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Gröbner–Shirshov basis for the singular part of the Brauer semigroup

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Abstract: In this paper, we obtain a Gröbner–Shirshov (noncommutative Gröbner) basis for the singular part of the Brauer semigroup. It gives an algorithm for getting normal forms and hence an algorithm for solving the word problem in these semigroups.

Key words: Gröbner–Shirshov bases, Brauer semigroup, normal form

1. Introduction and preliminaries

The theories of Gröbner and Gröbner–Shirshov bases were invented independently by Shirshov [27] for noncommutative and nonassociative algebras and by Hironaka [20] and Buchberger [14] for commutative algebras. In [27], the algorithmic decidability of the word problem and the Freiheitsatz theorem for any one-relator Lie algebra were proved. The technique of Gröbner–Shirshov bases has proved to be very useful in the study of presentations of associative algebras, Lie algebras, semigroups, groups, and Ω-algebras by considering generators and relations (see, for example, the book [11], written by Bokut and Kukin, and survey papers [7, 9, 10]). In [12], Bokut et al. defined the Gröbner–Shirshov basis for some braid groups. In [18], Gröbner–Shirshov bases for HNN-extensions of groups and for the alternating groups were considered. Furthermore, in [16] and [17], Gröbner–Shirshov bases for Schreier extensions of groups and for the Chinese monoid were defined separately. The reader is referred to [1, 5, 6, 8, 19, 21, 22] for some other recent papers about Gröbner–Shirshov bases.

The symmetric group $\mathcal{S}_n$ is a central object of study in many braches of mathematics. There exist several natural analogues (or generalizations) of $\mathcal{S}_n$ in the theory of semigroups. The most classical ones are the symmetric semigroup $\mathcal{T}_n$ and the inverse symmetric semigroup $\mathcal{IS}_n$. A less obvious semigroup generalization of $\mathcal{S}_n$ is the so-called *Brauer semigroup* $\mathcal{B}_n$, which appears in the context of centralizer algebras in representation theory (see [13]). $\mathcal{B}_n$ contains $\mathcal{S}_n$ as the subgroup of all invertible elements and has a geometric realization [25]. The reader can find semigroup properties of $\mathcal{B}_n$ in [23, 24, 26]. The deformation of the corresponding semigroup algebra, the so-called *Brauer algebra*, has been intensively studied by specialists in representation theory, knot theory, and theoretical physics. Brauer algebra is an algebra introduced by Brauer in 1937 and used in the representation theory of the orthogonal group. It plays the same role that the symmetric group

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does for the representation theory of the general linear group in Schur–Weyl duality.

In [25], the authors obtained a presentation for the *singular part of the Brauer semigroup*, $\mathcal{B}_n - \mathcal{S}_n$, which, by definition, is the set of all noninvertible elements. Thus, it is very natural to find a Gröbner–Shirshov basis of it. Hence, in this paper, we aim to obtain a Gröbner–Shirshov basis for $\mathcal{B}_n - \mathcal{S}_n$ and thus normal forms of words in this semigroup.

For $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, define $\sigma_{i,j}$ as follows:

$$\sigma_{i,j} = \{\{i, j\}, \{i', j'\}, \{k, k'\}_{k\neq i,j}\}.$$

We have $\sigma_{i,j} = \sigma_{j,i} = \sigma_{i,j}^2$ and $corank(\sigma_{i,j}) = 2$. We call these elements *atoms*. The following result was proved in [25].

Proposition 1 *The set of all atoms is an irreducible system of generators in* $\mathcal{B}_n - \mathcal{S}_n$.

Now let us denote by T_n the semigroup generated by $\tau_{i,j}$, $i, j \in \{1, 2, \ldots, n\}$, subject to the following relations (i, j, k, l are pairwise different):

$$\begin{aligned} \tau_{i,j} &= \tau_{j,i}, & \tau_{i,j}^2 = \tau_{i,j}, \\ \tau_{i,j}\tau_{i,l}\tau_{k,l} &= \tau_{i,j}\tau_{j,k}\tau_{k,l}, & \tau_{i,j}\tau_{i,k}\tau_{j,k} = \tau_{i,j}\tau_{j,k}, \\ \tau_{i,j}\tau_{k,l} &= \tau_{k,l}\tau_{i,j}, & \tau_{i,j}\tau_{j,l}\tau_{i,k} = \tau_{k,l}\tau_{i,j}\tau_{i,k}, \\ \tau_{i,j}\tau_{j,k}\tau_{i,j} &= \tau_{i,j}. \end{aligned}$$

In [25], the authors showed that there is an homomorphism $\varphi : T_n \to \mathcal{B}_n - \mathcal{S}_n$, sending $\tau_{i,j}$ to $\sigma_{i,j}$. Then they got the following main result.

