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Characterizations of *-DMP matrices over rings

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Abstract: Let R be a ring with involution $*$. $R^{m\times n}$ denotes the set of all $m\times n$ matrices over R. In this paper, we give a characterization of the pseudo core inverse of $A\in R^{n\times n}$ in the form of $A=GDH$, $N_r(G)=0$, $N_l(H)=0$, $D^2=D=D^*$, where $N_l(A)=\{x\in R^{1\times m}|xA=0\}$ and $N_r(A)=\{x\in R^{n\times 1}\mid Ax=0\}$. Then we obtain necessary and sufficient conditions for $A\in R^{n\times n}$, in the form of $A=GDH$, $N_r(G)=0$, $N_l(H)=0$, $D^2=D=D^*$, to be *-DMP. If R is a principal ideal domain (resp. semisimple Artinian ring), then matrices expressed as that form include all $n\times n$ matrices over R.

Key words: *-DMP matrix, pseudo core inverse, core-EP inverse, Drazin inverse, Moore–Penrose inverse, factorization

1. Introduction

Let R be a ring with involution $*$. $R^{m\times n}$ denotes the set of all $m\times n$ matrices over R. Suppose $A=(a_{ij})\in R^{m\times n}$. Put $A^*=(a^*_{ji})$. We consider the following equations:

$$(1)\ AXA=A,$$

$$(1^k)\ A^kXA=A^k \text{ for some positive integer } k,$$

$$(2)\ XAX=X,$$

$$(3)\ (AX)^*=AX,$$

$$(4)\ (XA)^*=XA,$$

$$(5)\ AX=XA.$$

The Moore–Penrose inverse of A, denoted by $A^\dagger$, is the unique matrix X satisfying the above (1), (2), (3), and (4); the {1,3}-inverse (resp. {1,4}-inverse) of A, denoted by $A^{(1,3)}$ (resp. $A^{(1,4)}$), is the matrix X satisfying the above (1) and (3) (resp. (1) and (4)). Let $A\in R^{n\times n}$, the group inverse of A, denoted by $A^\#$, be the unique matrix X satisfying the above (1), (2), and (5); the Drazin inverse of A, denoted by A^D, is the unique matrix X satisfying the above (1^k), (2), and (5); the smallest positive integer k satisfying the above (1^k), (2), and (5) is called the Drazin index of A, denoted by $i(A)$.

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The core inverse of $A \in R^{n\times n}$, denoted by $A^{\#}$, is the unique solution to equations

$$XA^2 = A, \quad AX^2 = X, \quad \text{and} \quad (AX)^* = AX \text{ (see [23])}.$$

We refer readers to [1] and [18] for a deep study of core inverses.

In [9], the authors introduced the notion of the pseudo core inverse, which extends the notion of core inverse to matrices of an arbitrary index. The pseudo core inverse of $A \in R^{n\times n}$, denoted by $A^{\textcircled{D}}$, is the unique solution to equations

$$XA^{m+1} = A^m \text{ for some positive integer } m,\ AX^2 = X \text{ and } (AX)^* = AX.$$

The smallest positive integer m satisfying the above equations is called the pseudo core index of A. If A is pseudo core invertible, then it must be Drazin invertible, and the pseudo core index coincides with the Drazin index (see [9]). Thus, here and subsequently, we denote the pseudo core index of A by $i(A)$. It is clear that if $i(A) = 1$, then the pseudo core inverse of A is the core inverse of A. Also, the pseudo core inverse extended the core-EP inverse [13] from complex matrices to matrices over rings in terms of equations (see [9]). For consistency and convenience, we use the terminology 'pseudo core inverse' throughout this paper.

Dually, the dual pseudo core inverse of $A \in R^{n\times n}$, denoted by $A_{\textcircled{D}}$, is the unique solution to equations $A^{m+1}X = A^m$ for some positive integer m, $X^2A = X$ and $(XA)^* = XA$ [9]. The smallest positive integer m satisfying the above equations is called the dual pseudo core index of A, denoted by $i(A)$ as well.

