

Conformable fractional Sturm–Liouville equation and some existence results on time scales

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Abstract: In this study, we analyze a conformable fractional (CF) Sturm–Liouville (SL) equation with boundary conditions on an arbitrary time scale \mathbb{T} . Then we extend the basic spectral properties of the classical SL equation to the CF case. Finally, some sufficient conditions are established to guarantee the existence of a solution for this CF-SL problem on \mathbb{T} by using certain fixed point theorems. For explaining these existence theorems, we give an example with appropriate choices.

Key words: Time scale calculus, conformable fractional derivative, existence theorem

1. Introduction

Fractional calculus means differentiation and integration of a noninteger order. This branch was introduced and developed by Leibniz, Liouville, Riemann, Letnikov, and Grünwald [29]. It is one of the most rapidly developing branches of mathematical physics. Fractional calculus has many applications in science and engineering, such as memory of a variety of materials, signal identification, temperature field problems in oil strata, or diffusion problems (see [8, 14, 24, 27, 31]).

Many researchers have started to deal with discrete versions of fractional calculus using the theory of time scales (see [1, 5, 17, 28, 32, 34]). For example, Benkhettou and his coworkers introduced the concept of the conformable fractional (CF) derivative on \mathbb{T} and explained all properties of CF derivative on \mathbb{T} . The CF derivative of a function order $\alpha \in (0, 1]$ defined on \mathbb{T} reduces to the Hilger derivative when $\alpha = 1$. Before expressing a CF derivative on \mathbb{T} , we should give historical development of time scale calculus.

Time scale calculus was first initiated by Stefan Hilger [20] in his doctoral dissertation under the supervision of Bernard Aulbach (see [6, 21]) to unify difference and differential equations. However, similar ideas had been used before Hilger and its history goes back at least to the introduction of the Riemann–Stieltjes integral, which unifies sums and integrals. More specifically, \mathbb{T} is an arbitrary, nonempty, closed subset of \mathbb{R} . Various results related to differential equations easily transfer to the related results for difference equations, while other results seem to be totally different in nature. Time scale calculus can be applied to any fields in which dynamic processes are described by discrete or continuous time models. Thus, time scale calculus has various applications including noncontinuous domains like the modeling of certain bug populations, chemical reactions, phytoremediation of metals, wound healing, maximization problems in economics, and traffic problems. In recent

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years, several authors have obtained important results about different subjects on time scales (see [3, 12, 13]). Although there are many studies on time scales in the literature, very few studies have been conducted about BVPs (see [2, 7, 11, 15, 16, 18, 19, 22, 30, 33, 35]).

In this study, we consider the following CF Sturm–Liouville (SL) dynamic equation:

$$L_\alpha y(t) = -T_\alpha (T_\alpha (y(t))) = \lambda \phi(t) f(t, y(t)), 0 < \alpha \leq 1, t \in [\rho(a), b] = \mathbb{J} \subset \mathbb{T}, \quad (1.1)$$

with boundary conditions

$$T_\alpha (y(\rho(a))) = 0, \quad (1.2)$$

$$\delta y(b) + \beta T_\alpha (y(b)) = 0, \quad (1.3)$$

where $\lambda > 0$ is a spectral parameter and $\phi : \mathbb{J} \rightarrow \mathbb{J}$ and $f : \mathbb{J} \times \mathbb{T} \rightarrow \mathbb{R}$ are continuous functions. Here, $T_\alpha (y(t))$ indicates the CF derivative of the function y order α and $(\delta^2 + \beta^2) \neq 0$. Moreover, $y(t, \lambda) \in C(\mathbb{J}, \mathbb{R})$ denotes the eigenfunction of the problem (1.1)–(1.3) where $C(\mathbb{J}, \mathbb{R})$ is the space of all continuous functions on \mathbb{J} . We look at the classical SL theory from a different perspective. Therefore, spectral properties and results on the existence of a solution for the problem (1.1)–(1.3) will be discussed for the first time in this study. By setting $\alpha = 1$ in (1.1)–(1.3), the problem is reduced to the SL boundary value problem, which includes the Hilger derivative.