Theorem 2 [25] $\varphi : T_n \to \mathcal{B}_n - \mathcal{S}_n$ *is an isomorphism.*

2. Gröbner–Shirshov bases and composition-diamond lemma

Let k be a field and $k\langle X\rangle$ be the free associative algebra over k generated by X. Denote by X^* the free monoid generated by X, where the empty word is the identity, which is denoted by 1. For a word $w \in X^*$, we denote the length of w by $|w|$. Let X^* be a well-ordered set. Then every nonzero polynomial $f \in k\langle X\rangle$ has the leading word $\overline{f}$. If the coefficient of $\overline{f}$ in f is equal to 1, then f is called monic.

Definition 3 *Let f and g be two monic polynomials in $k\langle X\rangle$. Then there are two kinds of compositions:*

1. If w is a word such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in X^*$ with $|\overline{f}| + |\overline{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the *intersection composition* of f and g with respect to w. The word w is called an *ambiguity* of intersection.

2. If $w = \overline{f} = a\overline{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the *inclusion composition* of f and g with respect to w. The word w is called an *ambiguity* of inclusion.

We denote the first and the second compositions by $f \wedge g$ and $f \vee g$, respectively.

Definition 4 *If g is monic, $\overline{f} = a\overline{g}b$, and α is the coefficient of the leading term $\overline{f}$, then the transformation $f \mapsto f - \alpha agb$ is called elimination of the leading word (ELW) of g in f.*

Definition 5 *Let $S \subseteq k\langle X\rangle$ with each $s \in S$ monic. Then the composition $(f,g)_w$ is called trivial modulo (S,w) if $(f,g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$, and $a_i\overline{s_i}b_i < w$. If this is the case, then we write*

$$(f,g)_w \equiv 0 \ mod(S,w).$$

In general, for $p, q \in k\langle X\rangle$, we write $p \equiv q \ mod(S,w)$, which means that $p - q = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$, and $a_i\overline{s_i}b_i < w$.

Definition 6 *We call the set S endowed with the well order $<$ a Gröbner–Shirshov basis for $k\langle X \mid S\rangle$ if any composition $(f,g)_w$ of polynomials in S is trivial modulo S and corresponding w.*

A well order $<$ on X^* is monomial if, for $u, v \in X^*$, we have $u < v \Rightarrow w_1uw_2 < w_1vw_2$, for all $w_1, w_2 \in X^*$.

The following lemma was proved by Shirshov [27] for free Lie algebras (with deg-lex ordering) in 1962 (see also [3]). In 1976, Bokut [4] specialized Shirshov's approach to associative algebras (see also [2]). Meanwhile, for commutative polynomials, this lemma is known as Buchberger's theorem (see [14, 15]).

Lemma 7 ***(Composition-diamond lemma)*** *Let k be a field,*

$$A = k\langle X \mid S\rangle = k\langle X\rangle/Id(S),$$

and $<$ a monomial order on X^*, where $Id(S)$ is the ideal of $k\langle X\rangle$ generated by S. Then the following statements are equivalent:

1. *S is a Gröbner–Shirshov basis.*
2. *$f \in Id(S) \Rightarrow \overline{f} = a\overline{s}b$ for some $s \in S$ and $a, b \in X^*$.*
3. *$Irr(S) = \{u \in X^* \mid u \neq a\overline{s}b, s \in S, a, b \in X^*\}$ is a basis of the algebra $A = k\langle X \mid S\rangle$.*

If a subset S of $k\langle X\rangle$ is not a Gröbner–Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner–Shirshov basis S^{comp}. Such a process is called the *Shirshov algorithm*.