Let $A \in \mathbb{C}^{n\times n}$, A be EP if and only if $N(A) = N(A^*)$ if and only if $A^\dagger$ and $A^\#$ exist with $A^\dagger = A^\#$ (see [2, 20]), where $N(A)$ denotes the null space of A and A^* denotes the conjugate transpose of A.

Meanwhile, suppose that $A \in R^{n\times n}$, $N(A) = N(A^*)$ may not imply that $A^\dagger$ and $A^\#$ exist with $A^\dagger = A^\#$. Hartwig [10] defined that an element a in a *-regular ring (a regular ring with involution such that $a^*a = 0$ implies $a = 0$) is EP if and only if $aR = a^*R$, and he also proved its equivalence with the existence of $a^\#$ together with $a^\# = a^\dagger$. Patricio and Puystjens [15] introduced the notions of *-EP and *-gMP in rings with involution. They said that a is *-EP if $aR = a^*R$; a is *-gMP if $a^\dagger$ and $a^\#$ exist with $a^\dagger = a^\#$. As a matter of convenience, we denote a *-gMP element (resp. matrix) as an EP element (resp. matrix) in this paper. A is *-DMP if there exists a positive integer m such that A^m is EP [15]; A is *-DMP with index m if m is the smallest positive integer such that A^m is EP [15]. We refer readers to [10, 12, 14, 16, 18, 19] for a deep study of EP. In [8], the authors gave several characterizations of the *-DMP elements, utilizing the pseudo core inverse and dual pseudo core inverse, in semigroups with involution. Those results are also true for matrices over rings.

Letting $A \in R^{m\times n}$, we will use the following notations:

$$N_l(A) = \{x \in R^{1\times m} \mid xA = 0\} \quad \text{and} \quad N_r(A) = \{x \in R^{n\times 1} \mid Ax = 0\}.$$

From [17] and [11], we find:

(1) if R is a principal ideal domain, then for arbitrary $A \in R^{m\times n}$, it follows that $A = GH$, $N_r(G) = 0$, $N_l(H) = 0$ for some matrices $G \in R^{m\times r}$, $H \in R^{r\times n}$;

(2) if R is a semisimple Artinian ring, then for arbitrary $A \in R^{m\times n}$, it follows that $A = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$ for some matrices $G \in R^{m\times r}, D \in R^{r\times r}$, $H \in R^{r\times n}$.

In [2, 21, 22], the authors pointed out respectively that a factorization of a matrix A leads to explicit formulae for its Moore–Penrose inverse, Drazin inverse, and generalized inverse $A^{(2)}_{T,S}$. In [3, 4, 7, 16], the authors gave some characterizations of real or complex EP matrices, EP Hilbert operators, EP Banach algebra elements, and weighted EP Banach space operators through factorizations respectively.

Chen [5] gave the existence criteria and formulae for the {1,3}-inverse and Moore–Penrose inverse of A in the form of $A = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$. Chen [6] gave the equivalent conditions for A in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D$ to have a Drazin inverse.

Motivated by the above papers, in Section 2 we give the necessary and sufficient conditions for A in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$ to have a pseudo core inverse. Namely, we give necessary and sufficient conditions for matrices over principal ideal domains (resp. semisimple Artinian rings) to have pseudo core inverses. As applications, in Section 3, we give several characterizations of *-DMP matrices in the form of $A = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$.

2. Characterizations of pseudo core invertible matrices

In this section, we characterize pseudo core invertibility of A in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$. We begin with some useful lemmas.

Lemma 2.1 [5] *Let A, G, D, H be matrices over R such that $A = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$. Then we have the following facts:*
(1) if $A\{1,3\} \neq \emptyset$, then $DG^*GD + I - D$ is invertible;
(2) if $A\{1,4\} \neq \emptyset$, then $DHH^*D + I - D$ is invertible;
(3) $A^\dagger$ exists if and only if both $DG^*GD + I - D$ and $DHH^*D + I - D$ are invertible.
In this case, $A^\dagger = (DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^*$.

Lemma 2.2 [6] *Let A, G, D, H be matrices over R such that $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D$ for some positive integer k. Then the following are equivalent:*
(1) A^D exists with $i(A) \leq k$;
(2) $DHAGD + I - D$ is invertible.
In this case, $A^D = GD(DHAGD + I - D)^{-1}DH$.