The remaining part of this study is organized as follows: In Section 2, we express some fundamental notations and definitions about CF calculus on \mathbb{T} . Using some methods, we get asymptotic estimates of the eigenfunction for the problem (1.1)–(1.3) in Section 3. In Section 4, we prove some existence theorems. Finally, the conclusions are given in Section 5.

2. Preliminaries

In this section, we express notations, lemmas, and theorems about CF calculus on \mathbb{T} . To give basic results for (1.1)–(1.3), we need to recall some fundamental notions on time scales. Forward and backward jump operators at $t \in \mathbb{T}$ for $t < \sup \mathbb{T}$ are defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \phi = \sup \mathbb{T}$, $\sup \phi = \inf \mathbb{T}$, and ϕ indicates the empty set. Thus, t is left dense, left scattered, right dense, and right scattered provided that $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. The distance from an arbitrary element $t \in \mathbb{T}$ to the closest element on the right is called the graininess of \mathbb{T} and is determined by

$$\mu(t) = \sigma(t) - t.$$

A closed interval on \mathbb{T} is denoted by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$, where a and b are fixed points of \mathbb{T} with $a < b$. In addition, we need to explain \mathbb{T}^κ along with the set \mathbb{T} to define the Δ -derivative of a function. If \mathbb{T} has left scattered maximum m , then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, we put $\mathbb{T}^\kappa = \mathbb{T}$ [13]. Let $h : \mathbb{T} \rightarrow \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then one can define $h^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of t for some $\delta > 0$ such that

$$|[h(\sigma(t)) - h(s)] - h^\Delta(t) [\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|, \quad (2.1)$$

for all $s \in U$. $h^\Delta(t)$ is called Δ or the Hilger derivative of h at t . h is regulated if its right sided limits exist (finite) at all right dense points on \mathbb{T} and its left sided limits exist (finite) at all left dense points on \mathbb{T} . Let h be a regulated function on \mathbb{T} . The indefinite Δ -integral of h is denoted by

$$\int h(t)\Delta t = H(t) + C,$$

where C is an arbitrary constant and H is the pre-antiderivative of h . Finally, the definite Δ -integral of h is defined by

$$\int_r^s h(t)\Delta t = H(s) - H(r),$$

for all $r, s \in \mathbb{T}$. For standard definitions and notations related to time scales theory, we refer to [13].

In 2016, Benkhattou et al. [10] defined the CF derivative and its properties on \mathbb{T} to generalize the Hilger derivative as follows: Let $h : \mathbb{T} \rightarrow \mathbb{R}$, $t \in \mathbb{T}^\kappa$ and $\alpha \in (0, 1]$. For $t > 0$, one can define $T_\alpha(h)(t)$ to be the number provided it exists with the property that, given any $\varepsilon > 0$, there is a δ -neighborhood $V_t \subset \mathbb{T}$ of t such that

$$|[h(\sigma(t)) - h(s)]t^{\alpha-1} - T_\alpha(h)(t) [\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|, \tag{2.2}$$

for all $s \in V_t$. $T_\alpha(h)(t)$ is the CF derivative of h of order α at t . If $\alpha = 1$ in (2.2), we obtain the Hilger derivative on \mathbb{T} , which is defined by (2.1). Benkhattou et al. [10] introduced the main properties of the CF derivative with the following lemmas:

Lemma 2.1 [10] *Let $\alpha \in (0, 1]$, $t \in \mathbb{T}^\kappa$, and $h : \mathbb{T} \rightarrow \mathbb{R}$ be a function. The following features hold:*

- (i) *If h is CF differentiable of order α at $t > 0$, then h is continuous at t .*
- (ii) *If h is continuous at t , which is right scattered, then h is CF differentiable of order α at t with*

$$T_\alpha(h)(t) = \frac{h(\sigma(t)) - h(t)}{\mu(t)} t^{1-\alpha}.$$

- (iii) *If t is right dense, then h is CF differentiable of order α at t if and only if $\lim_{s \rightarrow t} \frac{h(t) - h(s)}{t - s} t^{1-\alpha}$ exists as a finite number. In this instance,*

$$T_\alpha(h)(t) = \lim_{s \rightarrow t} \frac{h(t) - h(s)}{t - s} t^{1-\alpha}.$$

