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Differential subordination and radius estimates for starlike functions associated with the Booth lemniscate

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Abstract: We obtain several inclusions between the class of functions with positive real part and the class of starlike univalent functions associated with the Booth lemniscate. These results are proved by applying the well-known theory of differential subordination developed by Miller and Mocanu and these inclusions give sufficient conditions for normalized analytic functions to belong to some subclasses of Ma–Minda starlike functions. In addition, by proving an associated technical lemma, we compute various radii constants such as the radius of starlikeness, radius of convexity, radius of starlikeness associated with the lemniscate of Bernoulli, and other radius estimates for functions in the class of functions associated with the Booth lemniscate. The results obtained are sharp.

Key words: Starlike function, convex function, Booth lemniscate, radius estimate, differential subordination

1. Introduction

Let $\mathcal{A}_n$ denote the general class of the normalized analytic functions defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and having the Taylor series expansion given by $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$. In particular, let $\mathcal{A} := \mathcal{A}_1$. The subclass of $\mathcal{A}$ containing univalent functions is denoted by $\mathcal{S}$. For the analytic functions f and g defined on $\mathbb{D}$, we say that f is subordinate to g, written as $f \prec g$, if there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Among the several subclasses of $\mathcal{S}$, the classes of starlike and convex functions are most studied. Various classes of starlike and convex functions are characterized by the quantities $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$, respectively, by using the concept of subordination and the Hadamard product. The class $\mathcal{S}^*_g(\varphi)$ of all $f \in \mathcal{A}$ satisfying $z(f(z) * g(z))'/(f(z) * g(z)) \prec \varphi(z)$, where $\varphi(z)$ is a convex function and $g(z)$ is a fixed function in $\mathcal{A}$, was studied by Shanmugam [31]. For the special case $g(z) = z/(1-z)^\alpha$, the class $\mathcal{S}^*_g(\varphi)$ was studied in [23]. For the choice of function $g(z) = z/(1-z), z/(1-z)^2$ and analytic function φ with the positive real part mapping $\mathbb{D}$ onto a domain symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$, the class $\mathcal{S}^*_g(\varphi)$ reduces to classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$, respectively, studied by Ma and Minda [17]. They proved distortion, covering, and growth theorems. For special choices of the function φ, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ reduce to many well-known classes. For $\varphi(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$, these classes reduce respectively

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to the classes $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$ of Janowski starlike and convex functions [9]. The classes $\mathcal{S}^*[1,-1] := \mathcal{S}^*$ and $\mathcal{K}[1,-1] =: \mathcal{K}$ are respectively the well-known classes of starlike and convex functions. In recent years, several authors have defined many interesting subclasses of $\mathcal{S}^*$ by restricting the value of $\zeta(z) := zf'(z)/f(z)$ to lie in a certain precise domain in the right-half plane.

The class $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+z})$ is related to the right-half of the lemniscate of Bernoulli and was considered by Sokół and Stankiewicz [35]. In 2015, Mendiratta et al. [18, 19] introduced and studied the classes of starlike functions

$$\mathcal{S}^*_e = \mathcal{S}^*(e^z) \text{ and } \mathcal{S}^*_{RL} = \mathcal{S}^*\left(\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}\right).$$

Geometrically, $f \in \mathcal{S}^*_{RL}$ if $\zeta(z)$ lies in the interior of the left half of the displaced lemniscate of Bernoulli given by $|(\zeta-\sqrt{2})^2-1| < 1$. Similarly, Sharma et al. [32] studied various geometric properties of the class $\mathcal{S}^*_c := \mathcal{S}^*(\varphi_c(z))$, where $\varphi_c(z) := 1+(4z/3)+(2z^2/3)$. In 2015, Raina and Sokół [25] introduced an interesting class $\mathcal{S}^*_q := \mathcal{S}^*(\varphi_q(z))$, where $\varphi_q(z) := z+\sqrt{1+z^2}$, and proved that the class $\mathcal{S}^*_q$ is a subclass of the class consisting of functions $f \in \mathcal{A}$ such that

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 2\left|\frac{zf'(z)}{f(z)}\right|$$

and discussed several other properties of the class $\mathcal{S}^*_q$. In 2016, Kumar and Ravichandran [13] considered the class $\mathcal{S}^*_R := \mathcal{S}^*(\varphi_0(z))$, where $\varphi_0(z) := 1+(z/k)((k+z)/(k-z))$, $k=\sqrt{2}+1$. In a similar fashion, Cho et al. ["Radius problems for starlike functions associated with the sine function", preprint] defined and studied radius problems for the class $\mathcal{S}^*_s := \mathcal{S}^*(\varphi_s(z))$, where $\varphi_s(z) = 1+\sin z$.

Recently, Kargar et al. [10] introduced and studied a class of functions related to the Booth lemniscate. For $0 \le \alpha < 1$, they defined $\mathcal{BS}^*(\alpha) := \mathcal{S}^*(G_\alpha(z))$, where $G_\alpha(z) := 1+z/(1-\alpha z^2)$. They also obtained the bound for the initial coefficients and derived some subordination results. In [11], various radius problems and subordination results were also discussed for some subclasses of analytic functions. For more details, see [24]. The Booth lemniscate is a special case of the Persian curve [29] and it was named after Booth, an Irish mathematician, who studied it in 1873. In geometry, the Booth lemniscate is a plane algebraic curve of order 4.

If $f \in \mathcal{BS}^*(\alpha)$, then [28, Theorem 6, p. 195] yields $|f(z)| \le |z|K(\alpha)$, where

$$K(\alpha) = \exp\left(\frac{1}{2\sqrt{\alpha}}\log\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)$$

for all $|z|<1$. Therefore, there is a function f that belongs to the class $\mathcal{BS}^*(1/2)$ for which $\sup_{z\in\mathbb{D}}|f(z)| = K(1/2) \approx 3.478$.

The first two important results related to first-order differential subordination were introduced by Goluzin and Robinson in 1935 and 1947, respectively. These first-order differential subordinations have many applications in the theory of univalent functions. Later, Miller and Mocanu [20, 21] developed and discussed the general theory of differential subordination.

Using the theory of differential subordination, Tuneski [36] and Tuneski et al. [37] gave interesting criteria for normalized analytic functions to be Janowski starlike. They also studied certain properties of linear combinations of starlike functions. For analytic function $p:\mathbb{D}\to\mathbb{C}$ with $p(0)=1$, in 1989, Nunokawa et al. [22] proved that $p(z)\prec 1+z$, whenever the subordination $1+zp'(z)\prec 1+z$ holds. Later, Ali et al. [4] generalized this subordination implication and computed a bound on β in each case for which $1+\beta zp'(z)/p^j(z)\prec(1+Dz)/(1+Ez)$ $(j=0,1,2)$ implies $p(z)\prec(1+Az)/(1+Bz)$, where $A,B,D,E\in[-1,1]$. Further, Ali et al. [2] determined the estimates on β so that $p(z)\prec\sqrt{1+z}$, whenever $1+\beta zp'(z)/p^j(z)\prec\sqrt{1+z}$ $(j=0,1,2)$. In 2013, Kumar et al. [15] obtained the bound on β with $-1<E<1$ and $|D|\leq 1$ such that $1+\beta zp'(z)/p^j(z)\prec(1+Dz)/(1+Ez)$ $(j=0,1,2)$ implies $p(z)\prec\sqrt{1+z}$. These nonsharp results provide sufficient conditions for normalized analytic functions to be in the class of the Janowski starlike functions and in the class of functions associated with the lemniscate of Bernoulli. For more details, see [3, 5, 6, 21, 27, 33].