Lemma 2.3 [9] *Let $A \in R^{n\times n}$. Then we have the following facts:*
(1) $A^{\textcircled{D}}$ exists if and only if A^D and $(A^k)^{(1,3)}$ exist, where $k \geq i(A)$.
In this case, $A^{\textcircled{D}} = A^D A^k (A^k)^{(1,3)}$.
(2) $A_{\textcircled{D}}$ exists if and only if A^D and $(A^k)^{(1,4)}$ exist, where $k \geq i(A)$.
In this case, $A_{\textcircled{D}} = (A^k)^{(1,4)} A^k A^D$.
(3) $A^{\textcircled{D}}$ and $A_{\textcircled{D}}$ exist if and only if A^D and $(A^k)^\dagger$ exist, where $k \geq i(A)$.
In this case, $A^{\textcircled{D}} = A^D A^k (A^k)^\dagger$ and $A_{\textcircled{D}} = (A^k)^\dagger A^k A^D$.

Applying Lemmas 2.1–2.3, we derive the following result, which is a characterization of the pseudo core inverse of A, in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$.

Theorem 2.4 *Let A, G, D, H be matrices over R such that $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$ for some positive integer k. Then the following are equivalent:*

(1) $A^{\textcircled{D}}$ *exists with* $i(A) \leq k$ *if and only if both* $DG^*GD + I - D$ *and* $DHAGD + I - D$ *are invertible. In this case,* $A^{\textcircled{D}} = GD(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}(GD)^*$.

(2) $A_{\textcircled{D}}$ *exists with* $i(A) \leq k$ *if and only if both* $DHH^*D + I - D$ *and* $DHAGD + I - D$ *are invertible. In this case,* $A_{\textcircled{D}} = (DH)^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}DH$.

Proof (1). From Lemma 2.3, $A^{\textcircled{D}}$ exists if and only if A^D and $(A^k)^{(1,3)}$ exist, where $k \geq i(A)$. Moreover, $A^{\textcircled{D}} = A^D A^k (A^k)^{(1,3)}$. Thus, the necessity of (1) is clear by Lemmas 2.1 and 2.2.

Conversely, since $DHAGD + I - D$ is invertible, according to Lemma 2.2, A^D exists with $i(A) \leq k$ and

$$(A^D)^k = [GD(DHAGD + I - D)^{-1}DH]^k = GD(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}DH.$$

Then $GDH[GD(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}DH]GDH = A^k(A^D)^kA^k = A^k$
$= GDH$.

Since $N_r(G) = 0$ and $N_l(H) = 0$, we have $DHGD(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}DHGD = D$.

From $(DHAGD + I - D)(DHGD + I - D) = DHAGDHGD + I - D = DHGDHAGD + I - D$
$= (DHGD + I - D)(DHAGD + I - D)$,

it follows that $(DHAGD + I - D)^{-1}(DHGD + I - D) = (DHGD + I - D)(DHAGD + I - D)^{-1}$. Thus,

$$\begin{aligned} D &= DHGD(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}DHGD \\ &= D(DHGD + 1 - D)(DHGD + I - D)^{k-1}(DHAGD + I - D)^{-k}(DHGD + I - D)D \\ &= D(DHGD + I - D)^{k+1}(DHAGD + I - D)^{-k}D \\ &= (DHGD + I - D)^{k+1}D(DHAGD + I - D)^{-k}. \end{aligned}$$

Then $(DHAGD)^k = D(DHAGD + I - D)^k = (DHGD + I - D)^{k+1}D = (DHGD)^{k+1}$.

Therefore, $(DHAGD + I - D)^k = (DHAGD)^k + I - D = (DHGD)^{k+1} + I - D$
$= (DHGD + I - D)^{k+1}$.

Since $DHAGD + I - D$ is invertible, we conclude that $DHGD + I - D$ is invertible.

Observe that $GD(DHGD + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^*$ is a {1,3}-inverse of A^k,
and then $A^k(A^k)^{(1,3)} = GD(DG^*GD + I - D)^{-1}(GD)^*$.

Hence, $A^{\textcircled{D}} = GD(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}(GD)^*$.