If S is a set of "semigroup relations" (that is, the polynomials of the form $u - v$, where $u, v \in X^*$), then any nontrivial composition will have the same form. As a result, the set S^{comp} also consists of semigroup relations.

Let $M = sgp\langle X \mid S\rangle$ be a semigroup presentation. Then S is a subset of $k\langle X\rangle$ and hence one can find a Gröbner–Shirshov basis S^{comp}. The last set does not depend on k, and, as mentioned before, it consists of semigroup relations. We will call S^{comp} a Gröbner–Shirshov basis of M. This is the same as a Gröbner–Shirshov basis of the semigroup algebra $kM = k\langle X \mid S\rangle$. If S is a Gröbner–Shirshov basis of the semigroup $M = sgp\langle X \mid S\rangle$, then $Irr(S)$ is a normal form for M.

3. Main result

Let us order the generators lexicographically as

$$\tau_{i,j} > \tau_{k,l} \quad \text{if and only if} \quad (i,j) > (k,l)\,.$$

We order words in this alphabet in the deg-lex way comparing two words first by theirs degrees (lengths) and then lexicographically when the degrees are equal.

Let us assume that the following notation,

$$V_{[x_a,y_a]}\,, \text{ where } x_a > y_a \text{ for } 1 \leq a \leq 4\,,$$

is a reduced word obtained by generators depending on the restrictions on x_a and y_a. For example, we consider the reduced word $V_{[x_1,y_1]}$ for $j \geq x_1$ and $l \geq y_1$. This word can be represented as $\tau_{j,l}$ or $\tau_{k,p}$ or $\tau_{k,p}\tau_{k,r}$ or $\tau_{k,p}\tau_{l,r}\tau_{p,r}$, etc. We also note that the word $V_{[x_a,y_a]}$ $(1 \leq a \leq 4)$ can be empty word 1 as well. In this case, relations (5) and (6) given in Theorem 8 are the relations of the semigroup T_n as depicted in Section 1 of this paper.

We will also use the following notations,

$$\overline{V}_{[x_a,y_a]} \quad \text{and} \quad V_{[x_a,y_a]^2}\,,$$

where the first notation denotes the word that does not have the last generator of the word $V_{[x_a,y_a]}$ and the second notation denotes the word that has the last generator twice. For example, we can consider the word $V_{[x_2,y_2]}$ ($i \geq x_2$ and $l \geq y_2$) as the word $\tau_{j,p}\tau_{j,r}$, so we have $\overline{V}_{[x_2,y_2]} = \tau_{j,p}$ and $V_{[x_2,y_2]^2} = \tau_{j,p}\tau_{j,r}^2$.

We also note that throughout this section we will use the ordering $i > j > k > l > p > r$.

Now we give the main result of this paper.

Theorem 8 *A Gröbner–Shirshov basis for T_n consists of the following relations:*

(1) $\tau_{i,j}^2 = \tau_{i,j}\,,$

(2) $\tau_{i,j}\tau_{i,l}\tau_{k,l} = \tau_{i,j}\tau_{j,k}\tau_{k,l}\,,$

(3) $\tau_{i,j}\tau_{i,k}\tau_{j,k} = \tau_{i,j}\tau_{j,k}\,,$

(4) $\tau_{i,j}\tau_{k,l} = \tau_{k,l}\tau_{i,j}\,,$

(5) $\tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]} = \tau_{k,l}V_{[x_1{}^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}\,,$

(6) $\tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]} = V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\,,$

where $V_{[x_a,y_a]}$ $(1 \leq a \leq 4)$ are reduced words obtained by generators such that

$$j \geq x_1, \quad k \geq x_1{}^*, \quad l \geq y_1 \quad \textit{and} \quad i \geq x_2, \quad l \geq y_2,$$
$$k \geq x_3, \quad l \geq y_3 \quad \textit{and} \quad i \geq x_4, \quad k \geq y_4.$$