Recently, Kumar and Ravichandran [14] determined sharp estimates on β so that $p(z)\prec e^z$ whenever subordinations $1+\beta zp'(z)/p^j(z)\prec\varphi_0(z)$ $(j=0,1,2)$, $1+\beta zp'(z)/p^j(z)\prec(1+Az)/(1+Bz)$ $(j=0,2)$, $1+\beta zp'(z)/p^j(z)\prec\sqrt{1+z}$ $(j=0,2)$, and $1+\beta zp'(z)/p^j(z)\prec\varphi_s(z)$ $(j=0,1,2)$ hold. Ahuja et al. [1] found sharp estimates on β so that p is subordinate to some well-known starlike functions (for example, e^z, $\sqrt{1+z}$, $\varphi_s(z)$, $\varphi_c(z)$, $\varphi_q(z)$, and many more) whenever $1+\beta zp'(z)/p^k(z)$ $(k=0,1,2)$ is subordinate to $\sqrt{1+z}$.

It is well known that every convex function is starlike but not conversely. However, each starlike function is convex in the disk of radius $2-\sqrt{3}$. For two subfamilies T_1 and T_2 of $\mathcal{A}$, the T_1 radius of T_2 is the largest number $\rho\in(0,1)$ such that $r^{-1}f(rz)\in T_1$, $0<r\leq\rho$ for all $f\in T_2$. Grunsky [8] proved that the radius of starlikeness for functions in the class $\mathcal{S}$ is $\tanh\pi/4\approx 0.6558$. The radius of α-convexity and the α-starlikeness for $\mathcal{S}^*_L$ were recently obtained by Sokół [34]. In 2012, Ali et al. [3] obtained the $\mathcal{S}^*_L$-radius for certain well-known classes. Later, Mendiratta et al. [18, 19] computed the $\mathcal{S}^*_e$ and $\mathcal{S}^*_{RL}$-radii for certain classes. Subsequently, in [13], $\mathcal{S}^*_R$-radii were obtained for various well-known classes of starlike functions. For more results on radius problems, see [7, 12, 16, 38, 39].

Motivated by all these works, in Section 2, we consider the subordination inclusions, in which we compute the sharp bound on the parameter β so that a given differential subordination implication holds. We determine the sharp bound on β so that $p(z)\prec\mathcal{P}(z)$, where $\mathcal{P}(z)$ is a function with positive real part such as $\sqrt{1+z}$, $(1+Az)/(1+Bz)$, e^z, $\varphi_s(z)$, $\varphi_q(z)$, $\varphi_0(z)$, and $\varphi_c(z)$, whenever $1+\beta zp'(z)/p^j(z)\prec G_\alpha(z)$ $(j=0,1,2)$. In addition, we find the best possible bound on β so that p is subordinate to G_α, whenever $1+\beta zp'(z)$ is subordinate to $(1+Az)/(1+Bz)$ or some other well-known Carathéodory functions. As applications to these results, several sufficient conditions for normalized analytic functions to belong to certain well-known classes of starlike functions are obtained. In Section 3, we determine the radius of starlikeness and radius of convexity for the functions in the class $\mathcal{BS}^*(\alpha)$. We also determine the $\mathcal{BS}^*_n(\alpha):=\mathcal{BS}^*(\alpha)\cap\mathcal{A}_n$-radius for the functions belonging to several interesting classes. Furthermore, we compute $\mathcal{S}^*_L,\mathcal{S}^*_{RL},\mathcal{S}^*_e,\mathcal{S}^*_c$, and $\mathcal{S}^*_q$-radii for functions in the class $\mathcal{BS}^*(\alpha)$. The results obtained are sharp.

2. Differential subordination implications

The first result of this section gives a bound on β so that $1+\beta zp'(z)\prec G_\alpha(z)$ implies that p is subordinate to some well-known starlike functions.

Theorem 2.1 *Let the function p be analytic in $\mathbb{D}$, $p(0)=1$, and $1+\beta zp'(z) \prec G_\alpha(z)$. For $0<\alpha<1$, let*

$$l(\alpha)=\frac{1}{2\sqrt{\alpha}}\log\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

Then the following subordination results hold:

(a) $p(z)\prec e^z$ *for* $\beta \geq l(\alpha)e/(e-1)$.

(b) $p(z)\prec \sqrt{1+z}$ *for* $\beta \geq l(\alpha)/(\sqrt{2}-1)$.

(c) $p(z)\prec (1+Az)/(1+Bz)$ *for* $\beta \geq (1+|B|)l(\alpha)/(A-B)$, $(-1<B<A<1)$.

(d) $p(z)\prec \varphi_0(z)$ *for* $\beta \geq (3+2\sqrt{2})l(\alpha)$.

(e) $p(z)\prec \varphi_q(z)$ *for* $\beta \geq (2+\sqrt{2})l(\alpha)/2$.

(f) $p(z)\prec \varphi_c(z)$ *for* $\beta \geq 3l(\alpha)/2$.

(g) $p(z)\prec \varphi_s(z)$ *for* $\beta \geq l(\alpha)/\sin(1)$.

The bounds on β are sharp.

In proving our results, the following lemma will be needed:

Lemma 2.2 [21, Theorem 3.4h, p. 132] *Let q be analytic in $\mathbb{D}$ and let ψ and ν be analytic in a domain U containing $q(\mathbb{D})$ with $\psi(w)\neq 0$ when $w\in q(\mathbb{D})$. Set $Q(z):=zq'(z)\psi(q(z))$ and $h(z):=\nu(q(z))+Q(z)$. Suppose that (i) either h is convex or Q is starlike univalent in $\mathbb{D}$, and (ii)* $\operatorname{Re}(zh'(z)/Q(z))>0$ *for $z\in\mathbb{D}$. If p is analytic in $\mathbb{D}$, with $p(0)=q(0)$, $p(\mathbb{D})\subseteq U$ and*

$$\nu(p(z))+zp'(z)\psi(p(z))\prec \nu(q(z))+zq'(z)\psi(q(z)),$$

then $p(z)\prec q(z)$, and q is best dominant.

Proof of Theorem 2.1 The analytic function $q_\beta:\overline{\mathbb{D}}\to\mathbb{C}$ defined by

$$q_\beta(z)=1+\frac{1}{2\sqrt{\alpha}\beta}\log\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}$$

is a solution of the differential equation $1+\beta zq'_\beta(z)=G_\alpha(z)$. Consider the functions $\nu(w)=1$ and $\psi(w)=\beta$. The function $Q:\overline{\mathbb{D}}\to\mathbb{C}$ is defined by

$$Q(z)=zq'_\beta(z)\psi(q_\beta(z))=\beta zq'_\beta(z)=\frac{z}{1-\alpha z^2}.$$

Since

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right)=\operatorname{Re}\left(\frac{1+\alpha z^2}{1-\alpha z^2}\right)>0$$

in $\mathbb{D}$, it follows that function Q is starlike. Note that the function $h(z) = \nu(q_\beta(z)) + Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. Therefore, by Lemma 2.2, it follows that $1 + \beta zp'(z) \prec 1 + \beta zq'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. Each of the conclusions in all parts of this theorem is $p(z) \prec \mathcal{P}(z)$ for appropriate choice of $\mathcal{P}$ and this holds if the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds. If $q_\beta(z) \prec \mathcal{P}(z)$, then

$$\mathcal{P}(-1) < q_\beta(-1) < q_\beta(1) < \mathcal{P}(1).$$

This gives a necessary condition for $p \prec \mathcal{P}$ to hold. This necessary condition is also sufficient by looking at the graph of the respective functions.

(a) Let $\mathcal{P}(z) = e^z$. Then the inequalities $q_\beta(-1) \geq e^{-1}$ and $q_\beta(1) \leq e$ yield $\beta \geq \beta_1$ and $\beta \geq \beta_2$, respectively, where

$$\beta_1 = \frac{e}{2(e-1)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \quad \text{and} \quad \beta_2 = \frac{1}{2(e-1)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

A simple calculation gives

$$\beta_1 - \beta_2 = \frac{1}{2\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} > 0.$$

Therefore, the subordination $q_\beta(z) \prec e^z$ holds only if $\beta \geq \max\{\beta_1,\ \beta_2\} = \beta_1$.