(2). It is analogous. □

Let D be the identity matrix in Theorem 2.4; then we have the following result.

Corollary 2.5 *Let* A, G, H *be matrices over* R *such that* $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$ *for some positive integer* k. *Then the following are equivalent:*

(1) $A^{\textcircled{D}}$ *exists with* $i(A) \leq k$ *if and only if both* G^*G *and* HAG *are invertible. In this case,* $A^{\textcircled{D}} = G(HAG)^{-1}HG(G^*G)^{-1}G^*$.

(2) $A_{\textcircled{D}}$ *exists with* $i(A) \leq k$ *if and only if both* HH^* *and* HAG *are invertible. In this case,* $A_{\textcircled{D}} = H^*(HH^*)^{-1}HG(HAG)^{-1}H$.

Letting D be the identity matrix and $k = 1$ in Theorem 2.4, then we get the following result, which characterizes the core invertibility of A.

Corollary 2.6 *Let A,G,H be matrices over R such that $A=GH$, $N_r(G)=0$, $N_l(H)=0$. Then the following are equivalent:*

(1) *$A^{\oplus}$ exists if and only if both G^*G and HG are invertible.*

In this case, $A^{\oplus}=G(HG)^{-1}(G^*G)^{-1}G^*$.

(2) *$A_{\oplus}$ exists if and only if both HH^* and HG are invertible.*

In this case, $A_{\oplus}=H^*(HH^*)^{-1}(HG)^{-1}H$.

3. Characterizations of *-DMP matrices

Pearl [16] pointed out that if A, G, H are complex matrices with $A=GH$, $N_r(G)=0$, $N_l(H)=0$, then A is EP if and only if $G(G^*G)^{-1}G^*=H^*(HH^*)^{-1}H$. Drivaliaris et al. [7] obtained several characterizations of EP operators in Hilbert spaces through operator factorizations. Boasso [3] gave necessary and sufficient conditions for an operator T to be EP in Banach spaces under the assumptions that $T^{\dagger}$ exists and T is of a operator factorization.

Recall that A is symmetric if $A=A^*$. In what follows, we show several equivalent conditions for $A\in R^{n\times n}$, in the form of $A^k=GDH$, $N_r(G)=0$, $N_l(H)=0$, $D^2=D=D^*$, to be *-DMP. We begin with an auxiliary lemma.

Lemma 3.1 [8, 12, 15] *Let $A\in R^{n\times n}$. Then the following conditions are equivalent:*

(1) *A is *-DMP with index k;*

(2) *A^D exists with $i(A)=k$ and AA^D is symmetric;*

(3) *k is the smallest positive integer such that $(A^k)^{\dagger}$ exists with $A^k(A^k)^{\dagger}=(A^k)^{\dagger}A^k$;*

(4) *$A^{\textcircled{D}}$ exists with $i(A)=k$ and $A^{\textcircled{D}}=A^D$;*

(5) *$A_{\textcircled{D}}$ exists with $i(A)=k$ and $A_{\textcircled{D}}=A^D$;*

(6) *A^D exists with $i(A)=k$, $(A^k)^{\dagger}$ exist and $(A^D)^k=(A^k)^{\dagger}$;*

(7) *$A^{\textcircled{D}}$ and $A_{\textcircled{D}}$ exist with $i(A)=k$ and $A^{\textcircled{D}}=A_{\textcircled{D}}$.*

Applying Lemma 3.1, we obtain several characterizations of *-DMP matrices sequentially. First, we have the following result.

Theorem 3.2 *Let A,G,D,H be matrices over R such that $A^k=GDH$, $N_r(G)=0$, $N_l(H)=0$, $D^2=D$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if $DHAGD+I-D$ is invertible and one of the following equivalent conditions holds:*

(1) *$GD(DHGD+I-D)^{-1}DH$ is symmetric;*

(2) *$DH=DH[GD(DHGD+I-D)^{-1}DH]^*$;*

(3) *$GD=[GD(DHGD+I-D)^{-1}DH]^*GD$.*

Proof From Lemma 3.1, A is *-DMP with index $\leq k$ if and only if A^D exists with $i(A)\leq k$ and AA^D is symmetric. A^D exists with $i(A)\leq k$ if and only if $DHAGD+I-D$ is invertible by Lemma 2.2. According to the proof of Theorem 2.4, we have

$$(A^D)^k=[GD(DHAGD+I-D)^{-1}DH]^k=GD(DHGD+I-D)^{k-1}(DHAGD+I-D)^{-k}DH,$$

and $DHGD+I-D$ is invertible with $(DHAGD+I-D)^{-k}=(DHGD+I-D)^{-(k+1)}$.