Proof It is obvious that relations (1)–(6) are valid in T_n. We need to prove that all compositions of relations (1)–(6) are trivial. First we consider intersection compositions of relations (1)–(6). We will denote the intersection composition of polynomials f and g by $f \wedge g$. Let us consider compositions of (1) with all other relations. We start by listing all intersection ambiguities of (1):

$$
\begin{array}{llll}
(1)\wedge(1) & \tau_{i,j}^3, & & \\
(1)\wedge(2) & \tau_{i,j}^2\tau_{i,l}\tau_{k,l}, & (2)\wedge(1) & \tau_{i,j}\tau_{i,l}\tau_{k,l}^2, \\
(1)\wedge(3) & \tau_{i,j}^2,\tau_{i,k}\tau_{j,k}, & (3)\wedge(1) & \tau_{i,j}\tau_{i,k}\tau_{j,k}^2, \\
(1)\wedge(4) & \tau_{i,j}^2\tau_{k,l}, & (4)\wedge(1) & \tau_{i,j}\tau_{k,l}^2, \\
(1)\wedge(5) & \multicolumn{3}{l}{\tau_{i,j}^2\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]} \quad (j\geq x_1,\ l\geq y_1,\ i\geq x_2,\ l\geq y_2),} \\
(5)\wedge(1) & \multicolumn{3}{l}{\tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]^2} \quad (j\geq x_1,\ l\geq y_1,\ i\geq x_2,\ l\geq y_2),} \\
(1)\wedge(6) & \multicolumn{3}{l}{\tau_{i,j}^2\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]} \quad (k\geq x_3,\ l\geq y_3,\ i\geq x_4,\ k\geq y_4),} \\
(6)\wedge(1) & \multicolumn{3}{l}{\tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]^2} \quad (k\geq x_3,\ l\geq y_3,\ i\geq x_4,\ k\geq y_4).}
\end{array}
$$

It is seen that these compositions are trivial. Let us check one of them.

$$
\begin{aligned}
(1)\wedge(3):\quad w &= \tau_{i,j}^2,\tau_{i,k}\tau_{j,k}, \\
(f,g)_w &= (\tau_{i,j}^2-\tau_{i,j})\tau_{i,k}\tau_{j,k}-\tau_{i,j}(\tau_{i,j}\tau_{i,k}\tau_{j,k}-\tau_{i,j}\tau_{j,k}) \\
&= \tau_{i,j}^2\tau_{i,k}\tau_{j,k}-\tau_{i,j}\tau_{i,k}\tau_{j,k}-\tau_{i,j}^2\tau_{i,k}\tau_{j,k}+\tau_{i,j}^2\tau_{j,k} \\
&= \tau_{i,j}^2\tau_{j,k}-\tau_{i,j}\tau_{i,k}\tau_{j,k} \\
&\equiv \tau_{i,j}\tau_{j,k}-\tau_{i,j}\tau_{j,k}\equiv 0.
\end{aligned}
$$

We proceed with intersection compositions of (2) with (2)–(6). The ambiguities are the following:

$$
\begin{array}{llll}
(2)\wedge(2) & \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{k,r}\tau_{p,r}, & & \\
(2)\wedge(3) & \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{k,p}\tau_{l,p}, & (3)\wedge(2) & \tau_{i,j}\tau_{i,k}\tau_{j,k}\tau_{j,p}\tau_{l,p}, \\
(2)\wedge(4) & \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{p,r}, & (4)\wedge(2) & \tau_{i,j}\tau_{k,l}\tau_{k,r}\tau_{p,r}, \\
(2)\wedge(5) & \multicolumn{3}{l}{\tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{l,r}V_{[x_1,y_1]}\tau_{k,p}V_{[x_2,y_2]} \quad (l\geq x_1,\ r\geq y_1,\ k\geq x_2,\ r\geq y_2),} \\
(5)\wedge(2) & \multicolumn{3}{l}{\tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_2,y_3}\tau_{x_3,y_3} \quad (j\geq x_1,\ l\geq y_1,\ i\geq x_2,\ l\geq y_2,\ x_2>y_2>x_3>y_3),} \\
(2)\wedge(6) & \multicolumn{3}{l}{\tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{l,p}V_{[x_3,y_3]}\tau_{k,l}V_{[x_4,y_4]} \quad (p\geq x_3,\ r\geq y_3,\ k\geq x_4,\ p\geq y_4),} \\
(6)\wedge(2) & \multicolumn{3}{l}{\tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{x_4,y_5}\tau_{x_5,y_5} \quad (k\geq x_3,\ l\geq y_3,\ i\geq x_4,\ k\geq y_4,\ x_4>y_4>x_5>y_5).}
\end{array}
$$