(b) Let $\mathcal{P}(z) = \sqrt{1+z}$. Then the inequalities $q_\beta(-1) \geq 0$ and $q_\beta(1) \leq \sqrt{2}$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, respectively, where

$$\beta_1 = \frac{1}{2\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \quad \text{and} \quad \beta_2 = \frac{1}{2(\sqrt{2}-1)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

We also note that $\beta_1 - \beta_2 < 0$. Therefore, the subordination $q_\beta(z) \prec \sqrt{1+z}$ holds only if $\beta \geq \max\{\beta_1,\ \beta_2\} = \beta_2$.

(c) Let $\mathcal{P}(z) = (1+Az)/(1+Bz)$ $(-1 < B < A < 1)$. Then the inequalities $q_\beta(-1) \geq (1-A)/(1-B)$ and $q_\beta(1) \leq (1+A)/(1+B)$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, respectively, where

$$\beta_1 = \frac{1-B}{2(A-B)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \quad \text{and} \quad \beta_2 = \frac{1+B}{2(A-B)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

Therefore, the desired subordination $q_\beta(z) \prec (1+Az)/(1+Bz)$ holds if $\beta \geq \max\{\beta_1, \beta_2\}$.

(d) Let $\mathcal{P}(z) = \varphi_0(z) = 1+(z/k)((k+z)/(k-z))$, where $k = \sqrt{2}+1$. Then the inequalities $q_\beta(-1) \geq 2\left(\sqrt{2}-1\right)$ and $q_\beta(1) \leq 2$ become $\beta \geq \beta_1$ and $\beta \geq \beta_2$, respectively, where

$$\beta_1 = \frac{1}{2(3-2\sqrt{2})\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \quad \text{and} \quad \beta_2 = \frac{1}{2\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

We also note that $\beta_1 > \beta_2$. Therefore, if $\beta \geq \beta_1$, then $q_\beta(z) \prec \varphi_0$.

(e) Let $\mathcal{P}(z) = \varphi_q(z) = z+\sqrt{1+z^2}$. Then, on simplifying the inequalities $q_\beta(-1) \geq \sqrt{2}-1$ and $q_\beta(1) \leq \sqrt{2}+1$, we get $\beta \geq \beta_1$ and $\beta \geq \beta_2$, respectively, where

$$\beta_1 = \frac{1}{2(2-\sqrt{2})\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \quad \text{and} \quad \beta_2 = \frac{1}{2\sqrt{2\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

Thus, the subordination $q_\beta(z) \prec \varphi_q(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(f) Let $\mathcal{P}(z) = \varphi_c(z) = 1 + 4z/3 + 2z^2/3$. Then the inequalities $\varphi_c(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi_c(1)$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, respectively, where

$$\beta_1 = \frac{3}{4\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \quad \text{and} \quad \beta_2 = \frac{1}{4\sqrt{2\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

Note that $\beta_2/\beta_1 = 1/3\sqrt{2} < 1$. Therefore, the subordination $q_\beta(z) \prec \varphi_c(z)$ holds if $\beta \geq \beta_1$.

(g) Let $\mathcal{P}(z) = \varphi_s(z) = 1 + \sin z$. Then the inequalities $q_\beta(-1) \geq \varphi_s(-1)$ and $q_\beta(1) \leq \varphi_s(1)$ yield $\beta \geq \beta_1$ and $\beta \geq \beta_2$, respectively, where

$$\beta_1 = \beta_2 = \frac{1}{2\sin(1)\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

Therefore, if $\beta \geq \beta_1$, then $q_\beta(z) \prec \varphi_s(z)$.

□

Let the function $f \in \mathcal{A}$ satisfy the following subordination:

$$\beta \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{2z}{2-z^2}.$$

Then the following sufficient conditions are immediate consequences of Theorem 2.1:

(a) $f \in \mathcal{S}^*_e$ for $\beta \geq e \log(3+2\sqrt{2})/\sqrt{2}(e-1)$.

(b) $f \in \mathcal{S}^*_L$ for $\beta \geq \log(3+2\sqrt{2})/(2-\sqrt{2})$.

(c) $f \in \mathcal{S}^*[A,B]$ for $\beta \geq (1+|B|)\log(3+2\sqrt{2})/\sqrt{2}(A-B)$ $(-1 < B < A < 1)$.

(d) $f \in \mathcal{S}^*_{RL}$ for $\beta \geq (3+2\sqrt{2})\log(3+2\sqrt{2})/\sqrt{2}$.

(e) $f \in \mathcal{S}^*_q$ for $\beta \geq \log(3+2\sqrt{2})/2(\sqrt{2}-1)$.

(f) $f \in \mathcal{S}^*_c$ for $\beta \geq 3\log(3+2\sqrt{2})/2\sqrt{2}$.

(g) $f \in \mathcal{S}^*_s$ for $\beta \geq \log(3+2\sqrt{2})/\sqrt{2}\sin(1)$.

The next result gives a sharp bound on β so that $1+\beta zp'(z)/p(z) \prec G_\alpha(z)$ implies that p is subordinate to some well-known starlike functions.

Theorem 2.3 *Let p be an analytic function in $\mathbb{D}$, $p(0) = 1$ and satisfying the subordination*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec G_\alpha(z).$$

For $0 < \alpha < 1$, let

$$l(\alpha) = \frac{1}{2\sqrt{\alpha}}\log\frac{1-\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

Then the following subordination results hold:

(a) $p(z) \prec e^z$ *for* $\beta \geq l(\alpha)$.

(b) If $-1 < B < A < 1$, *then* $p(z) \prec (1 + Az)/(1 + Bz)$ *for* $\beta \geq \max\{\beta_1, \beta_2\}$, *where*

$$\beta_1 = \frac{l(\alpha)}{\log(1-B) - \log(1-A)} \quad \textit{and} \quad \beta_2 = \frac{l(\alpha)}{\log(1+A) - \log(1+B)}.$$

(c) $p(z) \prec \varphi_0(z)$ *for* $\beta \geq l(\alpha)/\log(2\sqrt{2} - 2)$.

(d) $p(z) \prec \varphi_q(z)$ *for* $\beta \geq l(\alpha)/\log(\sqrt{2} + 1)$.

(e) $p(z) \prec \varphi_c(z)$ *for* $\beta \geq l(\alpha)/\log(3)$.

(f) $p(z) \prec \varphi_s(z)$ *for* $\beta \geq l(\alpha)/\log(1 + \sin(1))$.

The bound on β in each case is the best possible.

Proof Define the function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$q_\beta(z) = \exp\left(\frac{1}{2\sqrt{\alpha}\beta}\log\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right).$$

Then the function q_β is analytic and a solution of the differential equation $1+\beta zq'_\beta(z)/q_\beta(z) = G_\alpha(z)$. Consider the functions $\nu(w) = 1$ and $\psi(w) = \beta/w$. Define the function

$$Q(z) := zq'_\beta(z)\psi(q_\beta(z)) = \frac{\beta zq'_\beta(z)}{q_\beta(z)} = \frac{z}{1-\alpha z^2}.$$

A simple calculation shows that the function Q is starlike in $\mathbb{D}$. Note that the function $h(z) := \nu(q_\beta(z))+Q(z) = 1 + Q(z)$ satisfies an inequality $\mathrm{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. Therefore, by Lemma 2.2, we see that the subordination $p(z) \prec q_\beta(z)$ holds if

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'_\beta(z)}{q_\beta(z)}.$$

On similar lines to those of the proof of Theorem 2.1, the proofs of parts (a)–(e) are completed. □

For the best possible value of β, let the function $f \in \mathcal{A}$ satisfy the following subordination:

$$1 + \beta\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \prec G_{1/2}(z).$$

Then the following sufficient conditions are immediate consequences of Theorem 2.3:

(a) $f \in \mathcal{S}_e^*$ for $\beta \geq \log(3+2\sqrt{2})/\sqrt{2}$.