Thus,

$$(A^D)^k = GD(DHGD + I - D)^{-2}DH.$$

Then $AA^D = A^k(A^D)^k = GDHGD(DHGD + I - D)^{-2}DH = GD(DHGD + I - D)^{-1}DH$. Applying Lemma 3.1, A is *-DMP if and only if (1) holds.

(1) $\Rightarrow$ (2). Since $[GD(DHGD + I - D)^{-1}DH]^* = GD(DHGD + I - D)^{-1}DH$, then

$$\begin{aligned} DH &= (DHGD + I - D)(DHGD + I - D)^{-1}DH = DHGD(DHGD + I - D)^{-1}DH \\ &= DH[GD(DHGD + I - D)^{-1}DH]^*. \end{aligned}$$

(2) $\Rightarrow$ (1). Equality $DH = DH[GD(DHGD + I - D)^{-1}DH]^*$ yields that $GD(DHGD+I-D)^{-1}DH = GD(DHGD+I-D)^{-1}DH[GD(DHGD+I-D)^{-1}DH]^*$. Hence $GD(DHGD+I-D)^{-1}DH$ is symmetric.

(1) $\Leftrightarrow$ (3). It is analogous. □

Let D be the identity matrix in Theorem 3.2, and then we have the following result.

Corollary 3.3 *Let A, G, H be matrices over R such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if HAG is invertible and one of the following equivalent conditions holds:*

(1) *$G(HG)^{-1}H$ is symmetric;*

(2) *$H = H[G(HG)^{-1}H]^*$;*

(3) *$G = [G(HG)^{-1}H]^*G$.*

Letting D be the identity matrix and $k = 1$ in Theorem 3.2, then we get the following result, which gives a characterization for A to be EP.

Corollary 3.4 *Let A, G, H be matrices over R such that $A = GH$, $N_r(G) = 0$, $N_l(H) = 0$. Then A is EP if and only if HG is invertible and one of the following equivalent conditions holds:*

(1) *$G(HG)^{-1}H$ is symmetric;*

(2) *$H = H[G(HG)^{-1}H]^*$;*

(3) *$G = [G(HG)^{-1}H]^*G$.*

The following result gives the second characterization for A, in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$, to be *-DMP.

Theorem 3.5 *Let A, G, D, H be matrices over R such that $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if $DG^*GD + I - D$ and $DHH^*D + I - D$ are invertible with*

$$GD(DG^*GD + I - D)^{-1}(GD)^* = (DH)^*(DHH^*D + I - D)^{-1}DH.$$

Proof According to Lemma 3.1, A is *-DMP with index $\leq k$ if and only if there exists a positive integer k such that $(A^k)^\dagger$ exists with $A^k(A^k)^\dagger = (A^k)^\dagger A^k$, by Lemma 2.1, which is equivalent to $DG^*GD + I - D$

and DHH^*D+I-D being invertible with $GDH(DH)^*(DHH^*D+I-D)^{-1}(DG^*GD+I-D)^{-1}(GD)^* = (DH)^*(DHH^*D+I-D)^{-1}(DG^*GD+I-D)^{-1}(GD)^*GDH$.
Note that

$$GDH(DH)^*(DHH^*D+I-D)^{-1}(DG^*GD+I-D)^{-1}(GD)^* = GD(DG^*GD+I-D)^{-1}(GD)^*$$

and

$$(DH)^*(DHH^*D+I-D)^{-1}(DG^*GD+I-D)^{-1}(GD)^*GDH = (DH)^*(DHH^*D+I-D)^{-1}DH,$$

and then we complete the proof. □

Let D be the identity matrix in Theorem 3.5, and then we get the following result.