- (iv) *If h is CF differentiable of order α at t , then $h(\sigma(t)) = h(t) + \mu(t)t^{1-\alpha}T_\alpha(h)(t)$.*

Lemma 2.2 [10] *Let $h, g : \mathbb{T} \rightarrow \mathbb{R}$ be CF differentiable functions of order α at $t \in \mathbb{T}^\kappa$. Then:*

- (i) $T_\alpha(h + g)(t) = T_\alpha(h)(t) + T_\alpha(g)(t)$,
- (ii) $T_\alpha(\lambda h)(t) = \lambda T_\alpha(h)(t)$ where $\lambda \in \mathbb{R}$,
- (iii) $T_\alpha(hg)(t) = T_\alpha(h)(t)g(t) + (h\sigma)(t)T_\alpha(g)(t) = T_\alpha(h)(t)(g\sigma)(t) + h(t)T_\alpha(g)(t)$,

$$(iv) T_\alpha \left(\frac{h}{g} \right) (t) = \frac{T_\alpha(h)(t)g(t) - h(t)T_\alpha(g)(t)}{g(t)(g\sigma)(t)}, \text{ where } g(t)(g\sigma)(t) \neq 0.$$

Lemma 2.3 [10] *Let $\alpha \in (0, 1]$ and $g : \mathbb{T} \rightarrow \mathbb{R}$ be a continuous and CF differentiable function of order α at $t \in \mathbb{T}^\kappa$ where $h : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable. Then there exists a $c \in \mathbb{J}$ such that*

$$T_\alpha(hog)(t) = h'(g(c))T_\alpha(g)(t).$$

Now let us recall the definition of the α -CF integral. Let $h : \mathbb{T} \rightarrow \mathbb{R}$ be a regulated function. Then the α -CF integral of h , $0 < \alpha \leq 1$, is defined by

$$\int h(t)\Delta^\alpha t = \int h(t)t^{\alpha-1}\Delta t. \tag{2.3}$$

The α -CF integral of h reduces to the classical CF integral, which is given by Khalil et al. [23] for $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$. Furthermore, we get the definition of the indefinite integral on \mathbb{T} for $\alpha = 1$ [13]. If indefinite α -CF integral of h order α is denoted by

$$H_\alpha(t) = \int h(t)\Delta^\alpha t,$$

then the Cauchy α -CF integral of h is defined by

$$\int_a^b h(t)\Delta^\alpha t = H_\alpha(b) - H_\alpha(a),$$

for all $a, b \in \mathbb{T}$.

3. Some spectral properties of the CF-SL equation on time scales

It is well known that (1.1)–(1.3) have only simple eigenvalues and the eigenfunctions are orthogonal when $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$ [25]. The following results generalize these basic consequences for the problem (1.1)–(1.3) to the CF

case by using the inner product on $L_2^\alpha \mathbb{J} = \left\{ f : \int_{\rho(a)}^b f^2(t)\Delta^\alpha t < \infty \right\}$. Let us first give some lemmas to be used in the proof of the main theorems.

Lemma 3.1 *Let $h, g : \mathbb{T} \rightarrow \mathbb{R}$ be continuous functions, $a, b \in \mathbb{T}$ and $\alpha \in (0, 1]$. Then:*

$$(i) \int_a^b T_\alpha(h)(t)g(t)\Delta^\alpha t = h(b)g(b) - h(a)g(a) - \int_a^b h^\sigma(t)T_\alpha(g)(t)\Delta^\alpha t;$$

$$(ii) \int_a^b h(t)T_\alpha(g)(t)\Delta^\alpha t = h(b)g(b) - h(a)g(a) - \int_a^b T_\alpha(h)(t)g^\sigma(t)\Delta^\alpha t.$$

Proof The proof can be easily obtained using a similar procedure as in [13]. □

Lemma 3.2 [10] *If $h : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is continuous and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} h(s) \Delta^\alpha s = h(t) \mu(t) t^{\alpha-1}.$$

