{\beta} - Y^{\alpha}\partial_{\alpha}X^{\beta} - Y^{\beta}\partial_{\beta}\theta_{\alpha}\partial_{\overline{\overline{\alpha}}}X^{\beta} \\
&= X^{\alpha}\partial_{\alpha}Y^{\beta} - Y^{\alpha}\partial_{\alpha}X^{\beta} \\
&= [X,Y]^{\beta}
\end{aligned}$$

by virtue of (1.5). Third, if $J = \overline{\overline{\beta}}$, then we have

$$\begin{aligned}
[CX,CY]^{\overline{\overline{\beta}}} &= (CX)^I\,\partial_I(CY)^{\overline{\overline{\beta}}} - (CY)^I\partial_I(CX)^{\overline{\overline{\beta}}} \\
&= (CX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(CY)^{\overline{\overline{\beta}}} + (CX)^{\alpha}\,\partial_{\alpha}(CY)^{\overline{\overline{\beta}}} + (CX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(CY)^{\overline{\overline{\beta}}} \\
&\quad -(CY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(CX)^{\overline{\overline{\beta}}} - (CY)^{\alpha}\partial_{\alpha}(CX)^{\overline{\overline{\beta}}} - (CY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(CX)^{\overline{\overline{\beta}}} \\
&= X^{\overline{\alpha}}\partial_{\overline{\alpha}}Y^{\gamma}\partial_{\gamma}\theta_{\beta} + X^{\alpha}\partial_{\alpha}Y^{\gamma}\partial_{\gamma}\theta_{\beta} + X^{\beta}\partial_{\beta}\theta_{\alpha}\partial_{\overline{\overline{\alpha}}}Y^{\gamma}\partial_{\gamma}\theta_{\beta} \\
&\quad -Y^{\overline{\alpha}}\partial_{\overline{\alpha}}X^{\gamma}\partial_{\gamma}\theta_{\beta} - Y^{\alpha}\partial_{\alpha}X^{\gamma}\partial_{\gamma}\theta_{\beta} - Y^{\beta}\partial_{\beta}\theta_{\alpha}\partial_{\overline{\overline{\alpha}}}X^{\gamma}\partial_{\gamma}\theta_{\beta}
\end{aligned}$$

$$
\begin{aligned}
&= X^{\alpha}\partial_{\alpha}Y^{\gamma}\partial_{\gamma}\theta_{\beta} - Y^{\alpha}\partial_{\alpha}X^{\gamma}\partial_{\gamma}\theta_{\beta} \\
&= \left(X^{\alpha}\partial_{\alpha}Y^{\gamma} - Y^{\alpha}\partial_{\alpha}X^{\gamma}\right)\partial_{\gamma}\theta_{\beta} \\
&= [X,Y]^{\gamma}\,\partial_{\gamma}\theta_{\beta}
\end{aligned}
$$

by virtue of (1.5). On the other hand, we know that $C[X,Y]$ have components

$$
C[X,Y] = \begin{pmatrix} [X,Y]^{\gamma}\,\partial_{\gamma}V^{\beta} \\ [X,Y]^{\beta} \\ [X,Y]^{\gamma}\,\partial_{\gamma}\theta_{\beta} \end{pmatrix}
$$

with respect to the coordinates in $t^{*}(M_n)$. Thus, we have $[CX,CY] = C[X,Y]$.

(ii) $X,Y \in \Im_0^1(T(M_n))$ and $\begin{pmatrix} [BX,BY]^{\overline{\beta}} \\ [BX,BY]^{\beta} \\ [BX,BY]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[BX,BY]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t^{*}(M_n)$, and then we have

$$
[BX,BY]^{J} = (BX)^{I}\,\partial_I(BY)^{J} - (BY)^{I}\partial_I(BX)^{J}.
$$

First, if $J=\overline{\beta}$, we have

$$
\begin{aligned}
[BX,BY]^{\overline{\beta}} &= (BX)^{I}\,\partial_I(BY)^{\overline{\beta}} - (BY)^{I}\partial_I(BX)^{\overline{\beta}} \\
&= (BX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(BY)^{\overline{\beta}} + (BX)^{\alpha}\,\partial_{\alpha}(BY)^{\overline{\beta}} + (BX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(BY)^{\overline{\beta}} \\
&\quad -(BY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(BX)^{\overline{\beta}} - (BY)^{\alpha}\partial_{\alpha}(BX)^{\overline{\beta}} - (BY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(BX)^{\overline{\beta}} \\
&= X^{\alpha}\partial_{\overline{\alpha}}Y^{\beta} - Y^{\alpha}\partial_{\overline{\alpha}}X^{\beta} \\
&= 0
\end{aligned}
$$

by virtue of (1.4). Second, if $J=\beta$, we have

$$
\begin{aligned}
[BX,BY]^{\beta} &= (BX)^{I}\,\partial_I(BY)^{\beta} - (BY)^{I}\partial_I(BX)^{\beta} \\
&= (BX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(BY)^{\beta} + (BX)^{\alpha}\,\partial_{\alpha}(BY)^{\beta} + (BX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(BY)^{\beta} \\
&\quad -(BY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(BX)^{\beta} - (BY)^{\alpha}\partial_{\alpha}(BX)^{\beta} - (BY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(BX)^{\beta} \\
&= 0
\end{aligned}
$$

by virtue of (1.4). Third, if $J=\overline{\overline{\beta}}$, then we have

$$
\begin{aligned}
[BX,BY]^{\overline{\overline{\beta}}} &= (BX)^{I}\,\partial_I(BY)^{\overline{\overline{\beta}}} - (BY)^{I}\partial_I(BX)^{\overline{\overline{\beta}}} \\
&= (BX)^{\overline{\alpha}}\,\partial_{\overline{\alpha}}(BY)^{\overline{\overline{\beta}}} + (BX)^{\alpha}\,\partial_{\alpha}(BY)^{\overline{\overline{\beta}}} + (BX)^{\overline{\overline{\alpha}}}\,\partial_{\overline{\overline{\alpha}}}(BY)^{\overline{\overline{\beta}}} \\
&\quad -(BY)^{\overline{\alpha}}\partial_{\overline{\alpha}}(BX)^{\overline{\overline{\beta}}} - (BY)^{\alpha}\partial_{\alpha}(BX)^{\overline{\overline{\beta}}} - (BY)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}(BX)^{\overline{\overline{\beta}}} \\
&= 0
\end{aligned}
$$

by virtue of (1.4). Thus, we have $[BX,BY]=0$.