(b) Let $-1 < B < A < 1$. $f \in \mathcal{S}^*[A,B]$, for $\beta \geq \max\{\beta_1, \beta_2\}$, where

$$\beta_1 = \frac{\log(3+2\sqrt{2})}{\sqrt{2}\log((1-B)/(1-A))} \text{ and } \beta_2 = \frac{\log(3+2\sqrt{2})}{\sqrt{2}\log((1+A)/(1+B))}.$$

(c) $f \in \mathcal{S}_{RL}^*$ for $\beta \geq \log(3+2\sqrt{2})/\sqrt{2}\log((\sqrt{2}+1)/2)$.

(d) $f \in \mathcal{S}_q^*$ for $\beta \geq \log(3+2\sqrt{2})/\sqrt{2}\log(\sqrt{2}+1)$.

(e) $f \in \mathcal{S}_c^*$ for $\beta \geq \log(3+2\sqrt{2})/\sqrt{2}\log 3$.

(f) $f \in \mathcal{S}_s^*$ for $\beta \geq \log(3+2\sqrt{2})/\sqrt{2}\log(1+\sin(1))$.

Next, the best possible bound on β is determined so that p is subordinate to several well-known starlike functions, whenever $1+\beta zp'(z)/p^2(z) \prec G_\alpha(z)$.

Theorem 2.4 *Let the function p be analytic in $\mathbb{D}$ with condition $p(0)=1$ and satisfying the subordination*

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec G_\alpha(z).$$

For $0<\alpha<1$, let

$$l(\alpha) = \frac{1}{2\sqrt{\alpha}}\log\frac{1-\sqrt{\alpha}}{1-\sqrt{\alpha}}.$$

Then the following subordination results hold:

(a) $p(z) \prec e^z$ *for* $\beta \geq el(\alpha)/(e-1)$.

(b) $p(z) \prec (1+Az)/(1+Bz)$ *for* $\beta \geq (1+|A|)l(\alpha)/(A-B) \quad (-1<B<A<1)$.

(c) $p(z) \prec \varphi_0(z)$ *for* $\beta \geq 2(2+\sqrt{2})l(\alpha)$.

(d) $p(z) \prec \varphi_q(z)$ *for* $\beta \geq (\sqrt{2}+1)l(\alpha)/\sqrt{2}$.

(e) $p(z) \prec \varphi_c(z)$ *for* $\beta \geq 2l(\alpha)$.

(f) $p(z) \prec \varphi_s(z)$ *for* $\beta \geq (1+\sin(1))l(\alpha)/\sin(1)$.

The results are sharp.

Proof The function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$q_\beta(z) = \left(1 - \frac{1}{2\sqrt{\alpha}\beta}\log\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{-1}$$

is clearly analytic and a solution of differential equation $1+\beta zq'_\beta(z)/q^2_\beta(z) = G_\alpha(z)$. Define $\nu(w)=1$, $\psi(w)=\beta/w^2$ and the function Q defined by

$$Q(z)=zq'_\beta(z)\psi(q_\beta(z))=\frac{\beta zq'_\beta(z)}{q^2_\beta(z)}=\frac{z}{1-\alpha z^2}.$$

A calculation reveals that the function Q is starlike in $\mathbb{D}$. We note that the function $h(z):=\nu(q_\beta(z))+Q(z)=\nu(q_\beta(z))+Q(z)$ satisfies the inequality $\mathrm{Re}(zh'(z)/Q(z))>0$ for all $z\in\mathbb{D}$. Therefore, by using Lemma 2.2, we see that the subordination

$$1+\beta\frac{zp'(z)}{p^2(z)}\prec 1+\beta\frac{zq'_\beta(z)}{q^2_\beta(z)} \text{ implies } p(z)\prec q_\beta(z).$$

The proofs of parts (a)–(f) are obtained by following lines similar to those of the proof of Theorem 2.1. This completes the proof. □

Let the function $f\in\mathcal{A}$ satisfy the following subordination for the best possible value of β:

$$\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right)\prec\frac{G_{1/2}(z)-1}{\beta}.$$

Then, from Theorem 2.3, we have the following results:

(a) $f\in\mathcal{S}^*_e$ for $\beta\geq e\log(3+2\sqrt{2})/\sqrt{2}(e-1)$.

(b) $f\in\mathcal{S}^*[A,B]$ for $\beta\geq(1+|A|)\log(3+2\sqrt{2})/\sqrt{2}(A-B)$, $(-1<B<A<1)$.

(c) $f\in\mathcal{S}^*_R$ for $\beta\geq(2+\sqrt{2})\log(3+2\sqrt{2})$.

(d) $f\in\mathcal{S}^*_q$ for $\beta\geq(\sqrt{2}+1)\log(3+2\sqrt{2})/2$.

(e) $f\in\mathcal{S}^*_c$ for $\beta\geq\sqrt{2}\log(3+2\sqrt{2})$.

(f) $f\in\mathcal{S}^*_s$ for $\beta\geq(1+\sin(1))\log(3+2\sqrt{2})/\sqrt{2}\sin(1)$.

Next, Theorem 2.5 provides the best possible bound on β so that p is subordinate to G_α, whenever $1+\beta zp'(z)$ is subordinate to $(1+Az)/(1+Bz)$.

Theorem 2.5 *Let $-1<B<A<1$, $B\neq 0$ and $p(z)$ be the analytic function with $p(0)=1$ satisfying the subordination $1+\beta zp'(z)\prec(1+Az)/(1+Bz)$ for $\beta\geq\max\{\beta_1,\beta_2\}$, where*

$$\beta_1=\frac{(1-\alpha)(A-B)\log(1-B)^{-1}}{B}\ \textit{and}\ \beta_2=\frac{(1-\alpha)(A-B)\log(1+B)}{B}.$$

Then $p(z)\prec G_\alpha(z)$. The bound on β is sharp.

Proof Let the function q_β be defined by

$$q_\beta(z)=1+\frac{(A-B)\log(1+Bz)}{B\beta}.$$

The function $q_\beta(z)$ is analytic and a solution of the differential equation $1+\beta z q_\beta^{'}(z)=(1+Az)/(1+Bz)$. Consider the functions ν and ψ as defined in the proof of Theorem 2.1. Consider the function Q defined by

$$Q(z)=zq_\beta'(z)\psi(q_\beta(z))=\frac{(A-B)z}{1+Bz}.$$

Then a computation shows that

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right)=\operatorname{Re}\left(\frac{1}{1+Bz}\right)>0 \text{ for all } z\in\mathbb{D}.$$

This inequality shows that Q is starlike in $\mathbb{D}$. We also note that the function $h:\mathbb{D}\to\mathbb{C}$ defined by $h(z):=\nu(q_\beta(z))+Q(z)=1+Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z))>0$ in $\mathbb{D}$. By Lemma 2.2, it is easy to see that the subordination

$$1+\beta zp'(z)\prec 1+\beta zq_\beta'(z) \text{ implies } p(z)\prec q_\beta(z).$$

The desired subordination $p(z)\prec G_\alpha(z)$ holds if $q_\beta(z)\prec G_\alpha(z)$ and this subordination holds provided

$$G_\alpha(-1)\le q_\beta(-1) \text{ and } q_\beta(1)\le G_\alpha(1).$$

Therefore, the subordination $p(z)\prec G_\alpha(z)$ holds if $\beta\ge\max\{\beta_1,\,\beta_2\}$ as in Theorem 2.1. A simple calculation gives that if $B<0$, then $\max\{\beta_1,\,\beta_2\}=\beta_2$, and if $B>0$, then $\max\{\beta_1,\,\beta_2\}=\beta_1$. This completes the proof. $\square$