Proposition 3.3 *Supposing that $a, b \in \mathbb{T}$, $a < b$, $\alpha \in (0, 1]$, and h is continuous at $t \in [a, b]$, then we have*

$$\int_a^b h(t) \Delta^\alpha t = [\sigma(a) - a] a^{\alpha-1} h(a) + \int_{\sigma(a)}^b h(t) \Delta^\alpha t.$$

Proof It can be easily proved by following a similar procedure as in [4]. □

Proposition 3.4 [4] *Let $[a, b] \subset \mathbb{T}$ and h be an increasing continuous function on $[a, b]$. If the extension of h is given in the below form:*

$$H(s) = \begin{cases} h(s), & s \in \mathbb{T} \\ h(t), & s \in (t, \sigma(t)) \notin \mathbb{T} \end{cases},$$

then we have

$$\int_a^b h(t) \Delta t \leq \int_a^b H(t) dt.$$

Through a similar procedure, we can rewrite the above inequality for the double case as follows:

$$\int_a^b \int_c^d h(t) \Delta t \Delta s \leq \int_a^b \int_c^d H(t) dt ds,$$

where $[a, b] \times [c, d] \subset \mathbb{T} \times \mathbb{T}$.

Theorem 3.5 *The CF-SL operator is self-adjoint on $L_2^\alpha \mathbb{J}$.*

Proof Let $x = x(t, \lambda)$ and $y = y(t, \lambda)$ be the solutions of (1.1)–(1.3), where $t \in \mathbb{J}$ is right dense. Using the inner product on $L_2^\alpha \mathbb{J}$ and integration by parts yields

$$\begin{aligned}
 \langle L_\alpha x, y \rangle &= \int_{\rho(a)}^b (L_\alpha x(t)) y(t) \Delta^\alpha t = - \int_{\rho(a)}^b T_\alpha [T_\alpha(x(t))] y(t) \Delta^\alpha t \\
 &= -T_\alpha(x(t))y(t) \Big|_{\rho(a)}^b + \int_{\rho(a)}^b T_\alpha(x(t))T_\alpha(y(t)) \Delta^\alpha t \\
 &= -T_\alpha(x(b))y(b) + T_\alpha(x(\rho(a)))y(\rho(a)) + T_\alpha(y(t))x(t) \Big|_{\rho(a)}^b - \int_{\rho(a)}^b T_\alpha [T_\alpha(y(t))] x(t) \Delta^\alpha t \\
 &= \int_{\rho(a)}^b (L_\alpha y(t)) x(t) \Delta^\alpha t \\
 &= \langle x, L_\alpha y \rangle .
 \end{aligned}$$

It yields the self-adjointness of the CF-SL operator on $L_2^\alpha \mathbb{J}$. □

Theorem 3.6 *All eigenvalues of the problem (1.1)–(1.3) are real.*

Proof Let $\bar{\lambda}$ be a complex eigenvalue and $\bar{y}(t)$ be an eigenfunction of the problem (1.1)–(1.3) related to $\bar{\lambda}$. Consider the below operators:

$$L_\alpha y = -T_\alpha (T_\alpha(y(t))) = \lambda \phi(t) f(t, y(t)), \tag{3.1}$$

$$L_\alpha \bar{y} = -T_\alpha (T_\alpha(\bar{y}(t))) = \bar{\lambda} \phi(t) f(t, \bar{y}(t)). \tag{3.2}$$

Multiplying (3.1) and (3.2) by \bar{y}^σ and y^σ , respectively, and then subtracting the resulting equality yields:

$$y^\sigma T_\alpha (T_\alpha(\bar{y}(t))) - \bar{y}^\sigma T_\alpha (T_\alpha(y(t))) = \phi(t) [\lambda f(t, y(t))\bar{y}^\sigma - \bar{\lambda} \phi(t) f(t, \bar{y}(t))y^\sigma]. \tag{3.3}$$

On the other hand, by using the multiplication rule for the CF derivative, we get:

$$T_\alpha [yT_\alpha(\bar{y}) - \bar{y}T_\alpha(y)] = \phi(t) [\lambda f(t, y(t))\bar{y}^\sigma - \bar{\lambda} f(t, \bar{y}(t))y^\sigma]. \tag{3.4}$$