(iii) If $\psi, \omega \in \Im_1^0(M_n)$ and $\begin{pmatrix} [E\psi, E\omega]^{\overline{\beta}} \\ [E\psi, E\omega]^{\beta} \\ [E\psi, E\omega]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[E\psi, E\omega]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t^*(M_n)$, then we have

$$\begin{aligned} [E\psi, E\omega]^J &= (E\psi)^I \partial_I (E\omega)^J - (E\omega)^I \partial_I (E\psi)^J \\ &= (E\psi)^{\overline{\alpha}} \partial_{\overline{\alpha}} (E\omega)^J + (E\psi)^{\alpha} \partial_{\alpha} (E\omega)^J + (E\psi)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (E\omega)^J \\ &\quad - (E\omega)^{\overline{\alpha}} \partial_{\overline{\alpha}} (E\psi)^J - (E\omega)^{\alpha} \partial_{\alpha} (E\psi)^J - (E\omega)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (E\psi)^J \\ &= \psi_\alpha \partial_{\overline{\overline{\alpha}}} (E\omega)^J - \omega_\alpha \partial_{\overline{\overline{\alpha}}} (E\psi)^J. \end{aligned}$$

First, if $J = \overline{\beta}$, we have

$$\begin{aligned} [E\psi, E\omega]^{\overline{\beta}} &= \psi_\alpha \partial_{\overline{\overline{\alpha}}} (E\omega)^{\overline{\beta}} - \omega_\alpha \partial_{\overline{\overline{\alpha}}} (E\psi)^{\overline{\beta}} \\ &= 0 \end{aligned}$$

by virtue of (1.7). Second, if $J = \beta$, we have

$$\begin{aligned} [E\psi, E\omega]^{\beta} &= \psi_\alpha \partial_{\overline{\overline{\alpha}}} (E\omega)^{\beta} - \omega_\alpha \partial_{\overline{\overline{\alpha}}} (E\psi)^{\beta} \\ &= 0 \end{aligned}$$

by virtue of (1.7). Third, if $J = \overline{\overline{\beta}}$, then we have

$$\begin{aligned} [E\psi, E\omega]^{\overline{\overline{\beta}}} &= \psi_\alpha \partial_{\overline{\overline{\alpha}}} (E\omega)^{\overline{\overline{\beta}}} - \omega_\alpha \partial_{\overline{\overline{\alpha}}} (E\psi)^{\overline{\overline{\beta}}} \\ &= \psi_\alpha \partial_{\overline{\overline{\alpha}}} \omega_\beta - \omega_\alpha \partial_{\overline{\overline{\alpha}}} \psi_\beta \\ &= 0 \end{aligned}$$

by virtue of (1.7). Thus, we have $[E\psi, E\omega] = 0$.

□

We consider in $\pi^{-1}(U)$ $3n$ local vector fields $B_{(\overline{\beta})}$, $C_{(\beta)}$, and $E_{(\overline{\overline{\beta}})}$ along $\beta_\theta(T(M_n))$, which are respectively represented by

$$B_{(\overline{\beta})} = B\frac{\partial}{\partial x^{\overline{\beta}}}, \quad C_{(\beta)} = C\frac{\partial}{\partial x^{\beta}}, \quad E_{(\overline{\overline{\beta}})} = Edx^{\beta}.$$

Theorem 2 *Let X be a vector field on $T(M_n)$. We have along $\beta_\theta(T(M_n))$ the formula*

$$^{cc}X = CX + B(L_V X) + E(-L_X\theta),$$

where $L_V X$ denotes the Lie derivative of X with respect to V, and $L_X\theta$ denotes the Lie derivative of θ with respect to X.

Proof Using (1.4), (1.5), and (1.7), we have

$$\begin{aligned} CX + B\left(L_V X\right) + E\left(-L_X\theta\right) &= \begin{pmatrix} X^\beta\partial_\beta V^\alpha \\ X^\alpha \\ X^\beta\partial_\beta\theta_\alpha \end{pmatrix} + \begin{pmatrix} V^\beta\partial_\beta X^\alpha - X^\beta\partial_\beta V^\alpha \\ 0 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ -X^\beta\partial_\beta\theta_\alpha - \theta_\beta\partial_\alpha X^\beta \end{pmatrix} \\ &= \begin{pmatrix} V^\beta\partial_\beta X^\alpha \\ X^\alpha \\ -\theta_\beta\partial_\alpha X^\beta \end{pmatrix} =^{cc} X. \end{aligned}$$

Thus, we have Theorem 2. □

On the other hand, on putting $C_{\left(\overline{\overline{\beta}}\right)} = E_{\left(\overline{\overline{\beta}}\right)}$, we write the adapted frame of $\beta_\theta\left(T(M_n)\right)$ as $\left\{B_{\left(\overline{\beta}\right)}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$. The adapted frame $\left\{B_{\left(\overline{\beta}\right)}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of $\beta_\theta\left(T(M_n)\right)$ is given by the matrix

$$\widetilde{A} = \left(\widetilde{A}_B^A\right) = \begin{pmatrix} \delta_\beta^\alpha & \partial_\beta V^\alpha & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & \partial_\beta\theta_\alpha & \delta_\alpha^\beta \end{pmatrix}. \tag{1.8}$$

Since the matrix $\widetilde{A}$ in (1.8) is nonsingular, it has the inverse. Denoting this inverse by $\left(\widetilde{A}\right)^{-1}$, we have

$$\left(\widetilde{A}\right)^{-1} = \left(\widetilde{A}_C^B\right)^{-1} = \begin{pmatrix} \delta_\theta^\beta & -\partial_\theta V^\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\partial_\theta\theta_\beta & \delta_\beta^\theta \end{pmatrix}, \tag{1.9}$$

where $\widetilde{A}\left(\widetilde{A}\right)^{-1} = (\widetilde{A}_B^A)\left(\widetilde{A}_C^B\right)^{-1} = \delta_C^A = \widetilde{I}$, where $A = \left(\overline{\alpha}, \alpha, \overline{\overline{\alpha}}\right)$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = \left(\overline{\theta}, \theta, \overline{\overline{\theta}}\right)$.