A simple calculation,

$$\frac{|(A-B)e^{i\theta}|}{|1+Be^{i\theta}|}\ge\frac{A-B}{1+|B|},$$

yields that the inequality $|w(z)|\le(A-B)/(1+|B|)$ implies $w(z)\prec(A-B)z/(1+Bz)$. By using this reasoning in the hypothesis of Theorem 2.5, we get the following sufficient condition for a function $f\in\mathcal{A}$ to be in the class $\mathcal{BS}^*(\alpha)$:

$$\left|\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right|\le\frac{(A-B)}{\max(\beta_1,\ \beta_2)(1+|B|)}.$$

The next result provides the estimates on β so that the subordination $p(z)\prec G_\alpha(z)$ holds whenever $1+\beta zp'(z)$ is subordinate to the functions $\varphi_0(z)$, $\varphi_s(z)$, e^z, $q(z)$, $\varphi_c(z)$, and $G_\alpha(z)$. Proof of this theorem is omitted as it is similar to that of Theorem 2.5.

Theorem 2.6 *Let p be an analytic function defined in $\mathbb{D}$ with $p(0)=1$. The subordination $p(z)\prec G_\alpha(z)$ holds if any one of the following differential subordinations holds:*

(a) $1+\beta zp'(z)\prec\varphi_0(z)$ *for* $\beta\ge(1-\alpha)(1-\sqrt{2}-2\log(2-\sqrt{2}))\approx 0.655386(1-\alpha)$.

(b) $1+\beta zp'(z)\prec\varphi_s(z)$ *for* $\beta\ge(1-\alpha)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2n+1)}\approx 0.946083(1-\alpha)$.

(c) $1+\beta zp'(z)\prec e^z$ *for* $\beta\ge(1-\alpha)\sum_{n=1}^{\infty}\frac{1}{n!n}\approx 1.3179(1-\alpha)$.

(d) $1+\beta zp'(z)\prec\varphi_q(z)$ *for* $\beta\ge(1-\alpha)(\sqrt{2}+\log(2)-\log(\sqrt{2}-1))\approx 1.22599(1-\alpha)$.

(e) $1+\beta zp'(z) \prec \varphi_c(z)$ *for* $\beta \geq 5(1-\alpha)/3$.

(f) $1+\beta zp'(z) \prec G_\alpha(z)$ *for* $\beta \geq \frac{1-\alpha}{2\sqrt{\alpha}} \log \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}$.

Theorem 2.6, in special cases, also provides several sufficient conditions for a normalized analytic function f to be in the class $\mathcal{BS}^*(\alpha)$.

3. Radius estimates

In 2015, Piejko and Sokół [24] proved that the function $z/(1-\alpha z^2)$ is convex univalent for $0 \leq \alpha < 3-2\sqrt{2}$ and also discussed some convolution properties related to the functions in the class $\mathcal{BS}^*(\alpha)$. Using the representation formula, we see that the function

$$F(z) = z\exp\left(\int_0^z \frac{1}{1-\alpha t^2}dt\right) = \begin{cases} ze^{\frac{\tanh^{-1}(\sqrt{\alpha}z)}{\sqrt{\alpha}}}, & 0<\alpha<1; \\ ze^z, & \alpha = 0 \end{cases} \tag{3.1}$$

belongs to the class $\mathcal{BS}^*(\alpha)$ and is not necessarily univalent in $\mathbb{D}$. In particular, the functions in the class $\mathcal{BS}^*(\alpha)$ are not necessarily starlike univalent. The following theorems give the radius of starlikeness and convexity, respectively.

Theorem 3.1 *The functions in the class* $\mathcal{BS}^*(\alpha)$ *are starlike univalent in the disk* $|z| < r_\alpha$, *where* $r_\alpha := 2/(\sqrt{4\alpha+1}+1)$.

Proof Since $f \in \mathcal{BS}^*(\alpha)$, it follows that $zf'(z)/f(z) \prec 1+z/(1-\alpha z^2)$. If $\alpha = 0$, then the result is obvious. Now if $\alpha \neq 0$, then for $|z| = r$, we have

$$\begin{aligned} \Re\left(\frac{zf'(z)}{f(z)}\right) &\geq 1-\frac{r}{1-\alpha r^2} \\ &> 0, \quad \text{for } r \leq \frac{\sqrt{4\alpha+1}-1}{2\alpha}. \end{aligned}$$

The result is sharp as the equality holds in the case of the function F defined by (3.1). □

Theorem 3.2 *The functions in the class* $\mathcal{BS}^*(\alpha)$ *are convex univalent in the disk* $|z| < \rho$, *with* $\rho = \min\{r^*, r_\alpha\}$, *where* r_α *is as defined in Theorem 3.1 and* r^* *is the smallest positive root of the equation*

$$1-\frac{r}{1-\alpha r^2} - \left(\frac{2\alpha r}{1-\alpha r^2} + \frac{2\alpha r+1}{1-\alpha r^2 - r}\right)\frac{r}{1-r^2} = 0.$$

Proof Since $f \in \mathcal{BS}^*(\alpha)$, it follows that there exists a Schwarz function w such that

$$\frac{zf'(z)}{f(z)} = 1+\frac{w(z)}{1-\alpha w^2(z)}. \tag{3.2}$$

Note that the Schwarz function w satisfies $w(0) = 0$, $|w(z)| \leq |z|$ and

$$|w'(z)| \leq \frac{1-|w(z)|^2}{(1-|z|^2)}.$$

Now a logarithmic differentiation of (3.2) gives

$$1+\frac{zf''(z)}{f'(z)}=1+\frac{w(z)}{1-\alpha w^2(z)}+\left(\frac{1-2\alpha w(z)}{1-\alpha w^2(z)+w(z)}-\frac{2\alpha w(z)}{1-\alpha w^2(z)}\right)zw'(z).$$

Using the properties of the Schwarz function, we have

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &\geq 1-\frac{|w(z)|}{1-\alpha|w^2(z)|}-\left(\frac{1+2\alpha|w(z)|}{1-\alpha|w^2(z)|-|w(z)|}+\frac{2\alpha|w(z)|}{1-\alpha|w^2(z)|}\right)|zw'(z)|\\ &\geq 1-\frac{r}{1-\alpha r^2}-\left(\frac{2\alpha r}{1-\alpha r^2}+\frac{2\alpha r+1}{1-\alpha r^2-r}\right)\frac{r}{1-r^2}=:H(r,\alpha). \qquad (3.3)\end{aligned}$$

It is easy to verify that $1-\alpha r^2-r>0$ for $r<r_\alpha$, where r_α is as defined in Theorem 3.1. Therefore, from (3.3), we have $H(r,\alpha)>0$, whenever $r=\rho<\min\{r^*,r_\alpha\}$, where r^* is the smallest positive root of

$$1-\frac{r}{1-\alpha r^2}-\left(\frac{2\alpha r}{1-\alpha r^2}+\frac{2\alpha r+1}{1-\alpha r^2-r}\right)\frac{r}{1-r^2}=0.$$

For $\alpha=0$, the result is sharp, as the equality holds in the case of the function defined by (3.1):

$$1+\frac{zf''(z)}{f'(z)}=1-z-\frac{z}{1-z}=0,\ \text{ for }\ z=\rho=\frac{1}{2}\left(3-\sqrt{5}\right).$$

□

Conjecture 3.3 *Let r_α be as defined in Theorem 3.1, and r' be the smallest positive root of*

$$1-\frac{r}{1-\alpha r^2}-\left(\frac{2\alpha r}{1-\alpha r^2}+\frac{2\alpha r+1}{1-\alpha r^2-r}\right)r=0.$$

Then sharp radius convexity for function $f\in\mathcal{BS}^*(\alpha)$ is $\rho=\min\{r',r_\alpha\}$.