Corollary 3.6 *Let A, G, H be matrices over R such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if G^*G and HH^* are invertible with $G(G^*G)^{-1}G^* = H^*(HH^*)^{-1}H$.*

Let D be the identity matrix and $k=1$ in Theorem 3.5, and then we get the following result, which gives the second characterization for A to be EP.

Corollary 3.7 *Let A, G, H be matrices over R such that $A = GH$, $N_r(G) = 0$, $N_l(H) = 0$. Then A is EP if and only if G^*G and HH^* are invertible with $G(G^*G)^{-1}G^* = H^*(HH^*)^{-1}H$.*

The following result gives the third characterization for A, in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$, to be *-DMP.

Theorem 3.8 *Let A, G, D, H be matrices over R such that $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if $DG^*GD+I-D$ and $DHAGD+I-D$ are invertible with*

$$(DG^*GD+I-D)^{-1}(GD)^* = (DHGD+I-D)^{-1}DH.$$

Proof According to Theorem 2.4, $DG^*GD+I-D$ and $DHAGD+I-D$ are invertible if and only if $A^{\textcircled{D}}$ exists with $i(A) \leq k$, in which case,

$$A^{\textcircled{D}} = GD(DHAGD+I-D)^{-1}DHGD(DG^*GD+I-D)^{-1}(GD)^*.$$

Notice that $A^{\textcircled{D}}$ existing implies that A^D exists by Lemma 2.3 and $A^D = GD(DHAGD+I-D)^{-1}DH$ by Lemma 2.2. Applying Lemma 3.1, A is *-DMP with $i(A) \leq k$ if and only if

$$\begin{aligned}
&G(DHAGD+I-D)^{-1}(DHGD+I-D)(DG^*GD+I-D)^{-1}(GD)^* \\
&= GD(DHAGD+I-D)^{-1}DHGD(DG^*GD+I-D)^{-1}(GD)^* = A^{\textcircled{D}} = A^D \qquad (3.1)\\
&= GD(DHAGD+I-D)^{-1}DH = G(DHAGD+I-D)^{-1}DH.
\end{aligned}$$

Since $N_r(G)=0$, by calculation, equality (3.1) is equivalent to

$$DG^*GD+I-D)^{-1}DG^*=(DHGD+I-D)^{-1}DH.$$

□

Let D be the identity matrix in Theorem 3.8, and then we get the following result.

Corollary 3.9 *Let A,G,H be matrices over R such that $A^k=GH$, $N_r(G)=0$, $N_l(H)=0$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if G^*G and HAG are invertible with $(G^*G)^{-1}G^*=(HG)^{-1}H$.*

Let D be the identity matrix and $k=1$ in Theorem 3.8, and then we get the following result, which gives the third characterization for A to be EP.

Corollary 3.10 *Let A,G,H be matrices over R such that $A=GH$, $N_r(G)=0$, $N_l(H)=0$. Then A is EP if and only if G^*G and HG are invertible with $(G^*G)^{-1}G^*=(HG)^{-1}H$.*

The following result gives the fourth characterization for A, in the form of $A^k=GDH$, $N_r(G)=0$, $N_l(H)=0$, $D^2=D=D^*$, to be *-DMP.

Theorem 3.11 *Let A,G,D,H be matrices over R such that $A^k=GDH$, $N_r(G)=0$, $N_l(H)=0$, $D^2=D=D^*$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if DHH^*D+I-D and $DGAGD+I-D$ are invertible with*

$$(DH)^*(DHH^*D+I-D)^{-1}=GD(DHGD+I-D)^{-1}.$$

Proof According to Theorem 2.4, DHH^*D+I-D and $DHAGD+I-D$ are invertible if and only if $A_{\textcircled{D}}$ exists with $i(A)\leq k$, in which case,

$$A_{\textcircled{D}}=(DH)^*(DHH^*D+I-D)^{-1}DHGD(DHAGD+I-D)^{-1}DH.$$

Notice that $A_{\textcircled{D}}$ existing implies that A^D exists by Lemma 2.3 and $A^D=GD(DHAGD+I-D)^{-1}DH$ by Lemma 2.2. Applying Lemma 3.1, A is *-DMP if and only if