Proof From (1.8) and (1.9), we easily see that

$$\begin{aligned} \widetilde{A}\left(\widetilde{A}\right)^{-1} &= (\widetilde{A}_B^A)\left(\widetilde{A}_C^B\right)^{-1} = \begin{pmatrix} \delta_\beta^\alpha & \partial_\beta V^\alpha & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & \partial_\beta\theta_\alpha & \delta_\alpha^\beta \end{pmatrix} \begin{pmatrix} \delta_\theta^\beta & -\partial_\theta V^\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\partial_\theta\theta_\beta & \delta_\beta^\theta \end{pmatrix} \\ &= \begin{pmatrix} \delta_\theta^\alpha & -\partial_\theta V^\alpha + \partial_\theta V^\alpha & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & \partial_\theta\theta_\alpha - \partial_\theta\theta_\alpha & \delta_\alpha^\theta \end{pmatrix} \\ &= \begin{pmatrix} \delta_\theta^\alpha & 0 & 0 \\ 0 & \delta_\theta^\alpha & 0 \\ 0 & 0 & \delta_\alpha^\theta \end{pmatrix} = \delta_C^A = \widetilde{I}. \end{aligned}$$

□

Then we see from Theorem 2 that the complete lift ${}^{cc}X$ of a vector field $X \in \Im_0^1(T(M_n))$ has along $\beta_\theta\left(T(M_n)\right)$ components of the form

$$ {}^{cc}X : \begin{pmatrix} L_V X^\alpha \\ X^\alpha \\ -L_X \theta_\alpha \end{pmatrix} $$

with respect to the adapted frame $\left\{ B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)} \right\}$.

BX, CX, and $E\omega$ also have components

$$ BX = \begin{pmatrix} X^\alpha \\ 0 \\ 0 \end{pmatrix}, \quad CX = \begin{pmatrix} 0 \\ X^\alpha \\ 0 \end{pmatrix}, \quad E\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix}, \tag{1.10} $$

respectively, with respect to the adapted frame $\left\{ B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)} \right\}$ of the cross-section $\beta_\theta\left(T(M_n)\right)$ determined by a 1-form θ on $T(M_n)$.

2. Complete lift of tensor fields of type (1,1) on a cross-section in a semi-cotangent bundle

Suppose now that $F \in \Im_1^1(T(M_n))$ and F has local components F_β^α in a neighborhood U of M_n, $F = F_\beta^\alpha \partial_\alpha \otimes dx^\beta$. Then the semi-cotangent (pull-back) bundle $t^*(M_n)$ of cotangent bundle $T^*(M_n)$ by using projection of the tangent bundle $T(M_n)$ admits the complete lift ${}^{cc}F$ of F with components [8]

$$ {}^{cc}F = ({}^{cc}F_J^I) = \begin{pmatrix} F_\beta^\alpha & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & F_\alpha^\beta \end{pmatrix}, \tag{2.1} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ on $t^*(M_n)$. Then ${}^{cc}F$ has components F_B^A given by

$$ {}^{cc}F = ({}^{cc}F_B^A) = \begin{pmatrix} F_\beta^\alpha & L_V F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & \phi_F \theta & F_\alpha^\beta \end{pmatrix} \tag{2.2} $$

with respect to the adapted frame $\left\{ B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)} \right\}$ of the cross-section $\beta_\theta\left(T(M_n)\right)$ determined by a 1-form θ in $T(M_n)$, where $A = \left(\overline{\alpha}, \alpha, \overline{\overline{\alpha}}\right)$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$. Also, the component ${}^{cc}F_\beta^{\overline{\overline{\alpha}}}$ of ${}^{cc}F_B^A$ is defined as the Tachibana operator $\phi_F \theta$ of F, i.e.

$$ {}^{cc}F_\beta^{\overline{\overline{\alpha}}} = \phi_F \theta = (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma, $$

and $L_V F_\beta^\alpha$ *denotes the Lie derivative of* F_β^α *with respect to* V, i.e.

$$ L_V F_\beta^\alpha = V^\gamma \partial_\gamma F_\beta^\alpha + F_\gamma^\alpha \partial_\beta V^\gamma - F_\beta^\gamma \partial_\gamma V^\alpha. $$

Proof Let $F \in \Im_1^1(T(M_n))$. Then we have by (1.8), (1.9), and (2.1):

$$\begin{aligned}
{}^{cc}F &= \left(\widetilde{A}_A^B\right)^{-1}\left({}^{cc}F_C^A\right)\left(\widetilde{A}_D^C\right) \\
&= \begin{pmatrix} \delta_\alpha^\beta & -\partial_\alpha V^\beta & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & -\partial_\alpha \theta_\beta & \delta_\beta^\alpha \end{pmatrix} \begin{pmatrix} F_\gamma^\alpha & V^\varepsilon \partial_\varepsilon F_\gamma^\alpha & 0 \\ 0 & F_\gamma^\alpha & 0 \\ 0 & \theta_\sigma(\partial_\gamma F_\alpha^\sigma - \partial_\alpha F_\gamma^\sigma) & F_\alpha^\gamma \end{pmatrix} \begin{pmatrix} \delta_\psi^\gamma & \partial_\psi V^\gamma & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \partial_\psi \theta_\gamma & \delta_\gamma^\psi \end{pmatrix} \\
&= \begin{pmatrix} F_\gamma^\beta & V^\varepsilon \partial_\varepsilon F_\gamma^\beta - F_\gamma^\alpha \partial_\alpha V^\beta & 0 \\ 0 & F_\gamma^\beta & 0 \\ 0 & -F_\gamma^\alpha \partial_\alpha \theta_\beta + \theta_\sigma \partial_\gamma F_\beta^\sigma - \theta_\sigma \partial_\beta F_\gamma^\sigma & F_\beta^\gamma \end{pmatrix} \begin{pmatrix} \delta_\psi^\gamma & \partial_\psi V^\gamma & 0 \\ 0 & \delta_\psi^\gamma & 0 \\ 0 & \partial_\psi \theta_\gamma & \delta_\gamma^\psi \end{pmatrix} \\
&= \begin{pmatrix} F_\psi^\beta & F_\gamma^\beta \partial_\psi V^\gamma + V^\varepsilon \partial_\varepsilon F_\psi^\beta - F_\psi^\alpha \partial_\alpha V^\beta & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & -F_\psi^\alpha \partial_\alpha \theta_\beta + \theta_\sigma \partial_\psi F_\beta^\sigma - \theta_\sigma \partial_\beta F_\psi^\sigma + F_\beta^\gamma \partial_\psi \theta_\gamma & F_\beta^\psi \end{pmatrix} \\
&= \begin{pmatrix} F_\psi^\beta & L_V F_\psi^\beta & 0 \\ 0 & F_\psi^\beta & 0 \\ 0 & \varphi_F \theta & F_\beta^\psi \end{pmatrix} = \left({}^{cc}F_D^B\right),
\end{aligned}$$