These intersection compositions are trivial. Let us check some of these compositions as examples:

$$
\begin{aligned}
(2)\wedge(3):\quad w &= \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{k,p}\tau_{l,p}, \\
(f,g)_w &= (\tau_{i,j}\tau_{i,l}\tau_{k,l}-\tau_{i,j}\tau_{j,k}\tau_{k,l})\tau_{k,p}\tau_{l,p}-\tau_{i,j}\tau_{i,l}(\tau_{k,l}\tau_{k,p}\tau_{l,p}-\tau_{k,l}\tau_{l,p})
\end{aligned}
$$

$$
\begin{aligned}
&= \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{k,p}\tau_{l,p} - \tau_{i,j}\tau_{j,k}\tau_{k,l}\tau_{k,p}\tau_{l,p} - \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{k,p}\tau_{l,p} + \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{l,p} \\
&= \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{l,p} - \tau_{i,j}\tau_{j,k}\tau_{k,l}\tau_{k,p}\tau_{l,p} \\
&\equiv \tau_{i,j}\tau_{i,l}\tau_{k,l}\tau_{l,p} - \tau_{i,j}\tau_{j,k}\tau_{k,l}\tau_{l,p} \\
&\equiv \tau_{i,j}\tau_{j,k}\tau_{k,l}\tau_{l,p} - \tau_{i,j}\tau_{j,k}\tau_{k,l}\tau_{l,p} \equiv 0.
\end{aligned}
$$

$$
\begin{aligned}
(6)\wedge(2):\quad w &= \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{x_4,y_5}\tau_{x_5,y_5} \\
&\quad \text{such that } k \geq x_3,\ l \geq y_3,\ i \geq x_4,\ k \geq y_4,\ x_4 > y_4 > x_5 > y_5, \\
(f,g)_w &= (\tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]} - V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]})\tau_{x_4,y_5}\tau_{x_5,y_5} \\
&- \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}\overline{V}_{[x_4,y_4]}(\tau_{x_4,y_4}\tau_{x_4,y_5}\tau_{x_5,y_5} - \tau_{x_4,y_4}\tau_{y_4,x_5}\tau_{x_5,y_5}) \\
&= \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{x_4,y_5}\tau_{x_5,y_5} - V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{x_4,y_5}\tau_{x_5,y_5} \\
&- \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{x_4,y_5}\tau_{x_5,y_5} + \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}\overline{V}_{[x_4,y_4]}\tau_{x_4,y_4}\tau_{y_4,x_5}\tau_{x_5,y_5} \\
&= \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}\underbrace{\overline{V}_{[x_4,y_4]}\tau_{x_4,y_4}}_{V_{[x_4,y_4]}}\tau_{y_4,x_5}\tau_{x_5,y_5} - V_{[x_3,y_3]}\tau_{i,j}\underbrace{V_{[x_4,y_4]}}_{\overline{V}_{[x_4,y_4]}\tau_{x_4,y_4}}\tau_{x_4,y_5}\tau_{x_5,y_5} \\
&\equiv V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{y_4,x_5}\tau_{x_5,y_5} - V_{[x_3,y_3]}\tau_{i,j}\underbrace{\overline{V}_{[x_4,y_4]}\tau_{x_4,y_4}}_{V_{[x_4,y_4]}}\tau_{y_4,x_5}\tau_{x_5,y_5} \equiv 0.
\end{aligned}
$$

Our next compositions will be (3) with (3)–(6). The ambiguities of these intersection compositions are the following:

$(3)\wedge(3)$ $\quad \tau_{i,j}\tau_{i,k}\tau_{j,k}\tau_{j,l}\tau_{k,l}$,

$(3)\wedge(4)$ $\quad \tau_{i,j}\tau_{i,k}\tau_{j,k}\tau_{l,p}$, $\qquad (4)\wedge(3)$ $\quad \tau_{i,j}\tau_{k,l}\tau_{k,p}\tau_{l,p}$,

$(3)\wedge(5)$ $\quad \tau_{i,j}\tau_{i,k}\tau_{j,k}\tau_{k,p}V_{[x_1,y_1]}\tau_{j,l}V_{[x_2,y_2]}$ $\quad (k \geq x_1,\ p \geq y_1,\ j \geq x_2,\ p \geq y_2)$,