Consider the function $w=G_\alpha(z)=1+z/(1-\alpha z^2)$, $0\leq\alpha<1$. Then we have

$$|w-1|=\frac{|z|}{|1-\alpha z^2|}.$$

It can be easily seen that

$$\min_{|z|=1}\frac{|z|}{|1-\alpha z^2|}=\frac{1}{1+\alpha}\quad\text{and}\quad\max_{|z|=1}\frac{|z|}{|1-\alpha z^2|}=\frac{1}{1-\alpha}.$$

Thus, the smallest disk centered at $(1,0)$ that contains $G_\alpha(\mathbb{D})$ and the largest disk centered at $(1,0)$ contained in $G_\alpha(\mathbb{D})$ are, respectively, given by:

$$|w-1|<\frac{1}{1+\alpha}\ \text{ and }\ |w-1|<\frac{1}{1-\alpha}.$$

On the basis of the above analysis, we have the following lemma:

Lemma 3.4 *Let* $G_\alpha(z) = 1 + z/(1 - \alpha z^2)$*. The inclusion relation is as follows:*

$$\left\{w \in \mathbb{C} : |w - 1| < \frac{1}{1+\alpha}\right\} \subset G_\alpha(\mathbb{D}) \subset \left\{w \in \mathbb{C} : |w - 1| < \frac{1}{1-\alpha}\right\}.$$

For further development in this section, we shall recall a few definitions and results. Let $\mathcal{P}$ be the class of analytic functions $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ and mapping $\mathbb{D}$ into the right half plane. The function $p_0(z) = (1 + z)/(1 - z)$, which maps $\mathbb{D}$ onto the right half plane conformally, is a leading example of a function with positive real part. Let

$$\mathcal{P}_n[A, B] := \left\{p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots : \; p(z) \prec \frac{1 + Az}{1 + Bz}, -1 \le B < A \le 1\right\}.$$

Let us denote $\mathcal{P}_n(\alpha) := \mathcal{P}_n[1 - 2\alpha, -1]$, $\mathcal{P}_n := \mathcal{P}_n(0)$, and $\mathcal{P}_1 =: \mathcal{P}$. For $f \in \mathcal{A}_n$, if we set $p(z) = zf'(z)/f(z)$, then the class $\mathcal{P}_n[A, B]$ is denoted by $\mathcal{S}^*_n[A, B]$ and $\mathcal{S}^*_n(\alpha) := \mathcal{S}^*_n[1-2\alpha, -1]$. Let $\mathcal{BS}^*_n(\alpha) := \mathcal{A}_n \cap \mathcal{BS}^*(\alpha)$. Ali et al. [3] considered the following classes:

$$\mathcal{S}_n := \{f \in \mathcal{A}_n : f(z)/z \in \mathcal{P}_n\}, \;\; \mathcal{CS}_n(\alpha) := \{f \in \mathcal{A}_n : f(z)/g(z) \in \mathcal{P}_n, g \in \mathcal{S}^*_n(\alpha)\}, \;\; \mathcal{S}^*_n[A, B].$$

They obtained the $\mathcal{S}^*_{L,n}$-radii for these classes.

The following lemmas will be used to establish our results related to radius estimates:

Lemma 3.5 [30] *If* $p \in \mathcal{P}_n(\alpha)$*, then, for* $|z| = r$*,*

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2(1-\alpha)nr^n}{(1 - r^n)(1 + (1 - 2\alpha)r^n)}.$$

Lemma 3.6 [26] *If* $p \in \mathcal{P}_n[A, B]$*, then, for* $|z| = r$*,*

$$\left|p(z) - \frac{1 - ABr^{2n}}{1 - B^2r^{2n}}\right| \le \frac{(A - B)r^n}{1 - B^2r^{2n}}.$$

In particular, if $p \in \mathcal{P}_n(\alpha)$*, then, for* $|z| = r$*,*

$$\left|p(z) - \frac{(1 + (1 - 2\alpha))r^{2n}}{1 - r^{2n}}\right| \le \frac{2(1 - \alpha)r^n}{1 - r^{2n}}.$$

Now we shall discuss the radius problem for the following classes. For brevity we shall denote them by

$$\mathcal{F}_1 := \left\{f \in \mathcal{A}_n : \operatorname{Re} \tfrac{f(z)}{g(z)} > 0 \text{ and } \operatorname{Re} \tfrac{g(z)}{z} > 0, \; g \in \mathcal{A}_n\right\},$$

$$\mathcal{F}_2 := \left\{f \in \mathcal{A}_n : \operatorname{Re} \tfrac{f(z)}{g(z)} > 0 \text{ and } \operatorname{Re} \tfrac{g(z)}{z} > 1/2, \; g \in \mathcal{A}_n\right\},$$

$$\mathcal{F}_3 := \left\{f \in \mathcal{A}_n : \left|\tfrac{f(z)}{g(z)} - 1\right| < 1 \text{ and } \operatorname{Re} \tfrac{g(z)}{z} > 0, \; g \in \mathcal{A}_n\right\}.$$

Since in the case when $\alpha = 0$ the situation becomes simple, hereafter in this section we restrict α as $0 < \alpha < 1$, unless stated specifically.

Theorem 3.7 *Sharp $\mathcal{BS}_n^*(\alpha)$-radii for functions in the classes $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_3$, respectively, are:*

1. $R_{\mathcal{BS}_n^*(\alpha)}(\mathcal{F}_1) = 2^{-1/n}\left(\frac{2}{\sqrt{4(\alpha+1)^2n^2+1}+2n(\alpha+1)}\right)^{1/n}.$

2. $R_{\mathcal{BS}_n^*(\alpha)}(\mathcal{F}_2) = 2^{-1/n}\left(\frac{4}{\sqrt{9(\alpha+1)^2n^2+4(\alpha n+n+1)}+3n(\alpha+1)}\right)^{1/n} = R_{\mathcal{BS}_n^*(\alpha)}(\mathcal{F}_3).$

Proof (1) Let us suppose $f \in \mathcal{F}_1$ and $g \in \mathcal{A}_n$. Let the functions $p, h : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = g(z)/z$ and $h(z) = f(z)/g(z)$. Then $p, h \in \mathcal{P}_n$. Since $f(z) = zp(z)h(z)$, it follows that

$$\frac{zf'(z)}{f(z)} - 1 = \frac{zh'(z)}{h(z)} + \frac{zp'(z)}{p(z)}.$$

Now from Lemma 3.8, we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\leq \left|\frac{zh'(z)}{h(z)}\right| + \left|\frac{zp'(z)}{p(z)}\right| \\ &\leq \frac{4nr^n}{1-r^{2n}} \leq \frac{1}{1+\alpha},\end{aligned}$$

for $r \leq 2^{-1/n}\left(\sqrt{(4\alpha+4)^2n^2+4} - 4\alpha n - 4n\right)^{1/n} = R_{\mathcal{BS}_n^*(\alpha)}(\mathcal{F}_1)$.

Consider the functions f_0 and g_0 defined by

$$f_0(z) = z\left(\frac{1+z^n}{1-z^n}\right)^2 \quad \text{and} \quad g_0(z) = z\left(\frac{1+z^n}{1-z^n}\right).$$

It is obvious that $\operatorname{Re}(f_0(z)/g_0(z)) > 0$ and $\operatorname{Re}(g_0(z)/z) > 0$, and therefore $f \in \mathcal{F}_1$. Now a computation shows that, for $z = R_{\mathcal{BS}_n^*(\alpha)}(\mathcal{F}_1)$,

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{4nz^n}{1-z^{2n}} = \frac{1}{1+\alpha}.$$

Hence, the result is sharp.