where $A = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = (\overline{\gamma}, \gamma, \overline{\overline{\gamma}})$, $D = \left(\overline{\psi}, \psi, \overline{\overline{\psi}}\right)$. □

Using (2.2), we have along $\beta_\theta(T(M_n))$:

Theorem 3 *If F and X are affinor and vector fields on $T(M_n)$, and $\omega \in \Im_1^0(M_n)$, then:*

(i) ${}^{cc}F(BX + CX) = B(FX) + C(FX) + B((L_V F)X) + E(P_X)$,

(ii) ${}^{cc}F(E\omega) = E(\omega \circ F)$,

where $P \in \Im_2^0(M_n)$ with local components

$$P_{\beta\alpha} = \phi_F \theta = (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma,$$

θ_β being local components of θ, and $P_X \in \Im_1^0(M_n)$ defined by $P_X(Y) = P(X,Y)$, for $Y \in \Im_0^1(T(M_n))$.

Proof (i) *If F and X are affinor and vector fields on $T(M_n)$,* then by (1.10) and (2.2), we have

$$\begin{aligned}
{}^{cc}F\,(BX+CX) &= \begin{pmatrix} F^{\alpha}_{\beta} & L_V F^{\alpha}_{\beta} & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & \varphi_F\theta & F^{\beta}_{\alpha} \end{pmatrix} \begin{pmatrix} X^{\beta} \\ X^{\beta} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} F^{\alpha}_{\beta} X^{\beta} + L_V F^{\alpha}_{\beta} X^{\beta} \\ F^{\alpha}_{\beta} X^{\beta} \\ X^{\beta}\partial_{\beta}F^{\sigma}_{\alpha}\theta_{\sigma} - X^{\beta}\partial_{\alpha}F^{\sigma}_{\beta}\theta_{\sigma} - F^{\gamma}_{\beta}X^{\beta}\partial_{\gamma}\theta_{\alpha} + F^{\gamma}_{\alpha}X^{\beta}\partial_{\beta}\theta_{\gamma} \end{pmatrix} \\
&= \begin{pmatrix} (FX)^{\alpha} + V^{\gamma}\partial_{\gamma}F^{\alpha}_{\beta}X^{\beta} + F^{\alpha}_{\gamma}\partial_{\beta}V^{\gamma}X^{\beta} - F^{\gamma}_{\beta}\partial_{\gamma}V^{\alpha}X^{\beta} \\ (FX)^{\alpha} \\ X^{\beta}\partial_{\beta}F^{\sigma}_{\alpha}\theta_{\sigma} - X^{\beta}\partial_{\alpha}F^{\sigma}_{\beta}\theta_{\sigma} - F^{\gamma}_{\beta}X^{\beta}\partial_{\gamma}\theta_{\alpha} + F^{\gamma}_{\alpha}X^{\beta}\partial_{\beta}\theta_{\gamma} \end{pmatrix} \\
&= \begin{pmatrix} (FX)^{\alpha} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (FX)^{\alpha} \\ 0 \end{pmatrix} + \begin{pmatrix} (L_V F)\,X \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ P_X \end{pmatrix} \\
&= B\,(FX) + C\,(FX) + B\,((L_V F)\,X) + E\,(P_X)\,.
\end{aligned}$$

Thus, we have ${}^{cc}F\,(BX+CX) = B\,(FX) + C\,(FX) + B\,((L_V F)\,X) + E\,(P_X)\,.$

(ii) If $\omega \in \Im^0_1(M_n)$, F is an affinor field on $T(M_n)$,and then by (1.10) and (2.2), we have

$${}^{cc}F\,(E\omega) = \begin{pmatrix} F^{\alpha}_{\beta} & L_V F^{\alpha}_{\beta} & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & \varphi_F\theta & F^{\beta}_{\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega_{\beta}F^{\beta}_{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (\omega\circ F)_{\alpha} \end{pmatrix} = E\,(\omega\circ F)\,,$$

which gives equation (ii) of Theorem 3. □

When ${}^{cc}F\,(BX+CX)$ is always tangent to $\beta_\theta\,(T(M_n))$ for any vector field $X \in \Im^1_0\,(T(M_n))$, ${}^{cc}F$ is said to leave the cross-section $\beta_\theta\,(T(M_n))$ invariant.

Thus, we have:

Theorem 4 *The complete lift ${}^{cc}F$ of an element of $F \in \Im^1_1(T(M_n))$ leaves the cross-section $\beta_\theta\,(T(M_n))$ invariant if and only if:*

(i) $(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta})\theta_{\sigma} - F^{\gamma}_{\beta}\partial_{\gamma}\theta_{\alpha} + F^{\gamma}_{\alpha}\partial_{\beta}\theta_{\gamma} = 0$ $(i.e.\phi_F\theta = 0)$,

(ii) $V^{\gamma}\partial_{\gamma}F^{\alpha}_{\beta} + F^{\alpha}_{\gamma}\partial_{\beta}V^{\gamma} - F^{\gamma}_{\beta}\partial_{\gamma}V^{\alpha} = 0$ $(i.e.L_V F = 0)$,

where F^{α}_{β}, θ_{β}, and V^{α} are local components of F, θ, and V, respectively.