$(5)\wedge(3)$ $\quad \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_2,y_3}\tau_{y_2,y_3}$ $\quad (j \geq x_1,\ l \geq y_1,\ i \geq x_2,\ l \geq y_2,\ x_2 > y_2 > y_3)$,

$(3)\wedge(6)$ $\quad \tau_{i,j}\tau_{i,k}\tau_{j,k}\tau_{k,l}V_{[x_3,y_3]}\tau_{j,k}V_{[x_4,y_4]}$ $\quad (l \geq x_3,\ p \geq y_3,\ j \geq x_4,\ l \geq y_4)$,

$(6)\wedge(3)$ $\quad \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{x_4,y_5}\tau_{y_4,y_5}$ $\quad (k \geq x_3,\ l \geq y_3,\ i \geq x_4,\ k \geq y_4,\ x_4 > y_4 > y_5)$.

It is easy to see that these compositions are trivial. Let us check one of them.

$$
\begin{aligned}
(5)\wedge(3):\quad w &= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_2,y_3}\tau_{y_2,y_3} \\
&\quad \text{such that } j \geq x_1,\ k \geq {x_1}^*,\ l \geq y_1,\ i \geq x_2,\ l \geq y_2,\ x_2 > y_2 > y_3.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(f,g)_w &= (\tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]} - \tau_{k,l}V_{[{x_1}^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]})\tau_{x_2,y_3}\tau_{y_2,y_3} \\
&- \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}(\tau_{x_2,y_2}\tau_{x_2,y_3}\tau_{y_2,y_3} - \tau_{x_2,y_2}\tau_{y_2,y_3}) \\
&= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_2,y_3}\tau_{y_2,y_3} - \tau_{k,l}V_{[{x_1}^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_2,y_3}\tau_{y_2,y_3} \\
&- \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}\tau_{x_2,y_3}\tau_{y_2,y_3} + \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}\tau_{y_2,y_3}
\end{aligned}
$$

$$\begin{aligned}
&= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\underbrace{\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}}_{V_{[x_2,y_2]}}\tau_{y_2,y_3} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}\underbrace{V_{[x_2,y_2]}}_{\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}}\tau_{x_2,y_3}\tau_{y_2,y_3} \\
&\equiv \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}\tau_{y_2,y_3} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}\underbrace{\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}}_{V_{[x_2,y_2]}}\tau_{y_2,y_3} \\
&\equiv 0.
\end{aligned}$$

Now we proceed with intersection compositions of (4) with (4)–(6). The ambiguities are the following:

$(4) \wedge (4)$ $\quad \tau_{i,j}\tau_{k,l}\tau_{p,r}$,

$(4) \wedge (5)$ $\quad \tau_{i,j}\tau_{k,l}\tau_{l,r}V_{[x_1,y_1]}\tau_{k,p}V_{[x_2,y_2]} \quad (l \geq x_1,\ r \geq y_1,\ k \geq x_2,\ r \geq y_2)$,

$(5) \wedge (4)$ $\quad \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_3,y_3} \quad (j \geq x_1,\ l \geq y_1,\ i \geq x_2,\ l \geq y_2,\ x_2 > y_2 > x_3 > y_3)$,

$(4) \wedge (6)$ $\quad \tau_{i,j}\tau_{k,l}\tau_{l,p}V_{[x_3,y_3]}\tau_{k,l}V_{[x_4,y_4]} \quad (p \geq x_3,\ r \geq y_3,\ k \geq x_4,\ p \geq y_4)$,

$(6) \wedge (4)$ $\quad \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{x_5,y_5} \quad (k \geq x_3,\ l \geq y_3,\ i \geq x_4,\ k \geq y_4,\ x_4 > y_4 > x_5 > y_5)$.