(2) Let $f \in \mathcal{F}_2$ and $g \in \mathcal{A}_n$. Define the functions $p, h : \mathbb{D} \to \mathbb{C}$ by $p(z) = g(z)/z$ and $h(z) = f(z)/g(z)$. Then $p \in \mathcal{P}_n$ and $h \in \mathcal{P}_n(1/2)$, and since $f(z) = zp(z)h(z)$, it follows that

$$\frac{zf'(z)}{f(z)} - 1 = \frac{zh'(z)}{h(z)} + \frac{zp'(z)}{p(z)},$$

and from Lemma 3.8, we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\leq \left|\frac{zh'(z)}{h(z)}\right| + \left|\frac{zp'(z)}{p(z)}\right| \\ &\leq \frac{4nr^n}{1-r^{2n}} + \frac{nr^n}{1-r^2} \\ &= \frac{3nr^n + nr^{2n}}{1-r^{2n}} \leq \frac{1}{1+\alpha}\end{aligned}$$

provided

$$r \le 2^{-1/n}\left(\frac{\sqrt{9(\alpha+1)^2n^2+4(\alpha n+n+1)}-3n(\alpha+1)}{\alpha n+n+1}\right)^{1/n}.$$

Thus, $f \in \mathcal{BS}^*_n(\alpha)$ for $r \le R_{\mathcal{BS}^*_n(\alpha)}(\mathcal{F}_2)$.

To see the sharpness of the result, consider the functions

$$f_0(z)=\frac{z(1+z^n)}{(1-z^n)^2} \text{ and } g_0(z)=\frac{z}{1-z^n}.$$

Then $\mathrm{Re}(f_0(z)/g_0(z))>0$ and $\mathrm{Re}(g_0(z)/z)>1/2$, and hence $f\in\mathcal{F}_2$. Now from the definition of f_0, we see that at $z=R_{\mathcal{BS}^*_n(\alpha)}(\mathcal{F}_2)$,

$$\frac{zf_0'(z)}{f_0(z)}-1=\frac{3nz^n+nz^{2n}}{1-z^{2n}}=\frac{1}{\alpha+1}.$$

(3) Now suppose $f\in\mathcal{F}_3$ and $g\in\mathcal{A}_n$. Further, define the functions $p,h:\mathbb{D}\to\mathbb{C}$ by $p(z)=g(z)/z$ and $h(z)=g(z)/f(z)$. Then $p\in\mathcal{P}_n$ and $h\in\mathcal{P}_n(1/2)$. Since $f(z)=zp(z)/h(z)$, in view of Lemma 3.8, we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-1\right| &\le \left|\frac{zp'(z)}{p(z)}\right|+\left|\frac{zh'(z)}{h(z)}\right|\\ &\le \frac{3nr^n+nr^{2n}}{1-r^{2n}}\le\frac{1}{\alpha+1},\end{aligned}$$

which holds for

$$r \le 2^{-1/n}\left(\frac{\sqrt{9(\alpha+1)^2n^2+4(\alpha n+n+1)}-3n(\alpha+1)}{\alpha n+n+1}\right)^{1/n}.$$

The result is sharp, since the equality in the result holds for the functions f and g defined by

$$f_0(z)=\frac{z(1+z^n)^2}{(1-z^n)} \text{ with } g_0(z)=\frac{z(1+z^n)}{1-z^n},$$

since at $z=R_{\mathcal{BS}^*_n(\alpha)}(\mathcal{F}_3)$, we have

$$\frac{zf_0'(z)}{f_0(z)}-1=\frac{1}{\alpha+1}.$$

This completes the proof. □

We now discuss radius estimates for the classes of starlike functions associated with lemniscate, reverse lemniscate, Booth lemniscate, exponential, and sine functions. To discuss these problems the prerequisite results are:

Lemma 3.8 [3] *Let* $Q_1(z)=\sqrt{1+z}$ *and* $\Omega_L:=Q_1(\mathbb{D})$. *Assume that* $0<a<\sqrt{2}$ *and*

$$r_a=\begin{cases}((1-(a)^2)^{1/2}-(1-(a)^2))^{1/2}, & 0<a\le 2\sqrt{2}/3;\\ \sqrt{2}-a, & 2\sqrt{2}/3\le a<\sqrt{2}.\end{cases}$$

Then $\{w\in\mathbb{C}:|w-a|<r_a\}\subset\Omega_L$.

Lemma 3.9 [18] *Let* $Q_2(z)=\sqrt{2}-\left(\sqrt{2}-1\right)\sqrt{\frac{1-z}{2\left(\sqrt{2}-1\right)z+1}}$ *and* $\Omega_R:=Q_2(\mathbb{D})$. *Assume that* $0<a<\sqrt{2}$ *and*

$$r_a=\begin{cases} a, & 0<a\leq \sqrt{2}/3; \\ ((1-(\sqrt{2}-a)^2)^{1/2}-(1-(\sqrt{2}-a)^2))^{1/2}, & \sqrt{2}/3\leq a<\sqrt{2}. \end{cases}$$

Then $\{w\in\mathbb{C}:|w-a|<r_a\}\subset\Omega_R$.

Lemma 3.10 [19] *Let* $Q_3(z)=e^z$ *and* $\Omega_e:=Q_3(\mathbb{D})$. *Assume that* $e^{-1}\leq a\leq e$ *and*

$$r_a=\begin{cases} a-e^{-1}, & e^{-1}<a\leq (e^{-1}+e)/2; \\ e-a, & (e^{-1}+e)/2\leq a<e. \end{cases}$$

Then $\{w\in\mathbb{C}:|w-a|<r_a\}\subset\Omega_e$.

Lemma 3.11 [32] *Let* $Q_4(z)=1+4z/3+2z^2/3$ *and* $\Omega_c:=Q_4(\mathbb{D})$. *Assume that* $1/3\leq a\leq 3$ *and*

$$r_a=\begin{cases} (3a-1)/3, & 1/3<a\leq 5/3; \\ 3-a, & 5/3\leq a<3. \end{cases}$$

Then $\{w\in\mathbb{C}:|w-a|<r_a\}\subset\Omega_c$.

Lemma 3.12 [7] *Let* $Q_5(z)=z+\sqrt{1+z^2}$ *and* $\Omega_q:=Q_5(\mathbb{D})$. *Assume that* $\sqrt{2}-1<a\leq\sqrt{2}+1$ *and* $r_a=1-|\sqrt{2}-a|$. *Then* $\{w\in\mathbb{C}:|w-a|<r_a\}\subset\Omega_q$.

Theorem 3.13 *Sharp* $\mathcal{S}^*_L,\mathcal{S}^*_{RL},\mathcal{S}^*_e,\mathcal{S}^*_c$, *and* $\mathcal{S}^*_q$*-radii for the class* $\mathcal{BS}^*(\alpha)$ *are:*

1. $R_{\mathcal{S}^*_L}(\mathcal{BS}^*(\alpha))=\frac{2}{(\sqrt{2}+1)(\left(1+(12-8\sqrt{2})\alpha\right)^{1/2}+1)}$.
2. $R_{\mathcal{S}^*_{RL}}(\mathcal{BS}^*(\alpha))=\frac{2}{\left(4\left(\sqrt{2\left(\sqrt{2}-1\right)}-2\sqrt{2}+2\right)\alpha+1\right)^{1/2}+1}$.
3. $R_{\mathcal{S}^*_e}(\mathcal{BS}^*(\alpha))=\frac{e-1}{e+\sqrt{(4-8e+4e^2)\alpha+e^2}}$.
4. $R_{\mathcal{S}^*_c}(\mathcal{BS}^*(\alpha))=\frac{4}{\sqrt{16\alpha+9}+3}$.
5. $R_{\mathcal{S}^*_q}(\mathcal{BS}^*(\alpha))=\frac{2(2-\sqrt{2})}{\left(\left(8\left(3-2\sqrt{2}\right)\alpha+1\right)^{1/2}+1\right)}$.