3. Adapted frames and diagonal lifts of affinor fields

Let ∇ be a symmetric affine connection in M_n. In each coordinate neighborhood $\{U, x^{\alpha}\}$ of M_n, we put

$$X_{(\alpha)} = \frac{\partial}{\partial x^{\alpha}}, \quad \theta^{(\alpha)} = dx^{\alpha}.$$

Then $3n$ local vector fields $Y_{(\alpha)}$, ${}^{HH}X_{(\alpha)}$, and ${}^{vv}\theta^{(\alpha)}$ have respectively components of the form

$$Y_{(\alpha)}:\begin{pmatrix}\delta_\alpha^\beta\\0\\0\end{pmatrix},\quad {}^{HH}X_{(\alpha)}:\begin{pmatrix}-\Gamma_\beta^\alpha\\\delta_\alpha^\beta\\\Gamma_{\beta\alpha}\end{pmatrix},\quad {}^{vv}\theta^{(\alpha)}:\begin{pmatrix}0\\0\\\delta_\beta^\alpha\end{pmatrix} \tag{3.1}$$

with respect to the induced coordinates $(x^{\overline{\alpha}},x^{\alpha},x^{\overline{\overline{\alpha}}})$ in $\pi^{-1}(U)$, where we have used (1.6). We call the set $\{Y_{(\alpha)},{}^{HH}X_{(\alpha)},{}^{vv}\theta^{(\alpha)}\}$ the frame adapted to the symmetric affine connection ∇ in $\pi^{-1}(U)$. On putting

$$\widehat{e}_{(\overline{\alpha})}=Y_{(\alpha)},\quad \widehat{e}_{(\alpha)}={}^{HH}X_{(\alpha)},\quad \widehat{e}_{(\overline{\overline{\alpha}})}={}^{vv}\theta^{(\alpha)} \tag{3.2}$$

we write the adapted frame as

$$\{\widehat{e}_{(B)}\}=\left\{\widehat{e}_{(\overline{\alpha})},\widehat{e}_{(\alpha)},\widehat{e}_{(\overline{\overline{\alpha}})}\right\}. \tag{3.3}$$

The adapted frame $\{\widehat{e}_{(B)}\}=\left\{\widehat{e}_{(\overline{\alpha})},\widehat{e}_{(\alpha)},\widehat{e}_{(\overline{\overline{\alpha}})}\right\}$ is given by the matrix

$$\widehat{A}=\left(\widehat{A}_B^A\right)=\begin{pmatrix}\delta_\beta^\alpha & -\Gamma_\beta^\alpha & 0\\ 0 & \delta_\beta^\alpha & 0\\ 0 & \Gamma_{\beta\alpha} & \delta_\alpha^\beta\end{pmatrix}. \tag{3.4}$$

Since the matrix $\widehat{A}$ in (3.4) is nonsingular, it has the inverse. Denoting this inverse by $\left(\widehat{A}\right)^{-1}$, we have

$$\left(\widehat{A}\right)^{-1}=\left(\widehat{A}_C^B\right)^{-1}=\begin{pmatrix}\delta_\theta^\beta & \Gamma_\theta^\beta & 0\\ 0 & \delta_\theta^\beta & 0\\ 0 & -\Gamma_{\theta\beta} & \delta_\beta^\theta\end{pmatrix}, \tag{3.5}$$

where $\widehat{A}\left(\widehat{A}\right)^{-1}=(\widehat{A}_B^A)\left(\widehat{A}_C^B\right)^{-1}=\delta_C^A=\widetilde{I}$, where $A=(\overline{\alpha},\alpha,\overline{\overline{\alpha}})$, $B=\left(\overline{\beta},\beta,\overline{\overline{\beta}}\right)$, $C=\left(\overline{\theta},\theta,\overline{\overline{\theta}}\right)$.

Proof From (3.4) and (3.5), we easily see that

$$\begin{aligned}\widehat{A}\left(\widehat{A}\right)^{-1} &= (\widehat{A}_B^A)\left(\widehat{A}_C^B\right)^{-1}=\begin{pmatrix}\delta_\beta^\alpha & -\Gamma_\beta^\alpha & 0\\ 0 & \delta_\beta^\alpha & 0\\ 0 & \Gamma_{\beta\alpha} & \delta_\alpha^\beta\end{pmatrix}\begin{pmatrix}\delta_\theta^\beta & \Gamma_\theta^\beta & 0\\ 0 & \delta_\theta^\beta & 0\\ 0 & -\Gamma_{\theta\beta} & \delta_\beta^\theta\end{pmatrix}\\ &= \begin{pmatrix}\delta_\theta^\alpha & \Gamma_\theta^\alpha-\Gamma_\theta^\alpha & 0\\ 0 & \delta_\theta^\alpha & 0\\ 0 & \Gamma_{\theta\alpha}-\Gamma_{\theta\alpha} & \delta_\alpha^\theta\end{pmatrix}=\begin{pmatrix}\delta_\theta^\alpha & 0 & 0\\ 0 & \delta_\theta^\alpha & 0\\ 0 & 0 & \delta_\alpha^\theta\end{pmatrix}\\ &= \delta_C^A=\widehat{I}.\end{aligned}$$

□

If we take account of (3.3), we see that the diagonal lift ${}^{DD}F$ of $F\in\Im_1^1(T(M_n))$ has components

$${}^{DD}F=({}^{DD}F_J^I)=\begin{pmatrix}-F_\beta^\alpha & -\Gamma_\varepsilon^\alpha F_\beta^\varepsilon-\Gamma_\beta^\varepsilon F_\varepsilon^\alpha & 0\\ 0 & F_\beta^\alpha & 0\\ 0 & \Gamma_{\beta\sigma}F_\alpha^\sigma+\Gamma_{\alpha\sigma}F_\beta^\sigma & -F_\alpha^\beta\end{pmatrix}, \tag{3.6}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ on $t^*(M_n)$, where

$$\Gamma^{\alpha}_{\varepsilon} = y^{\gamma}\Gamma^{\alpha}_{\gamma\varepsilon}, \quad \Gamma_{\alpha\sigma} = p_{\gamma}\Gamma^{\gamma}_{\alpha\sigma}.$$