Now we consider compositions of intersection of (5) with (5)–(6) and (6) with (6). We have the ambiguities as follows:

$(5) \wedge (5)$ $\quad \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{y_2,y_3}V_{[x_4,y_4]}\tau_{x_2,x_3}V_{[x_5,y_5]}$
$(j \geq x_1,\ l \geq y_1,\ i \geq x_2,\ l \geq y_2,\ x_2 > y_2 > x_3 > y_3,\ y_2 \geq x_4,\ x_2 \geq x_5,\ y_3 \geq y_4,\ y_3 \geq y_5)$,

$(6) \wedge (6)$ $\quad \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{y_4,y_5}V_{[x_6,y_6]}\tau_{x_4,y_4}V_{[x_7,y_7]}$
$(k \geq x_3,\ l \geq y_3,\ i \geq x_4,\ k \geq y_4,\ x_4 > y_4 > y_5 > t,\ y_5 \geq x_6,\ t \geq y_6,\ x_4 \geq x_7,\ y_5 \geq y_7)$,

$(5) \wedge (6)$ $\quad \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{y_2,y_3}V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]}$
$(j \geq x_1,\ l \geq y_1,\ i \geq x_2,\ l \geq y_2,\ x_2 > y_2 > y_3 > t,\ y_3 \geq x_4,\ t \geq y_4,\ x_2 \geq x_5,\ y_3 \geq y_5)$,

$(6) \wedge (5)$ $\quad \tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}\tau_{y_4,y_5}V_{[x_6,y_6]}\tau_{x_4,x_5}V_{[x_7,y_7]}$
$(k \geq x_3,\ l \geq y_3,\ i \geq x_4,\ k \geq y_4,\ x_4 > y_4 > x_5 > y_5,\ y_4 \geq x_6,\ y_5 \geq y_6,\ x_4 \geq x_7,\ y_5 \geq y_7)$.

These compositions are trivial. Let us check some of them as examples:

$$\begin{aligned}
(4) \wedge (4): \quad w \quad &= \tau_{i,j}\tau_{k,l}\tau_{p,r}, \\
(f,g)_w \quad &= (\tau_{i,j}\tau_{k,l} - \tau_{k,l}\tau_{i,j})\tau_{p,r} - \tau_{i,j}(\tau_{k,l}\tau_{p,r} - \tau_{p,r}\tau_{k,l}) \\
&= \tau_{i,j}\tau_{k,l}\tau_{p,r} - \tau_{k,l}\tau_{i,j}\tau_{p,r} - \tau_{i,j}\tau_{k,l}\tau_{p,r} + \tau_{i,j}\tau_{p,r}\tau_{k,l} \\
&= \tau_{i,j}\tau_{p,r}\tau_{k,l} - \tau_{k,l}\tau_{i,j}\tau_{p,r} \\
&\equiv \tau_{p,r}\tau_{i,j}\tau_{k,l} - \tau_{k,l}\tau_{p,r}\tau_{i,j} \\
&\equiv \tau_{p,r}\tau_{k,l}\tau_{i,j} - \tau_{p,r}\tau_{k,l}\tau_{i,j} \equiv 0.
\end{aligned}$$

$$
\begin{aligned}
(5)\wedge(4): \quad w \quad &= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_3,y_3}\\
&(j \geq x_1,\ l \geq y_1,\ i \geq x_2,\ l \geq y_2,\ x_2 > y_2 > x_3 > y_3),\\
(f,g)_w \quad &= (\tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]})\tau_{x_3,y_3}\\
&- \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}(\tau_{x_2,y_2}\tau_{x_3,y_3} - \tau_{x_3,y_3}\tau_{x_2,y_2})\\
&= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_3,y_3} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_3,y_3}\\
&- \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}\tau_{x_3,y_3} + \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}\tau_{x_3,y_3}\tau_{x_2,y_2}\\
&= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}\tau_{x_3,y_3}\tau_{x_2,y_2} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}\tau_{x_3,y_3}\\
&\equiv \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\tau_{x_3,y_3}\underbrace{\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}}_{V_{[x_2,y_2]}} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}\tau_{x_3,y_3}V_{[x_2,y_2]}\\
&\equiv \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{x_3,y_3}\tau_{i,k}V_{[x_2,y_2]} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{x_3,y_3}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}\\
&\equiv \tau_{k,l}V_{[x_1^*,y_1]}\tau_{x_3,y_3}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{x_3,y_3}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}\\
&\equiv 0.
\end{aligned}
$$