Finally, if we take the α -CF integral of the last equality from $\rho(a)$ to b , we get

$$[yT_\alpha(\bar{y}) - \bar{y}T_\alpha(y)] \Big|_{\rho(a)}^b = \int_{\rho(a)}^b \phi(t) [\lambda f(t, y(t))\bar{y}^\sigma - \bar{\lambda} f(t, \bar{y}(t))y^\sigma] \Delta^\alpha t,$$

and by using the conditions (1.2)–(1.3),

$$\int_{\rho(a)}^b \phi(t) [\lambda f(t, y(t))\bar{y}^\sigma - \bar{\lambda} f(t, \bar{y}(t))y^\sigma] \Delta^\alpha t = 0, \tag{3.5}$$

which holds if and only if $\lambda = \bar{\lambda}$ where $\phi(t) \neq 0$ for all $t \in \mathbb{J}$. Hence, eigenvalues of the problem (1.1)–(1.3) are purely real. □

Theorem 3.7 Let $x, y \in C_\alpha(\mathbb{J}, \mathbb{R})$ be the eigenfunctions of the problem (1.1)–(1.3). Then we have:

a) $(L_\alpha x)y^\sigma - (L_\alpha y)x^\sigma = T_\alpha(W(x, y))$ on $\mathbb{J} \cap \mathbb{T}$.

b) $\langle L_\alpha x, y^\sigma \rangle - \langle L_\alpha y, x^\sigma \rangle = W(x, y)(b) - W(x, y)(\rho(a))$, where $W(x, y) = xT_\alpha(y) - yT_\alpha(x)$ is the Wronskian of x and y .

Proof Here, $C_\alpha(\mathbb{J}, \mathbb{R})$ is the set of all functions whose α -derivatives are all continuous on \mathbb{J} .

a) By definition of the Wronskian and the CF derivative, we get

$$\begin{aligned} T_\alpha(W(x, y)) &= T_\alpha[xT_\alpha(y) - yT_\alpha(x)] \\ &= x^\sigma T_\alpha(T_\alpha(y)) - y^\sigma T_\alpha(T_\alpha(x)) \\ &= (L_\alpha x)y^\sigma - (L_\alpha y)x^\sigma. \end{aligned}$$

b) Using the definition of the Wronskian and the inner product on $L_2^\alpha \mathbb{J}$, we have

$$\begin{aligned} \langle L_\alpha x, y^\sigma \rangle - \langle L_\alpha y, x^\sigma \rangle &= \int_{\rho(a)}^b [(L_\alpha x)y^\sigma - (L_\alpha y)x^\sigma] \Delta^\alpha t \\ &= \int_{\rho(a)}^b \{-T_\alpha(T_\alpha(x))y^\sigma + T_\alpha(T_\alpha(y))x^\sigma\} \Delta^\alpha t \\ &= \int_{\rho(a)}^b T_\alpha(W(x, y)) \Delta^\alpha t \\ &= W(x, y)(b) - W(x, y)(\rho(a)). \end{aligned}$$

Hence, the proof is complete. □

Theorem 3.8 The eigenfunctions $x(t, \lambda_1)$ and $y(t, \lambda_2)$ of the problem (1.1)–(1.3) corresponding to distinct eigenvalues λ_1 and λ_2 are orthogonal on $L_2^\alpha \mathbb{J}$, i.e.

$$\int_{\rho(a)}^b \phi(t) [\lambda_2 f(t, y(t, \lambda_2))x^\sigma(t, \lambda_1) - \lambda_1 f(t, x(t, \lambda_1))y^\sigma(t, \lambda_2)] \Delta^\alpha t = 0.$$

Proof Let us use the below equality:

$$T_\alpha[T_\alpha(x(t, \lambda_1))y(t, \lambda_2) - T_\alpha(y(t, \lambda_2))x(t, \lambda_1)] = \phi(t) [\lambda_2 f(t, y(t, \lambda_2))x^\sigma(t, \lambda_1) - \lambda_1 f(t, x(t, \lambda_1))y^\sigma(t, \lambda_2)].$$