Proof Let $F \in \Im^1_1(T(M_n))$. Then we have by (3.4), (3.5), and (3.6):

$$\begin{aligned}
{}^{DD}F &= \left(\widehat{A}\right)\left({}^{DD}F\right)\left(\widehat{A}\right)^{-1} \\
&= \begin{pmatrix} \delta^{\beta}_{\alpha} & -\Gamma^{\beta}_{\alpha} & 0 \\ 0 & \delta^{\beta}_{\alpha} & 0 \\ 0 & \Gamma_{\alpha\beta} & \delta^{\alpha}_{\beta} \end{pmatrix} \begin{pmatrix} -F^{\alpha}_{\gamma} & -\Gamma^{\alpha}_{\varepsilon}F^{\varepsilon}_{\gamma} - \Gamma^{\varepsilon}_{\gamma}F^{\alpha}_{\varepsilon} & 0 \\ 0 & F^{\alpha}_{\gamma} & 0 \\ 0 & \Gamma_{\gamma\sigma}F^{\sigma}_{\alpha} + \Gamma_{\alpha\sigma}F^{\sigma}_{\gamma} & -F^{\gamma}_{\alpha} \end{pmatrix} \begin{pmatrix} \delta^{\gamma}_{\psi} & \Gamma^{\gamma}_{\psi} & 0 \\ 0 & \delta^{\gamma}_{\psi} & 0 \\ 0 & -\Gamma_{\psi\gamma} & \delta^{\psi}_{\gamma} \end{pmatrix} \\
&= \begin{pmatrix} -F^{\beta}_{\gamma} & -\Gamma^{\beta}_{\varepsilon}F^{\varepsilon}_{\gamma} - \Gamma^{\varepsilon}_{\gamma}F^{\beta}_{\varepsilon} - \Gamma^{\beta}_{\alpha}F^{\alpha}_{\gamma} & 0 \\ 0 & F^{\beta}_{\gamma} & 0 \\ 0 & \Gamma_{\alpha\beta}F^{\alpha}_{\gamma} + \Gamma_{\gamma\sigma}F^{\sigma}_{\beta} + \Gamma_{\beta\sigma}F^{\sigma}_{\gamma} & -F^{\gamma}_{\beta} \end{pmatrix} \begin{pmatrix} \delta^{\gamma}_{\psi} & \Gamma^{\gamma}_{\psi} & 0 \\ 0 & \delta^{\gamma}_{\psi} & 0 \\ 0 & -\Gamma_{\psi\gamma} & \delta^{\psi}_{\gamma} \end{pmatrix} \\
&= \begin{pmatrix} -F^{\beta}_{\psi} & -\Gamma^{\gamma}_{\psi}F^{\beta}_{\gamma} - \Gamma^{\beta}_{\varepsilon}F^{\varepsilon}_{\psi} - \Gamma^{\varepsilon}_{\psi}F^{\beta}_{\varepsilon} - \Gamma^{\beta}_{\alpha}F^{\alpha}_{\psi} & 0 \\ 0 & F^{\beta}_{\psi} & 0 \\ 0 & \Gamma_{\alpha\beta}F^{\alpha}_{\psi} + \Gamma_{\psi\sigma}F^{\sigma}_{\beta} + \Gamma_{\beta\sigma}F^{\sigma}_{\psi} + \Gamma_{\psi\gamma}F^{\gamma}_{\beta} & -F^{\psi}_{\beta} \end{pmatrix} \\
&= \begin{pmatrix} -F^{\beta}_{\psi} & -\Gamma^{\beta}_{\rho}F^{\rho}_{\psi} - \Gamma^{\rho}_{\psi}F^{\beta}_{\rho} & 0 \\ 0 & F^{\beta}_{\psi} & 0 \\ 0 & \Gamma_{\psi\mu}F^{\mu}_{\beta} + \Gamma_{\beta\mu}F^{\mu}_{\psi} & -F^{\psi}_{\beta} \end{pmatrix},
\end{aligned}$$

which proves (3.6). □

We now see, from (3.3), that the diagonal lift ${}^{DD}F$ of $F \in \Im^1_1(T(M_n))$ has components of the form

$${}^{DD}F = ({}^{DD}F^{A}_{B}) = \begin{pmatrix} -F^{\alpha}_{\beta} & 0 & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & 0 & -F^{\beta}_{\alpha} \end{pmatrix}$$

with respect to the adapted frame $\left\{\widehat{e}_{(B)}\right\}$ in $t^*(M_n)$.

Proof Let $F \in \Im^1_1(T(M_n))$. Then we have by (3.4), (3.5), and (3.6):

