Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1361 – 1365

doi:10.3906/mat-1706-42

On strongly autinertial groups

Cansu BETİN ONUR*
Department of Mathematics, Atılım University, Ankara, Turkey

Received: 12.06.2017 • **Accepted/Published Online:** 29.11.2017 • **Final Version:** 08.05.2018

Abstract: A subgroup X of G is said to be inert under automorphisms (autinert) if $|X : X^{\alpha} \cap X|$ is finite for all $\alpha \in Aut(G)$ and it is called strongly autinert if $| < X, X^{\alpha} >: X|$ is finite for all $\alpha \in Aut(G)$. A group is called strongly autinertial if all subgroups are strongly autinert. In this article, the strongly autinertial groups are studied. We characterize such groups for a finitely generated case. Namely, we prove that a finitely generated group G is strongly autinertial if and only if one of the following hold:

i) G is finite;

ii) $G = \langle a \rangle \ltimes F$ where F is a finite subgroup of G and $\langle a \rangle$ is a torsion-free subgroup of G.

Moreover, in the preliminary part, we give basic results on strongly autinert subgroups.

Key words: Autinert subgroups, inertial groups, FC groups, VTA groups, virtually cyclic groups

1. Introduction

Let X be a subgroup of a group G. If the index $|X : X^g \cap X|$ is finite for all $g \in G$, then X is said to be an inert subgroup of G. A group G is called totally inert, in other words inertial, if every subgroup of G is inert. The inert subgroups were introduced by Belyaev [1] in 1993 and studied by many authors in a variety of aspects. In [2] Belyaev et al. established inertial groups and these were later studied by Dixon et al. [7] and Robinson [10].

In 2013, De Facto et al. [4] introduced the concept of a strongly inert subgroup of a group and investigated the structure of strongly inertial groups. A subgroup X of G is called strongly inert if $| < X, X^g >: X|$ is finite for all $g \in G$. A group G is called strongly inertial if all subgroups are strongly inert. One may see that every strongly inert subgroup is also an inert subgroup. However, the converse is not true.

In this article, the concept of inert subgroups is extended by means of automorphisms. Let us call a subgroup X of G inert under automorphisms, in short autinert, if $|X : X^{\alpha} \cap X|$ is finite for all $\alpha \in Aut(G)$, and it is called strongly autinert if $| < X, X^{\alpha} >: X|$ is finite for all $\alpha \in Aut(G)$. In a similar manner, we say that G is strongly autinertial if all subgroups are strongly autinert.

It is obvious that every autinert subgroup is inert and every strongly autinert subgroup is strongly inert. In the literature, a similar generalization is studied for FC-groups, called VTA-groups. A group G is said to be a virtually trivial automorphism group (VTA-group) if $|G : C_G(\alpha)|$ is finite for all $\alpha \in Aut(G)$. Menegazzo and Robinson gave in [8] the necessary and sufficient conditions for a group to be a VTA-group.

*Correspondence: cansu.betin@atilim.edu.tr
2010 *AMS Mathematics Subject Classification:* 20F24, 20E36, 20F50

In [5], de Giovanni proved that every FC-group is strongly inertial. It is natural to ask whether the result of de Giovanni can be drawn for VTA-groups. We answer this question positively in this article.

Theorem 1.1 *Let G be a VTA-group. Then for any subgroup X of G and for any automorphism $\alpha \in Aut(G)$, G has an α invariant subgroup H containing X such that $|H : X|$ is finite.*

Corollary 1.2 *Every VTA-group is strongly autinertial.*

In general, the converse is not true. A cyclic torsion-free group $G = \langle x \rangle$ has the automorphism group $Aut(G) = \{1, \alpha\}$ where $x^{\alpha} = x^{-1}$. Here $C_G(\alpha)$ is the trivial subgroup and so G is not a VTA-group. On the other hand, any subgroup of G is α-invariant. Therefore, G is strongly autinertial. Furthermore, the Prüfer p-group C_{p^∞} is an example of a locally finite infinitely generated strongly autinertial group that is not a VTA-group.