Taking the α -CF integral of the last equality from $\rho(a)$ to b , we get

$$\int_{\rho(a)}^b \phi(t) [\lambda_2 f(t, y(t, \lambda_2))x^\sigma(t, \lambda_1) - \lambda_1 f(t, x(t, \lambda_1))y^\sigma(t, \lambda_2)] \Delta^\alpha t = 0,$$

for $\lambda_1 \neq \lambda_2$. It shows that the eigenfunctions $x(t, \lambda_1)$ and $y(t, \lambda_2)$ are always orthogonal on $L_2^\alpha \mathbb{J}$. □

4. Existence of the solution for the CF-SL equation on time scales

In this section, we get the asymptotic estimates of the eigenfunction of the problem (1.1)–(1.3) on \mathbb{T} . Then sufficient conditions are presented for the existence of the solution of the CF-SL problem (1.1)–(1.3) on \mathbb{T} where $y(\rho(a), \lambda) = \gamma \in \mathbb{R}$ and $t \in \mathbb{J}$ is right dense.

Theorem 4.1 *The eigenfunction $y(t, \lambda)$ is a solution of the problem (1.1)–(1.3) if and only if it satisfies the below equation:*

$$y(t, \lambda) = \gamma - \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} \phi(s) f(s, y(s)) \Delta^\alpha s \Delta^\alpha t_0.$$

Proof Let us consider the equation

$$T_\alpha (T_\alpha (y(t))) = -\lambda \phi(t) f(t, y(t)).$$

After taking the α -CF integral of this equation from $\rho(a)$ to t_0 , we get

$$\int_{\rho(a)}^{t_0} T_\alpha (T_\alpha (y(s))) \Delta^\alpha s = -\lambda \int_{\rho(a)}^{t_0} \phi(s) f(s, y(s)) \Delta^\alpha s.$$

Boundary condition (1.2) yields

$$T_\alpha (y(t_0)) = -\lambda \int_{\rho(a)}^{t_0} \phi(s) f(s, y(s)) \Delta^\alpha s. \tag{4.1}$$

By taking the α -CF integral of (4.1) from $\rho(a)$ to t , we obtain

$$\int_{\rho(a)}^t T_\alpha (y(t_0)) \Delta^\alpha t_0 = -\lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} \phi(s) f(s, y(s)) \Delta^\alpha s \Delta^\alpha t_0,$$

and with boundary condition (1.3),

$$y(t, \lambda) = \gamma - \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} \phi(s) f(s, y(s)) \Delta^\alpha s \Delta^\alpha t_0.$$

Thus, the proof is completed. Now we can give some existence results for the problem (1.1)–(1.3) using a similar way as in [4, 9, 26]. □

Theorem 4.2 *Let $f(t, y(t))$ be a continuous function such that $|f(t, y(t))| < M$ for $M > 0$, and the Lipschitz condition,*

$$\exists L > 0; \forall t \in \mathbb{J} \text{ and } x, y \in \mathbb{R}, \|f(t, x) - f(t, y)\| \leq L \|x - y\|,$$

holds. The problem (1.1)–(1.3) has a unique solution on \mathbb{J} when $\lambda \in [c, d)$ if there exists $R > N > 0$ such that

$$0 < \frac{N}{\min_{t \in \mathbb{J}} f(t, N\varpi(t))} = c < \frac{R}{\|\varpi(t)\| \max_{\substack{t \in \mathbb{J} \\ N\varpi(t) \leq y(t) \leq R}} f(t, y(t))} = d,$$

and $\phi(t)$ is bounded on \mathbb{J} , for $K > 0$, $|\phi(t)| \leq K$.

Proof Let us consider (1.1)–(1.3) and $S \subset C(\mathbb{J}, \mathbb{R})$ on \mathbb{J} . For $y \in S$, define

$$\|y(t)\| = \sup_{t \in \mathbb{J}} |y(t)|.$$

Here it can be easily seen that S is a Banach space with this norm. Define the subset $S(\mu)$ and operator A as below:

$$S(\mu) = \{x \in S : \|x\| \leq \mu\},$$

and

$$A(y(t)) = \gamma - \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} \phi(s) f(s, y(s)) \Delta^\alpha s \Delta^\alpha t_0, \tag{4.2}$$

respectively. By using Proposition 3.3, we get

$$\begin{aligned} |A(y(t))| &\leq \gamma + \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} |\phi(s)| |f(s, y(s))| \Delta^\alpha s \Delta^\alpha t_0 \\ &= \gamma + \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} |\phi(s)| |f(s, y(s))| s^{\alpha-1} t_0^{\alpha-1} \Delta s \Delta t_0 \\ &\leq \gamma + dKM \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} s^{\alpha-1} t_0^{\alpha-1} \Delta s \Delta t_0. \end{aligned}$$