$$
\begin{aligned}
(5)\wedge(6): \quad w \quad &= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{y_2,y_3}V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]}\\
&(j \geq x_1,\ l \geq y_1,\ i \geq x_2,\ l \geq y_2,\ x_2 > y_2 > y_3 > t,\ y_3 \geq x_4,\ t \geq y_4,\ x_2 \geq x_5,\ y_3 \geq y_5),\\
(f,g)_w \quad &= (\tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]})\tau_{y_2,y_3}V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]}\\
&- \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}(\tau_{x_2,y_2}\tau_{y_2,y_3}V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]} - V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]})\\
&= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}\tau_{y_2,y_3}V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}\tau_{y_2,y_3}V_{[x_4,y_4]}\\
&\quad \tau_{x_2,y_2}V_{[x_5,y_5]}\\
&- \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}\tau_{y_2,y_3}V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]} + \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}V_{[x_4,y_4]}\\
&\quad \tau_{x_2,y_2}V_{[x_5,y_5]}\\
&= \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}\overline{V}_{[x_2,y_2]}V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}\underbrace{V_{[x_2,y_2]}}_{\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}}\tau_{y_2,y_3}V_{[x_4,y_4]}\\
&\quad \tau_{x_2,y_2}V_{[x_5,y_5]}\\
&\equiv \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}V_{[x_4,y_4]}\tau_{i,k}\overline{V}_{[x_2,y_2]}\tau_{x_2,y_2}V_{[x_5,y_5]} - \tau_{k,l}V_{[x_1^*,y_1]}\tau_{i,j}\tau_{i,k}\overline{V}_{[x_2,y_2]}V_{[x_4,y_4]}\tau_{x_2,y_2}V_{[x_5,y_5]}\\
&\equiv \tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}V_{[x_4,y_4]}\tau_{i,k}V_{[x_2,y_2]}V_{[x_5,y_5]} - \tau_{k,l}V_{[x_1^*,y_1]}V_{[x_4,y_4]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}V_{[x_5,y_5]}\\
&\equiv \tau_{k,l}V_{[x_1^*,y_1]}V_{[x_4,y_4]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}V_{[x_5,y_5]} - \tau_{k,l}V_{[x_1^*,y_1]}V_{[x_4,y_4]}\tau_{i,j}\tau_{i,k}V_{[x_2,y_2]}V_{[x_5,y_5]}\\
&\equiv 0.
\end{aligned}
$$

Now we consider the left-hand sides of relations (1)–(6). These words are $\tau_{i,j}^2$, $\tau_{i,j}\tau_{i,l}\tau_{k,l}$, $\tau_{i,j}\tau_{i,k}\tau_{j,k}$, $\tau_{i,j}\tau_{k,l}$, $\tau_{i,j}\tau_{j,l}V_{[x_1,y_1]}\tau_{i,k}V_{[x_2,y_2]}$, $\tau_{i,j}\tau_{j,k}V_{[x_3,y_3]}\tau_{i,j}V_{[x_4,y_4]}$. We see that no word contains other words as a subword. By Definition 3, it is seen that there are not any inclusion compositions. Consequently, since all intersection compositions of relations (1)–(6) are trivial and there are no inclusion compositions, by Definition 6, relations (1)–(6) are a Gröbner–Shirshov basis for the singular part of the semigroup. □

Two generators $\tau_{i,j}$ and $\tau_{k,l}$ are said to be *connected* if $\{i,j\} \cap \{k,l\} \neq \emptyset$. A word $\tau_{i_1,j_1}\tau_{i_2,j_2}\cdots\tau_{i_s,j_s}$ is said to be connected if τ_{i_t,j_t} and $\tau_{i_{t+1},j_{t+1}}$ are connected for all $1 \leq t \leq s-1$.

Now let R be the set of relations (1)–(6) and $C(u)$ be a normal form of a word $u \in T_n$. By using the composition-diamond lemma, the normal form for the singular part of the Brauer monoid can be given as follows:

Corollary 9 [25] *$C(u)$ has a form*

$$W\tau_{i_1,j_1}\tau_{i_2,j_2}\cdots\tau_{i_s,j_s},$$

where W is an R-irreducible word, $W\tau_{i_1,j_1}$ is connected, and all sets $\{i_t, j_t\}$, $1 \leq t \leq s$ are pairwise disjoint.

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