Proof (1) Let $f\in\mathcal{BS}^*(\alpha)$. Then for $|z|=r$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{r}{1-\alpha r^2}. \tag{3.4}$$

Therefore, the function $f\in\mathcal{S}^*_L$, if

$$\frac{r}{1-\alpha r^2}\leq\sqrt{2}-1.$$

Therefore, from Lemma 3.8, we see that $\mathcal{S}_L^*$-radius for the class $\mathcal{BS}^*(\alpha)$ is the root r_0 of the equation

$$(\sqrt{2}-1)\alpha r^2 + r + 1 - \sqrt{2} = 0,$$

which is given by

$$r_0 = \frac{\sqrt{1+(12-8\sqrt{2})\alpha}-1}{2\left(\sqrt{2}-1\right)\alpha}.$$

To check sharpness, we consider the function defined in (3.1). From the definition of F, we see that, at $z_0 = R_{\mathcal{S}_L^*}(\mathcal{BS}^*(\alpha))$,

$$\frac{z_0F'(z_0)}{F(z_0)} = 1 + \frac{z_0}{1-\alpha z_0^2} = \sqrt{2}.$$

Therefore, the result is the best possible.

(2) Letting $f \in \mathcal{BS}^*(\alpha)$ and proceeding as in the proof of part (1) of Theorem 3.13, we have (3.4). Therefore, in view of Lemma 3.9, we conclude that $f \in \mathcal{S}_{RL}^*$, if

$$\frac{r}{1-\alpha r^2} \leq \left((2\sqrt{2}-2)^{1/2} - \left(2\sqrt{2}-2\right)\right)^{1/2},$$

or equivalently if the following inequality holds:

$$\left((2\sqrt{2}-2)^{1/2} - \left(2\sqrt{2}-2\right)\right)^{1/2}\alpha r^2 + r - \left((2\sqrt{2}-2)^{1/2} - \left(2\sqrt{2}-2\right)\right)^{1/2} \leq 0.$$

Therefore, the $\mathcal{S}_{RL}^*$-radius for the class $\mathcal{BS}^*(\alpha)$ is the smallest positive root

$$R_{\mathcal{S}_{RL}^*}(\mathcal{BS}^*(\alpha)) = \frac{\left(4\left(\sqrt{2\left(\sqrt{2}-1\right)}-2\sqrt{2}+2\right)\alpha+1\right)^{1/2}-1}{2\left(\sqrt{2\left(\sqrt{2}-1\right)}-2\sqrt{2}+2\right)^{1/2}\alpha}$$

of the equation

$$\left((2\sqrt{2}-2)^{1/2} - \left(2\sqrt{2}-2\right)\right)^{1/2}\alpha r^2 + r - \left((2\sqrt{2}-2)^{1/2} - \left(2\sqrt{2}-2\right)\right)^{1/2} = 0.$$

Now we check for sharpness of the result. For this, we consider the function defined by (3.1). From the definition of F, at $z_1 = R_{\mathcal{S}_{RL}^*}(\mathcal{BS}^*(\alpha))$, we have

$$\frac{z_1F'(z_1)}{F(z_1)} - 1 = \left((2\sqrt{2}-2)^{1/2} - \left(2\sqrt{2}-2\right)\right)^{1/2}.$$

This indicates that the result is the best possible.

(3) Let $f \in \mathcal{BS}^*(\alpha)$. As in the above cases, from (3.4) and Lemma 3.10, we see that the function $f \in \mathcal{S}_e^*$ if

$$\frac{r}{1-\alpha r^2} \leq 1 - \frac{1}{e}$$

or

$$\alpha(e-1)r^2 + er + 1 - e \leq 0.$$

Therefore, the $\mathcal{S}_e^*$-radius of the function $f \in \mathcal{BS}^*(\alpha)$ is the smallest positive root

$$R_{\mathcal{S}_e^*}(\mathcal{BS}^*(\alpha)) = \frac{e - \sqrt{(4-8e+4e^2)\,\alpha + e^2}}{2(1-e)\alpha}$$

of the equation $\alpha(e-1)r^2 + er + 1 - e = 0$.

The result is sharp since for the function F defined in (3.1), we see that

$$\frac{z_2 F'(z_2)}{F(z_2)} - 1 = -\frac{1}{e},$$

for $z_2 = R_{\mathcal{S}_e^*}(\mathcal{BS}^*(\alpha))$.

(4) Letting $f \in \mathcal{BS}^*(\alpha)$ and proceeding as in the proof of part (1), we have (3.4). Now from Lemma 3.11 it is easy to see that the function $f \in \mathcal{S}_c^*$ if $r/(1-\alpha r^2) \leq 2/3$ or equivalently if the inequality $2\alpha r^2 + 3r - 2 \leq 0$ holds. Therefore, the $\mathcal{S}_c^*$-radius of the function $f \in \mathcal{BS}^*(\alpha)$ is the smallest positive root $R_{\mathcal{S}_c^*}(\mathcal{BS}^*(\alpha)) = (\sqrt{16\alpha+9} - 3)/(4\alpha)$ of the equation $2\alpha r^2 + 3r - 2 = 0$.

For the function F defined in (3.1), we see that

$$\frac{z_3 F'(z_3)}{F(z_3)} - 1 = \frac{2}{3},$$

where $z_3 = R_{\mathcal{S}_c^*}(\mathcal{BS}^*(\alpha))$. The result is sharp

(5) Proceeding as in the above cases from (3.4), in view of Lemma 3.12, we see that the function $f \in \mathcal{S}_q^*$ if $r/(1-\alpha r^2) \leq 2 - \sqrt{2}$, or equivalently if $(2-\sqrt{2})\alpha r^2 + r + \sqrt{2} - 2 \leq 0$. Thus, the $\mathcal{S}_q^*$-radius of the function $f \in \mathcal{BS}^*(\alpha)$ is the smallest positive root

$$R_{\mathcal{S}_q^*}(\mathcal{BS}^*(\alpha)) = \frac{(2+\sqrt{2})\left(\left(8\left(3-2\sqrt{2}\right)\alpha + 1\right)^{1/2} - 1\right)}{4\alpha}$$

of the equation $(2-\sqrt{2})\alpha r^2 + r + \sqrt{2} - 2 = 0$. Sharpness can be verified in the case of the function defined in (3.1). □

Theorem 3.14 *The sharp* $\mathcal{BS}^*(\alpha)$*-radius for the class* $\mathcal{S}_s^*$ *is* $R_{\mathcal{BS}^*(\alpha)}(\mathcal{S}_s^*) = \sinh^{-1}\left(\frac{1}{\alpha+1}\right)$.

Proof Let $f \in \mathcal{S}_s^*$. Then

$$\frac{zf'(z)}{f(z)} \prec 1 + \sin z$$

and so we can write

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = |\sin z| \leq \sinh r, \quad |z| = r.$$

Thus, for function $f \in \mathcal{BS}^*(\alpha)$, in view of Lemma 3.4, we must have $\sinh r \leq 1/(\alpha+1)$, which holds for $r \leq \operatorname{arcsinh}(1/(\alpha+1)) = R_{\mathcal{BS}^*(\alpha)}(\mathcal{S}_s^*)$.

The result is sharp as the equality holds in the case of the function defined by

$$f_0(z) = z \exp\left(\int_0^z \frac{i \sinh t}{t} dt\right).$$

Thus, we have

$$\left|\frac{z f_0'(z)}{f_0(z)} - 1\right| = |\sinh z| = \frac{1}{\alpha+1} \text{ for } z = R_{\mathcal{BS}^*(\alpha)}(\mathcal{S}_s^*).$$

□

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