By Proposition 3.4, we have

$$\begin{aligned} |A(y(t))| &\leq \gamma + dKM \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} s^{\alpha-1} t_0^{\alpha-1} ds dt_0 \\ &\leq \gamma + dKM \frac{b^{2\alpha}}{\alpha^2}. \end{aligned}$$

With the equality $\gamma + dKM \frac{b^{2\alpha}}{\alpha^2} = \mu$, we conclude that A is an operator from $S(\mu)$ to $S(\mu)$. In addition, for $x, y \in S(\mu)$, we obtain

$$\begin{aligned} \|A(x) - A(y)\| &= \left\| \gamma - \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} \phi(s) f(s, x(s)) \Delta^\alpha s \Delta^\alpha t_0 - \gamma + \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} \phi(s) f(s, y(s)) \Delta^\alpha s \Delta^\alpha t_0 \right\| \\ &\leq \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} |\phi(s)| |f(s, x(s)) - f(s, y(s))| s^{\alpha-1} t_0^{\alpha-1} \Delta s \Delta t_0 \\ &\leq dK \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} |f(s, x(s)) - f(s, y(s))| s^{\alpha-1} t_0^{\alpha-1} ds dt_0 \\ &\leq dKL \|x - y\| \frac{b^{2\alpha}}{\alpha^2}. \end{aligned}$$

Here, if $dKL \frac{b^{2\alpha}}{\alpha^2} < 1$, this will be a contraction map. It implies that (1.1)–(1.3) has a unique solution by the Banach fixed point theorem. \square

Theorem 4.3 *Let $f : \mathbb{J} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and bounded such that $|f(t, y(t))| \leq M$, $M > 0$ for all $t \in \mathbb{J}$ and $y \in \mathbb{R}$. Then (1.1)–(1.3) has a solution on \mathbb{J} .*

Proof The proof is given step by step using Schauder’s fixed point theorem to prove that A , which is defined in (4.2), has a fixed point.

Step1: A is continuous. Let y_n be a sequence such that $y_n \rightarrow y$ in $C(\mathbb{J}, \mathbb{R})$. Then, for each $t \in \mathbb{J}$,

$$\begin{aligned} |A(y_n(t)) - A(y(t))| &\leq \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} |\phi(s)| |f(s, y_n(s)) - f(s, y(s))| \Delta^\alpha s \Delta^\alpha t_0 \\ &\leq \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} |\phi(s)| \sup_{\rho(a) \leq t \leq b} |f(s, y_n(s)) - f(s, y(s))| s^{\alpha-1} t_0^{\alpha-1} \Delta s \Delta t_0 \\ &\leq dK \frac{b^{2\alpha}}{\alpha^2} \|f(s, y_n(s)) - f(s, y(s))\|. \end{aligned}$$

Since f is continuous function, we have

$$\|A(y_n(t)) - A(y(t))\| \leq dK \frac{b^{2\alpha}}{\alpha^2} \|f(s, y_n(s)) - f(s, y(s))\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Step2: The map A sends bounded sets into bounded sets in $C(\mathbb{J}, \mathbb{R})$. It is sufficient to denote that there exists a positive constant l , such that $y \in B_\mu = \{y \in C(\mathbb{J}, \mathbb{R}) : \|y\| \leq \mu\}$, and we have $\|A(y)\| \leq l$ for any μ .

$$\begin{aligned} |A(y(t))| &\leq \gamma + \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} |\phi(s)| |f(s, y_n(s))| \Delta^\alpha s \Delta^\alpha t_0 \\ &= \gamma + \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} |\phi(s)| |f(s, y_n(s))| s^{\alpha-1} t_0^{\alpha-1} ds dt_0 \\ &\leq \gamma + dKM \frac{b^{2\alpha}}{\alpha^2} = l. \end{aligned}$$