$$\begin{aligned}
{}^{DD}F &= \left(\widehat{A}\right)^{-1}\left({}^{DD}F\right)\left(\widehat{A}\right) \\
&= \begin{pmatrix} \delta^{\beta}_{\alpha} & \Gamma^{\beta}_{\alpha} & 0 \\ 0 & \delta^{\beta}_{\alpha} & 0 \\ 0 & -\Gamma_{\alpha\beta} & \delta^{\alpha}_{\beta} \end{pmatrix} \begin{pmatrix} -F^{\alpha}_{\gamma} & -\Gamma^{\alpha}_{\varepsilon}F^{\varepsilon}_{\gamma} - \Gamma^{\varepsilon}_{\gamma}F^{\alpha}_{\varepsilon} & 0 \\ 0 & F^{\alpha}_{\gamma} & 0 \\ 0 & \Gamma_{\gamma\sigma}F^{\sigma}_{\alpha} + \Gamma_{\alpha\sigma}F^{\sigma}_{\gamma} & -F^{\gamma}_{\alpha} \end{pmatrix} \begin{pmatrix} \delta^{\gamma}_{\psi} & -\Gamma^{\gamma}_{\psi} & 0 \\ 0 & \delta^{\gamma}_{\psi} & 0 \\ 0 & \Gamma_{\psi\gamma} & \delta^{\psi}_{\gamma} \end{pmatrix} \\
&= \begin{pmatrix} -F^{\beta}_{\gamma} & -\Gamma^{\beta}_{\varepsilon}F^{\varepsilon}_{\gamma} - \Gamma^{\varepsilon}_{\gamma}F^{\beta}_{\varepsilon} + \Gamma^{\beta}_{\alpha}F^{\alpha}_{\gamma} & 0 \\ 0 & F^{\beta}_{\gamma} & 0 \\ 0 & -\Gamma_{\alpha\beta}F^{\alpha}_{\gamma} + \Gamma_{\gamma\sigma}F^{\sigma}_{\beta} + \Gamma_{\beta\sigma}F^{\sigma}_{\gamma} & -F^{\gamma}_{\beta} \end{pmatrix} \begin{pmatrix} \delta^{\gamma}_{\psi} & -\Gamma^{\gamma}_{\psi} & 0 \\ 0 & \delta^{\gamma}_{\psi} & 0 \\ 0 & \Gamma_{\psi\gamma} & \delta^{\psi}_{\gamma} \end{pmatrix} \\
&= \begin{pmatrix} -F^{\beta}_{\psi} & \Gamma^{\gamma}_{\psi}F^{\beta}_{\gamma} - \Gamma^{\beta}_{\varepsilon}F^{\varepsilon}_{\psi} - \Gamma^{\varepsilon}_{\psi}F^{\beta}_{\varepsilon} + \Gamma^{\beta}_{\alpha}F^{\alpha}_{\psi} & 0 \\ 0 & F^{\beta}_{\psi} & 0 \\ 0 & -\Gamma_{\alpha\beta}F^{\alpha}_{\psi} + \Gamma_{\psi\sigma}F^{\sigma}_{\beta} + \Gamma_{\beta\sigma}F^{\sigma}_{\psi} - \Gamma_{\psi\gamma}F^{\gamma}_{\beta} & -F^{\psi}_{\beta} \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} -F_{\psi}^{\beta} & 0 & 0 \\ 0 & F_{\psi}^{\beta} & 0 \\ 0 & 0 & -F_{\beta}^{\psi} \end{pmatrix}.$$

This completes the proof. □

We now obtain from (3.6) that the diagonal lift ${}^{DD}F$ of an affinor field $F \in \Im_1^1(T(M_n))$ has along $\beta_\theta\left(T(M_n)\right)$ components of the form

$$^{DD}F : \begin{pmatrix} -F_{\beta}^{\alpha} & -\left(\nabla_{\varepsilon}V^{\alpha}\right)F_{\beta}^{\varepsilon} - \left(\nabla_{\beta}V^{\varepsilon}\right)F_{\varepsilon}^{\alpha} & 0 \\ 0 & F_{\beta}^{\alpha} & 0 \\ 0 & -\left(\nabla_{\beta}\theta_{\sigma}\right)F_{\alpha}^{\sigma} - \left(\nabla_{\alpha}\theta_{\sigma}\right)F_{\beta}^{\sigma} & -F_{\alpha}^{\beta} \end{pmatrix}, \tag{3.7}$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$.

Proof Let $F \in \Im_1^1(T(M_n))$. Then we have by (1.8), (1.9), and (3.7):

$$\begin{aligned}
{}^{DD}F &= \left(\widetilde{A}\right)^{-1}\left({}^{DD}F\right)\left(\widetilde{A}\right) \\
&= \begin{pmatrix} \delta_{\alpha}^{\beta} & -\partial_{\alpha}V^{\beta} & 0 \\ 0 & \delta_{\alpha}^{\beta} & 0 \\ 0 & -\partial_{\alpha}\theta_{\beta} & \delta_{\beta}^{\alpha} \end{pmatrix}\begin{pmatrix} -F_{\gamma}^{\alpha} & -\Gamma_{\varepsilon}^{\alpha}F_{\gamma}^{\varepsilon} - \Gamma_{\gamma}^{\varepsilon}F_{\varepsilon}^{\alpha} & 0 \\ 0 & F_{\gamma}^{\alpha} & 0 \\ 0 & \Gamma_{\gamma\sigma}F_{\alpha}^{\sigma} + \Gamma_{\alpha\sigma}F_{\gamma}^{\sigma} & -F_{\alpha}^{\gamma} \end{pmatrix}\begin{pmatrix} \delta_{\psi}^{\gamma} & \partial_{\psi}V^{\gamma} & 0 \\ 0 & \delta_{\psi}^{\gamma} & 0 \\ 0 & \partial_{\psi}\theta_{\gamma} & \delta_{\gamma}^{\psi} \end{pmatrix} \\
&= \begin{pmatrix} -F_{\gamma}^{\beta} & -\Gamma_{\varepsilon}^{\beta}F_{\gamma}^{\varepsilon} - \Gamma_{\gamma}^{\varepsilon}F_{\varepsilon}^{\beta} - \partial_{\alpha}V^{\beta}F_{\gamma}^{\alpha} & 0 \\ 0 & F_{\gamma}^{\beta} & 0 \\ 0 & -\partial_{\alpha}\theta_{\beta}F_{\gamma}^{\alpha} + \Gamma_{\gamma\sigma}F_{\beta}^{\sigma} + \Gamma_{\beta\sigma}F_{\gamma}^{\sigma} & -F_{\beta}^{\gamma} \end{pmatrix}\begin{pmatrix} \delta_{\psi}^{\gamma} & \partial_{\psi}V^{\gamma} & 0 \\ 0 & \delta_{\psi}^{\gamma} & 0 \\ 0 & \partial_{\psi}\theta_{\gamma} & \delta_{\gamma}^{\psi} \end{pmatrix} \\
&= \begin{pmatrix} -F_{\psi}^{\beta} & -\partial_{\psi}V^{\gamma}F_{\gamma}^{\beta} - \Gamma_{\varepsilon}^{\beta}F_{\psi}^{\varepsilon} - \Gamma_{\psi}^{\varepsilon}F_{\varepsilon}^{\beta} - \partial_{\alpha}V^{\beta}F_{\psi}^{\alpha} & 0 \\ 0 & F_{\psi}^{\beta} & 0 \\ 0 & -\partial_{\alpha}\theta_{\beta}F_{\psi}^{\alpha} + \Gamma_{\psi\sigma}F_{\beta}^{\sigma} + \Gamma_{\beta\sigma}F_{\psi}^{\sigma} - \partial_{\psi}\theta_{\gamma}F_{\beta}^{\gamma} & -F_{\beta}^{\psi} \end{pmatrix} \\
&= \begin{pmatrix} -F_{\psi}^{\beta} & -\left(\nabla_{\gamma}V^{\beta}\right)F_{\psi}^{\gamma} - \left(\nabla_{\psi}V^{\gamma}\right)F_{\gamma}^{\beta} & 0 \\ 0 & F_{\psi}^{\beta} & 0 \\ 0 & -\left(\nabla_{\psi}\theta_{\gamma}\right)F_{\beta}^{\gamma} - \left(\nabla_{\beta}\theta_{\sigma}\right)F_{\psi}^{\sigma} & -F_{\beta}^{\psi} \end{pmatrix}.
\end{aligned}$$