$$\begin{aligned}
&(DH)^*(DHH^*D+I-D)^{-1}(DHGD+I-D)(DHAGD+I-D)^{-1}H\\
&=(DH)^*(DHH^*D+I-D)^{-1}(DHGD+I-D)D(DHAGD+I-D)^{-1}H\\
&=(DH)^*(DHH^*D+I-D)^{-1}DHGD(DHAGD+I-D)^{-1}DH=A_{\textcircled{D}}=A^D\\
&=GD(DHAGD+I-D)^{-1}DH=GD(DHAGD+I-D)^{-1}H.
\end{aligned}\tag{3.2}$$

Since $N_l(H)=0$, by calculation, equality (3.2) is equivalent to

$$(DH)^*(DHH^*D+I-D)^{-1}=GD(DHGD+I-D)^{-1}.$$

□

Let D be the identity matrix in Theorem 3.11, and then we get the following result.

Corollary 3.12 *Let A, G, H be matrices over R such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if G^*G and HAG are invertible with $H^*(HH^*)^{-1} = G(HG)^{-1}$.*

Let D be the identity matrix and $k = 1$ in Theorem 3.11, and then we get the following result, which gives the fourth characterization for A to be EP.

Corollary 3.13 *Let A, G, H be matrices over R such that $A = GH$, $N_r(G) = 0$, $N_l(H) = 0$. Then A is EP if and only if G^*G and HG are invertible with $H^*(HH^*)^{-1} = G(HG)^{-1}$.*

The following result gives the fifth characterization for A, in the form of $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$, to be *-DMP.

Theorem 3.14 *Let A, G, D, H be matrices over R such that $A^k = GDH$, $N_r(G) = 0$, $N_l(H) = 0$, $D^2 = D = D^*$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if $DG^*GD + I - D$, $DHH^*D + I - D$, and $DHAGD + I - D$ are invertible and one of the following equivalent conditions holds:*
(1) $(DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^* = GD(DHGD + I - D)^{-2}DH$;
*(2) $G(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}G^* = H^*(DHH^*D + I - D)^{-1}DHGD(D$
$HAGD + I - D)^{-1}H$.*

Proof $DG^*GD + I - D$, $DHH^*D + I - D$, and $DHAGD + I - D$ are invertible if and only if $(A^k)^\dagger$ and A^D exist with $i(A) \leq k$ by Lemmas 2.1 and 2.2, which is equivalent to $A^{\textcircled{D}}$ and $A_{\textcircled{D}}$ existing with $i(A) \leq k$ by Lemma 2.3. Observe that $(A^D)^k = GD(DHGD + I - D)^{-2}DH$ by the proof of Theorem 3.2; $(A^k)^\dagger = (DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^*$ by Lemma 2.1; and, by Theorem 2.4,

$$A^{\textcircled{D}} = GD(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}(GD)^*,$$
$$A_{\textcircled{D}} = (DH)^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}DH.$$

From Lemma 3.1, A is *-DMP with index $\leq k$ if and only if

$$\begin{aligned} &(DH)^*(DHH^*D + I - D)^{-1}(DG^*GD + I - D)^{-1}(GD)^* \\ &= (A^k)^\dagger = (A^D)^k = GD(DHGD + I - D)^{-2}DH, \end{aligned}$$

if and only if

$$\begin{aligned} &G(DHAGD + I - D)^{-1}DHGD(DG^*GD + I - D)^{-1}G^* \\ &= A^{\textcircled{D}} = A_{\textcircled{D}} \\ &= H^*(DHH^*D + I - D)^{-1}DHGD(DHAGD + I - D)^{-1}H. \end{aligned}$$

□

Let D be the identity matrix in Theorem 3.14, and then we get the following result.

Corollary 3.15 *Let A, G, H be matrices over R such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$ for some positive integer k. Then A is *-DMP with index $\leq k$ if and only if G^*G, HH^* and HAG are invertible and one of the following equivalent conditions holds:*

(1) $H^*(HH^*)^{-1}(G^*G)^{-1}G^* = G(HG)^{-2}H$;
(2) $G(HAG)^{-1}HG(G^*G)^{-1}G^* = H^*(HH^*)^{-1}HG(HAG)^{-1}H$.