For finitely generated VTA-groups, Menegazzo and Robinson gave the following characterization.

Theorem 1.3 *[8] A finitely generated group G is a VTA-group if and only if it is either finite or the split extension of a finite group F by an infinite cyclic group $\langle x \rangle$ such that $\widehat{x}$, the image of x in $OutF$, is not conjugate to its inverse.*

We obtain using the technique of [8] the following result for finitely generated strongly autinertial groups.

Theorem 1.4 *A finitely generated group G is strongly autinertial if and only if one of the following hold:*

i) G is finite;

ii) $G = \langle a \rangle \ltimes F$ where F is a finite subgroup of G and $\langle a \rangle$ is a torsion-free subgroup of G.

Note that in the latter case, $F = \{1\}$ gives an infinite cyclic group. One consequence of Theorem 1.4 is that the class of finitely generated strongly autinertial groups is a proper subclass of virtually cyclic groups.

The generalization of the inert subgroup via homomorphisms has also attracted the attention of other authors. In [6], Dikranjan et al. gave such a generalization by endomorphisms. Namely, they defined the fully inert subgroups. For a group G, a subgroup X is called fully inert if $| < X, X^{\alpha} >: X|$ is finite for all endomorphisms α of G. However, their definition and results are on abelian groups. In addition, the finiteness conditions on characteristic closures were studied by Cutolo et al. [3]. In this article, a property called $\mathcal{P}^{\#}$ was established. A group G is said to satisfy the property $\mathcal{P}^{\#}$ if every subgroup has finite index in its characteristic closure. Clearly, if a group satisfies the property $\mathcal{P}^{\#}$, then it is strongly autinertial. However, a strongly autinertial group need not satisfy $\mathcal{P}^{\#}$. For example, $(\mathbb{Q}, +)$ is strongly autinertial (see [6]). On the other hand, it is a characteristically simple group and it has no proper finite index subgroup. Therefore, it does not satisfy $\mathcal{P}^{\#}$.

2. Preliminaries

In this section we present some basic properties of autinert subgroups.

Lemma 2.1 *Every strongly autinert subgroup is autinert.*

Proof Let X be a strongly autinert subgroup of a group G. Take any $\alpha \in Aut(G)$. Then X^α is also strongly autinert in G. Therefore, $|\langle X, X^\alpha\rangle : X|$ and $|\langle X, X^\alpha\rangle : X^\alpha|$ are finite indexes. Hence, $|X : X \cap X^\alpha| \leq |\langle X, X^\alpha\rangle : X \cap X^\alpha|$ is finite. □

Lemma 2.2 *Let G be a group and $\alpha \in Aut(G)$ such that $|G : C_G(\alpha)|$ is finite. Then the index $|X : X \cap X^\alpha|$ is finite for any subgroup X of G.*

Proof Set $C_X(\alpha) = \{a \in X | a^\alpha = a\}$. Then $C_X(\alpha) = C_G(\alpha) \cap X$. As $|G : C_G(\alpha)|$ is finite, $|X : C_X(\alpha)| = |X : X \cap C_G(\alpha)|$ is finite. Note that $C_X(\alpha) \leq X \cap X^\alpha$. Thus, $|X : X \cap X^\alpha|$ is finite. □

Lemma 2.3 *Let G be a group and X be an autinert subgroup of G. If $XX^\alpha = X^\alpha X$ for all α in $Aut(G)$, then X is a strongly autinert subgroup of G.*

Proof Let X be an autinert subgroup of G and take any α in $Aut(G)$. Then X^α is also autinert. Hence, $|\langle X, X^\alpha\rangle : X| = |XX^\alpha : X| = |X^\alpha : X \cap X^\alpha| < \infty$. □

Lemma 2.4 *Let G be a group. If two permutable subgroups X, Y of G are autinert in G, then $\langle X, Y\rangle$ is a strongly autinert subgroup of G.*