Step3: The map A sends bounded sets into equicontinuous sets of $C(\mathbb{J}, \mathbb{R})$. Let $t_1, t_2 \in \mathbb{J}, t_1 < t_2$ and B_μ be a bounded subset of $C(\mathbb{J}, \mathbb{R})$ as in step 2, and let $y \in B_\mu$. Then

$$\begin{aligned} |A(y(t_2)) - A(y(t_1))| &\leq \lambda KM \left[\int_{\rho(a)}^{t_1} \int_{\rho(a)}^{t_0} s^{\alpha-1} t_0^{\alpha-1} ds dt_0 - \int_{\rho(a)}^{t_2} \int_{\rho(a)}^{t_0} s^{\alpha-1} t_0^{\alpha-1} ds dt_0 \right] \\ &\leq \frac{dKM}{\alpha^2} \left[\frac{(t_1^{2\alpha} - t_2^{2\alpha})}{2} - (\rho(a))^\alpha (t_1^\alpha - t_2^\alpha) \right] \rightarrow 0, \text{ as } t_1 \rightarrow t_2. \end{aligned}$$

By step 1, step 2, and the Arzela–Ascoli theorem, we see that $A : C(\mathbb{J}, \mathbb{R}) \rightarrow C(\mathbb{J}, \mathbb{R})$ is continuous and completely continuous.

Step 4: Now it remains to show that the set $\Omega = \{y \in C(\mathbb{J}, \mathbb{R}) : y(t) = \eta A(y(t)), 0 < \eta < 1\}$ is bounded. Let $y \in \Omega$. Thus, for each $t \in \mathbb{J}$, we have

$$\begin{aligned} |y(t)| &\leq \eta \left| \gamma - \lambda \int_{\rho(a)}^t \int_{\rho(a)}^{t_0} \phi(s) f(s, y(s)) \Delta^\alpha s \Delta^\alpha t_0 \right| \\ &\leq \eta \gamma + \eta \lambda KM \frac{b^{2\alpha}}{\alpha^2} \\ &< \gamma + dKM \frac{b^{2\alpha}}{\alpha^2}. \end{aligned}$$

By Schauder’s fixed point theorem, we can conclude that A has a fixed point, which represents a solution for the problem (1.1)–(1.3). As an explanation for these theorems, we give an example below. □

Example 4.4 Consider the following CF-SL boundary value problem:

$$-T_{0.5}(T_{0.5}(y(t))) = \frac{t}{18} \frac{y}{y+3}, t \in [0, 1], \tag{4.3}$$

$$\begin{aligned} T_{0.5}(y(0)) &= 0, \\ \delta y(1) + \beta T_{0.5}(y(1)) &= 0, \end{aligned} \tag{4.4}$$

where $\lambda = 1$, $f(t, y(t)) = \frac{t}{18} \frac{y}{y+3}$, $\phi(t) = 1$, and $d = 1, K = 1$. Because of these choices, $f(t, y(t))$ is continuous on $[0, 1]$ and $|f(t, y(t))| < \frac{1}{18} = M$. Let $x, y \in \mathbb{T}$. Then we obtain

$$|f(t, x) - f(t, y)| \leq \frac{1}{6} |x - y|,$$

and $L = \frac{1}{6}$. Therefore, the Lipschitz condition holds and $A : S(\frac{11}{9}) \rightarrow S(\frac{11}{9})$ is an operator mentioned in Theorem 4.2 for $\gamma + dKM \frac{b^{2\alpha}}{\alpha^2} = \frac{11}{9} = \mu$. Furthermore, it is easy to prove that

$$dKL \frac{b^{2\alpha}}{\alpha^2} = \frac{2}{3} < 1,$$

where $\gamma = 1$. As a result, all conditions of Theorem 4.2 are satisfied and the problem (4.3)–(4.4) has a unique solution.

5. Conclusion

Fractional SL problems attract the attention of many authors. In this study, we have considered the CF-SL dynamic equation with boundary conditions on \mathbb{T} in order to obtain the results of the eigenfunctions for this problem. Then we have examined the existence of the solution. Finally, we have tried to explain these existence theorems with an example.

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