Thus, the proof is complete. □

Then we see from (1.6) that the horizontal lift ${}^{HH}X$ of a vector field $X \in \Im_0^1(T(M_n))$ has along $\beta_\theta\left(T(M_n)\right)$ components of the form

$$^{HH}X : \begin{pmatrix} -X^{\beta}\left(\nabla_{\beta}V^{\alpha}\right) \\ X^{\alpha} \\ -\left(\nabla_{\beta}\theta_{\alpha}\right)X^{\beta} \end{pmatrix} \tag{3.8}$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$.

Proof Let $X \in \Im_0^1(T(M_n))$. Then we have by (1.6) and (1.9):

$$
\begin{aligned}
{}^{HH}X &= \left(\widetilde{A}\right)^{-1}\left({}^{HH}X\right) = \begin{pmatrix} \delta_\beta^\alpha & -\partial_\beta V^\alpha & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & -\partial_\beta \theta_\alpha & \delta_\alpha^\beta \end{pmatrix} \begin{pmatrix} -V^\varepsilon \Gamma_{\varepsilon\,\alpha}^{\beta} X^\alpha \\ X^\beta \\ X^\alpha \theta_\varepsilon \Gamma_{\alpha\beta}^\varepsilon \end{pmatrix} \\
&= \begin{pmatrix} -V^\varepsilon \Gamma_{\varepsilon\,\theta}^{\beta} X^\theta - \partial_\beta V^\alpha X^\beta \\ X^\alpha \\ -\partial_\beta \theta_\alpha X^\beta + X^\theta \theta_\varepsilon \Gamma_{\theta\alpha}^\varepsilon \end{pmatrix} = \begin{pmatrix} -X^\beta \left(\nabla_\beta V^\alpha\right) \\ X^\alpha \\ -\left(\nabla_\beta \theta_\alpha\right) X^\beta \end{pmatrix},
\end{aligned}
$$

which gives (3.8). □

Using (1.6), (3.7), and (3.8), we have along $\beta_\theta(T(M_n))$:

Theorem 5 *If F and X are affinor and vector fields on $T(M_n)$, and $\omega \in \Im_1^0(M_n)$, then with respect to a symetric affine connection ∇ in M_n, we have*

(i) ${}^{DD}F\left({}^{HH}X\right) = {}^{HH}(FX)$,

(ii) ${}^{DD}F\left({}^{vv}\omega\right) = -{}^{vv}(\omega \circ F)$.

Proof

(i) *If* $F \in \Im_1^1(T(M_n))$ *and* $X \in \Im_0^1(T(M_n))$, then by (3.7) and (3.8), we have

$$
\begin{aligned}
{}^{DD}F\left({}^{HH}X\right) &= \begin{pmatrix} -F_\beta^\alpha & -\left(\nabla_\varepsilon V^\alpha\right) F_\beta^\varepsilon - \left(\nabla_\beta V^\varepsilon\right) F_\varepsilon^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -\left(\nabla_\beta \theta_\sigma\right) F_\alpha^\sigma - \left(\nabla_\alpha \theta_\sigma\right) F_\beta^\sigma & -F_\alpha^\beta \end{pmatrix} \begin{pmatrix} -X^\varepsilon\left(\nabla_\varepsilon V^\beta\right) \\ X^\beta \\ -\left(\nabla_\sigma \theta_\beta\right) X^\sigma \end{pmatrix} \\
&= \begin{pmatrix} F_\beta^\alpha X^\varepsilon\left(\nabla_\varepsilon V^\beta\right) - \left(\nabla_\varepsilon V^\alpha\right) F_\beta^\varepsilon X^\beta - \left(\nabla_\beta V^\varepsilon\right) F_\varepsilon^\alpha X^\beta \\ (FX)^\alpha \\ -\left(\nabla_\alpha \theta_\sigma\right) F_\beta^\sigma X^\beta - \left(\nabla_\beta \theta_\sigma\right) F_\alpha^\sigma X^\beta + \left(\nabla_\sigma \theta_\beta\right) X^\sigma F_\alpha^\beta \end{pmatrix} \\
&= \begin{pmatrix} -\left(\nabla_\varepsilon V^\alpha\right)(FX)^\varepsilon \\ (FX)^\alpha \\ -\left(\nabla_\sigma \theta_\alpha\right)(FX)^\sigma \end{pmatrix} = {}^{HH}(FX).
\end{aligned}
$$

Thus, we have ${}^{DD}F\left({}^{HH}X\right) = {}^{HH}(FX)$.

(ii) If $\omega \in \Im_1^0(M_n)$ and $F \in \Im_1^1(T(M_n))$, then by (1.6), (1.10), and (3.7), we have

$$
\begin{aligned}
{}^{DD}F\left({}^{vv}\omega\right) &= \begin{pmatrix} -F_\beta^\alpha & -\left(\nabla_\varepsilon V^\alpha\right) F_\beta^\varepsilon - \left(\nabla_\beta V^\varepsilon\right) F_\varepsilon^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & -\left(\nabla_\beta \theta_\sigma\right) F_\alpha^\sigma - \left(\nabla_\alpha \theta_\sigma\right) F_\beta^\sigma & -F_\alpha^\beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_\beta \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ -\omega_\beta F_\alpha^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(\omega \circ F)_\alpha \end{pmatrix} = -{}^{vv}(\omega \circ F).
\end{aligned}
$$

Thus, we have (ii) of Theorem 5. □

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