Proof By Lemma 2.3 the subgroups X and Y are strongly autinert in G. Take any automorphism α of G. Then $|\langle X, X^\alpha\rangle : X|$ and $|\langle Y, Y^\alpha\rangle : Y|$ are finite. Observe that for subgroups A, B, N of G, $B \leq A$ and N is permutable such that $|A : B| < \infty$ satisfy $|AN : BN| < \infty$. Therefore, we obtain $|XX^\alpha YY^\alpha : XY| < \infty$. □

Lemma 2.5 *Let G be a group and X be a strongly autinert subgroup of G. If X has a finite indexed subgroup Y, then Y is a strongly autinert subgroup of G.*

Proof $|\langle Y, Y^\alpha\rangle : Y| \leq |\langle X, X^\alpha\rangle : Y| = |\langle X, X^\alpha\rangle : X||X : Y| < \infty$. □

Lemma 2.6 *Let G be a group and H be a finite characteristic subgroup of G such that G/H is strongly autinertial. Then G is strongly autinertial.*

Proof Take any subgroup X of G and any G-automorphism α. Then α induces an automorphism $\overline{\alpha}$ on G/H such that $(XH/H)^{\overline{\alpha}} = X^\alpha H/H$. Thus, $|\langle X^\alpha H/H, XH/H\rangle : XH/H|$ is finite. As H is finite, $|\langle X^\alpha, X\rangle : X|$ is finite. □

Lemma 2.7 *Let G be a group and let X_1 and X_2 be two strongly autinert subgroups of G. Then $X_1 \cap X_2$ is a strongly autinert subgroup of G.*

Proof Take any G-automorphism α and set $K = X_1 \cap X_2$. Then $\langle K^\alpha, K\rangle$ is a subgroup of $\langle {X_i}^\alpha, X_i\rangle$ for $i = 1, 2$. As X_1 and X_2 are strongly autinert, the indexes $|\langle K^\alpha, K\rangle : \langle K^\alpha, K\rangle \cap X_i|$ are finite for $i = 1, 2$. Thus, $|\langle K^\alpha, K\rangle : K| = |\langle K^\alpha, K\rangle : \langle K^\alpha, K\rangle \cap X_1 \cap X_2|$ is finite. □

3. Proofs of the main results

The proof of Theorem 1.1:

Proof Let G be a VTA-group. Observe that every VTA-group is an FC-group (also noted in [8]). Take any subgroup X and any automorphism α of G. Then by Corollary 2 of [8] α has a finite order, say m. Set $H =< X, X^{\alpha}, \dots, X^{\alpha^{m-1}} >$ and $N = Core_H(C_H(\alpha))$. Then H is invariant under α and N is a finite indexed subgroup of H. Moreover, $X^{\alpha^i} \cap N = X \cap N$ for $1 \leq i \leq m-1$. Hence, $H/(X \cap N) = \langle X/(X \cap N), X^{\alpha}/(X \cap N), \dots, X^{\alpha^{m-1}}/(X \cap N)\rangle$ is a finitely generated FC-group generated by periodic elements. As periodic elements form a fully invariant subgroup of an FC-group, $H/(X \cap N)$ is periodic and so finite. Therefore, $|H : X|$ is finite.

□

The proof of Theorem 1.4:

Proof Let G be an infinite noncyclic finitely generated strongly autinertial group. Then, by Theorem 24 of [5], $G/Z(G)$ is finite. Hence, the torsion subgroup $T(G)$ of G is finite and $Z(G)$ is a finitely generated infinite abelian group. Therefore, $Z(G)$ is a direct sum of a finite subgroup and a finitely generated free abelian subgroup. Let A be a free abelian subgroup of $Z(G)$ such that $|G : A| = n$ and $rank(A) = k$ for some natural numbers n and k. Suppose that $k > 1$. Say $A = \langle a_1\rangle \times \cdots \times \langle a_k\rangle$. Then the map $\alpha_0 : A \to A$ defined by ${a_1}^{\alpha_0} = a_1 a_2^n$ and ${a_i}^{\alpha_0} = a_i$ for all $i \neq 1$ is an automorphism of A, which acts trivially on A/A^n. Let η be the endomorphism of A defined by $(a^{\eta})^n = [a, \alpha_0] = a^{-1}a^{\alpha_0}$. Then the map $\alpha : G \to G$ defined by $g^{\alpha} = g(g^n)^{\eta}$ gives an automorphism of G (see also the proof of Theorem 3.3 of [9]). Now set two infinite subgroups $X = \langle a_1\rangle$ and $Y = \langle {a_2}^n\rangle$ of G.

Then $X \cap Y$ is the trivial subgroup and $\langle X, X^{\alpha}\rangle = \langle a_1, a_2^n\rangle = XY$. As G is strongly autinertial, we get the contradicting result $|Y| = |Y : Y \cap X| = |XY : X| < \infty$. Consequently, we observe that A is cyclic. Furthermore, the transfer map $\tau : G \to A$ defined by $g^{\tau} = g^n$ is a homomorphism where $Ker(\tau)$ is the torsion subgroup of G. Hence, $G/T(G)$ is an infinite cyclic group and so G is finite-by-cyclic as required.

Conversely, let G satisfy one of the above conditions. Obviously every finite and every cyclic group is strongly autinertial. Assume that $G = \langle a\rangle \ltimes F$ where F is a nontrivial finite group and $\langle a\rangle$ is an infinite cyclic group. Take any subgroup X of G and any element α in $Aut(G)$. Then $\langle X, X^{\alpha}\rangle$ is a finitely generated group. Since G/F is abelian, G' is finite. Consequently, G is a finitely generated FC-group and so it is central-by-finite. Indeed, there exists a natural number m such that $Z(G)^m$ is a characteristic infinite cyclic subgroup of G containing G^n for some natural number n. Now $\langle X, X^{\alpha}\rangle/\langle X, X^{\alpha}\rangle^n$ is a finitely generated periodic FC-group. Therefore, it is a finite group. Furthermore, as the only automorphisms of $Z(G)^m$ are the trivial map and the map sending each element to its inverse, $\langle X, X^{\alpha}\rangle^n = \langle X^n, X^{\alpha n}\rangle = X^n \leq X$. It follows that $|\langle X, X^{\alpha}\rangle : X| < \infty$ as required. □

Acknowledgment

The author would like to thank the referees for their valuable comments.

References

[1] Belyaev VV. Inert subgroups in infinite simple groups. Siber Math J 1993; 34: 606-611.

[2] Belyaev VV, Kuzucuoglu M, Seckin M. Totally inert groups. Rend Sem Mat Univ Padova 1999; 102: 151-156.

[3] Cutolo G, Smith H, Wiegold J. Finiteness conditions on characteristic closures and cores of subgroups. J Group Theory 2009; 12: 591-610.

[4] De Falco M, De Giovanni F, Musella C, Trabelsi N. Strongly inertial groups. Comm Algebra 2013; 41: 2213-2227

[5] De Giovanni F. Some trends in the theory of groups with restricted conjugacy classes. Note Mat 2013; 33: 71-87.

[6] Dikranjan D, Giordano Bruno A, Salce L, Virili S. Fully inert subgroups of divisible abelian groups. J Group Theory 2013; 16: 915-939.

[7] Dixon MR, Evans MJ, Tortora A. On totally inert simple groups. Centr Eur J Math 2010; 8: 22-25

[8] Menegazzo F, Robinson DJS. A finiteness condition on automorphism groups. Rend Sem Mat Univ Padova 1987; 78: 267-277.

[9] Pettet MR. Finitely generated groups with virtually free automorphism groups. Proc Edinburgh Math Soc 1995; 38: 475-484.

[10] Robinson DJS. On inert subgroups of a group. Rend Sem Mat Univ Padova 2006; 115: 137-159