Let D be the identity matrix and $k = 1$ in Theorem 3.14, and then we get the following result, which gives the fifth characterization for A to be EP.

Corollary 3.16 *Let A, G, H be matrices over R such that $A^k = GH$, $N_r(G) = 0$, $N_l(H) = 0$. Then A is EP if and only if G^*G, HH^*, and HG are invertible and one of the following equivalent conditions holds:*
(1) $H^*(HH^*)^{-1}(G^*G)^{-1}G^* = G(HG)^{-2}H$;
(2) $G(HAG)^{-1}HG(G^*G)^{-1}G^* = H^*(HH^*)^{-1}HG(HAG)^{-1}H$.

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References

[1] Baksalary OM, Trenkler G. Core inverse of matrices. Linear Multilinear A 2010; 58: 681-697.

[2] Ben-Israel A, Greville TNE. Generalized Inverses: Theory and Applications. New York, NY, USA: Wiley-Interscience, 1974.

[3] Boasso E. Factorizations of EP Banach operators and EP Banach algebra elements. J Math Anal Appl 2011; 379: 245-255.

[4] Boasso E, Djordjević DS, Mosić D. Factorizations of weighted EP Banach space operators and Banach algebra elements. Banach J Math Anal 2015; 9: 114-127.

[5] Chen JL. Generalized inverses of matrices over rings. Acta Math Sinica 1991; 34: 622-630.

[6] Chen JL. Group inverses and Drazin inverses of matrices over rings. Acta Math Sinica 1994; 37: 373-380.

[7] Drivaliaris D, Karanasios S, Pappas D. Factorizations of EP operators. Linear Algebra Appl 2008; 429: 1555-1567.

[8] Gao YF, Chen JL. *-DMP elements in $*$-semigroups and $*$-rings. arXiv: 1701.00621v1 [math.RA], 2017.

[9] Gao YF, Chen JL. Pseudo core inverses in rings with involution. Commun Algebra 2018; 46: 38-50.

[10] Hartwig RE. Block generalized inverses. Arch Ration Mech An 1976; 61: 197-251.

[11] Huylebrouck D, Puystjens R, Van Geel J. The Moore-Penrose inverse of a matrix over a semi-simple artinian ring. Linear Multilinear A 1984; 16: 239-246.

[12] Koliha JJ, Patrício P. Elements of rings with equal spectral idempotents. J Aust Math Soc 2002; 72: 137-152.

[13] Manjunatha Prasad K, Mohana KS. Core-EP inverse. Linear Multilinear A 2014; 62: 792-802.

[14] Mosić D, Djordjević DS, Koliha JJ. EP elements in rings. Linear Algebra Appl 2009; 431: 527-535.

[15] Patrício P, Puystjens R. Drazin-Moore-Penrose invertibility in rings. Linear Algebra Appl 2004; 389: 159-173.

[16] Pearl MH. On generalized inverses of matrices. P Camb Philos Soc 1966; 62: 673-677.

[17] Puystjens R. The Moore-Penrose inverses for matrices over some noetherian rings. J Pure Appl Algebra 1984; 31: 191-198.

[18] Rakić DS, Dinčić NČ, Djordjević DS. Group, Moore-Penrose, core and dual core inverse in rings with involution. Linear Algebra Appl 2014; 463: 115-133.

[19] Robinson DW, Puystjens R. EP morphisms. Linear Algebra Appl 1985; 64: 157-174.

[20] Schwerdtfeger H. Introduction to Linear Algebra and the Theory of Matrices. Groningen, the Netherlands: Noordhoff, 1950.

[21] Sheng XP, Chen GL. Full-rank representation of generalized inverse $A_{T,S}^{(2)}$ and its application. Comput Math Appl 2007; 54: 1422-1430.

[22] Stanimirovic PS. Block representations of {2}, {1,2} inverses and the Drazin inverses of matrices. Indian J Pure Ap Mat 1998; 29: 1159-1176.

[23] Xu SZ, Chen JL, Zhang XX. New characterizations for core inverses in rings with involution. Front Math China 2017; 12: 231-246.