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On the local superconvergence of the fully discretized multiprojection method for weakly singular Volterra integral equations of the second kind

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Abstract: In this paper, we extend the well-known multiprojection method for solving the second kind of weakly singular Volterra integral equations. We apply this method based on the collocation projection and develop a fully discretized version using appropriate quadrature rules. This method has a superconvergence property that the classic collocation method lacks. Although the new approach results in a significant increase in computational cost, when performing the related matrix-matrix products in parallel the computational time can be reduced. We provide a rigorous mathematical discussion about error analysis of this method. Finally, we present some numerical examples to confirm our theoretical results.

Key words: Fully discretized multiprojection method, collocation method, weakly singular Volterra integral equations, local convergence, error analysis

1. Introduction

In this paper, we consider the weakly singular Volterra integral equations of the second kind,

$$y(x) - \int_0^x \frac{K(x,t)}{(x-t)^\alpha} y(t)\mathrm{d}t = f(x), \quad x \in [0,T], \quad 0 < \alpha < 1, \tag{1.1}$$

where $f(x)$ and $K(x,t)$, with $K(x,x) \neq 0$ for $x \in [0,T]$, are given and continuous functions on $[0,T]$, and $D = \{(x,t) : 0 \leq t \leq x \leq T\}$, respectively. Over the past few decades, the solution of singular integral equations was the research topic for many researchers [1, 4, 21]. Weakly singular Volterra integral equations of the second kind are an important category of the singular integral equations and appear in numerous applications. Since an analytical solution of this kind of equation is not available, numerical solution methods play an important role. One example is the collocation method [6]. The convergence rate of the collocation method in the piecewise polynomial space on uniform mesh with mesh diameter h for solving (1.1) is $O(h^{1-\alpha})$, independent of the degree of the polynomials [6, 10], where α is the denominator power of integral kernel. Brunner [6, Th. 6.2.14] showed that the discretized collocation method preserves the convergence rate of the exact collocation method on the collocation points. He also showed that the collocation method on the graded mesh with $r = m/(1-\alpha)$ grading exponent has $O(h^m)$ global convergence rate [5, Th. 2.2], where the degree of the approximating

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polynomials does not exceed $m-1$. Since graded mesh points have more dense distribution near the left end point of the interval domain, the collocation method on graded mesh loses its efficiency in practice, and an accurate solution is not accessible with higher degree polynomials using more collocation points. In this paper, we extend the multiprojection method [14] and apply it to the collocation method for solving weakly singular Volterra integral equations (1.1). We also implement a fully discretized version and indicate that the discretized version has superconvergence rate $O(h^{m+1-\alpha})$ at the collocation points, which the classic collocation method lacks. Asymptotic analysis of computational complexity indicates that this superconvergence order is obtained by additional computations.

The outline of this paper is as follows: in Section 2, we recall some basic concepts. In Section 3, we introduce the exact multiprojection method and implement the fully discretized version. At the end of this section we compare the computational complexity of the proposed fully discretized multiprojection with the ordinary collocation method in an asymptotic sense. In Section 4, we study the local convergence estimate at the collocation points. Some numerical examples are given in Section 5. Finally, Section 6 is dedicated to a brief conclusion.

2. Multiprojection method

In this section, some basic concepts and notations are recalled. Let $\mathcal{K}$ denote the weakly singular Volterra integral operator from Banach space $\mathbb{X}=L^2[0,T]$ into its subspace $\mathbb{V}=C[0,T]$ defined by

$$(\mathcal{K}y)(x)=\int_0^x \frac{K(x,t)}{(x-t)^\alpha}y(t)\mathrm{d}t. \tag{2.1}$$

Then (1.1) can be written as

$$(\mathcal{I}-\mathcal{K})y=f. \tag{2.2}$$

To describe exactly the regularity properties of solutions to (2.2) assume that $f\in C^m[0,T]$, $K\in C^m(D)$ ($m\geq 1$), with $K(x,x)\neq 0$ on $[0,T]$, and then the solution lies in Hölder space $C^{1-\alpha}[0,T]$, i.e. $y\in C^m(0,T]\bigcap C[0,T]$ with $|y'(x)|\leq C_\alpha x^{-\alpha}$ for $x\in(0,T]$ (see [6, Th. 6.1.8]). For the sequence $\{\mathbb{X}_n\}$ ($n\in\mathbb{N}$) of finite-dimensional subspaces of $\mathbb{X}$ satisfying

$$\mathbb{V}\subseteq\overline{\bigcup_{n=1}^{\infty}\mathbb{X}_n}\subseteq\mathbb{X},$$

let $\{\mathcal{P}_n\}$ denote a sequence of linear projection operators from $\mathbb{X}$ into $\mathbb{X}_n$, which satisfy the following conditions [8]:

(H1) The set of operators $\{\mathcal{P}_n : n\in\mathbb{N}\}$ is uniformly bounded, i.e. there exists a positive constant p such that $\|\mathcal{P}_n\|\leq p$ for every $n\in\mathbb{N}$.

(H2) Operators $\mathcal{P}_n$ converge pointwise to the identity operator $\mathcal{I}$ on $\mathbb{V}$, i.e. for any $y\in\mathbb{V}$, $\|\mathcal{P}_n y-y\|\to 0$, as $n\to\infty$.

In the standard collocation method, we use $\mathcal{K}_n=\mathcal{P}_n\mathcal{K}\mathcal{P}_n$ as an approximation of $\mathcal{K}$ and find $y_n\in\mathbb{X}_n$ such that

$$(\mathcal{I}-\mathcal{K}_n)y_n=\mathcal{P}_n f. \tag{2.3}$$

Chen et al. [7] represented the solution of (1.1) as $y = y_n^L + y_n^H$, where $y_n^L = \mathcal{P}_n y$ and $y_n^H = (\mathcal{I} - \mathcal{P}_n)y$ correspond to lower and higher resolutions of the solution y. Using this decomposition, they rewrote the operator $\mathcal{K}$ as

$$\mathcal{K}_L^H = \begin{bmatrix} \mathcal{K}_n^{LL} & \mathcal{K}_n^{HL} \\ \mathcal{K}_n^{LH} & \mathcal{K}_n^{HH} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_n\mathcal{K}\mathcal{P}_n & \mathcal{P}_n\mathcal{K}(\mathcal{I} - \mathcal{P}_n) \\ (\mathcal{I} - \mathcal{P}_n)\mathcal{K}\mathcal{P}_n & (\mathcal{I} - \mathcal{P}_n)\mathcal{K}(\mathcal{I} - \mathcal{P}_n) \end{bmatrix}.$$

Hence, (1.1) has the following representation:

$$\begin{bmatrix} y_n^L \\ y_n^H \end{bmatrix} - \begin{bmatrix} \mathcal{K}_n^{LL} & \mathcal{K}_n^{HL} \\ \mathcal{K}_n^{LH} & \mathcal{K}_n^{HH} \end{bmatrix} \begin{bmatrix} y_n^L \\ y_n^H \end{bmatrix} = \begin{bmatrix} f_n^L \\ f_n^H \end{bmatrix},$$

where $f = f_n^L + f_n^H$, with $f_n^L = \mathcal{P}_n f$, and $f_n^H = (\mathcal{I} - \mathcal{P}_n)f$. In order to remove the interconnection between y_n^L and y_n^H, they assumed $\mathcal{K}_n^{HH} = 0$ and defined the multiprojection operator

$$\mathcal{K}_n^M = \mathcal{P}_n\mathcal{K}\mathcal{P}_n + \mathcal{P}_n\mathcal{K}(\mathcal{I} - \mathcal{P}_n) + (\mathcal{I} - \mathcal{P}_n)\mathcal{K}\mathcal{P}_n. \tag{2.4}$$

To find the approximate solution $u_n \in \mathbb{X}_n$, we use the above notation in the multiprojection method and then equation (1.1) can be written in the following form:

$$(\mathcal{I} - \mathcal{K}_n^M)u_n = f. \tag{2.5}$$

It follows from (2.4) and (H1) that

$$\|\mathcal{K}_n^M - \mathcal{K}\| = \|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\| \leq (1 + p)\|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\|. \tag{2.6}$$

Since $\mathcal{K}$ is a compact operator from $\mathbb{X}$ into $\mathbb{V}$ [17, Lem. 4.3], from (H2) we conclude that $\|\mathcal{K}_n^M - \mathcal{K}\| \to 0$ as $n \to \infty$. Therefore, $(\mathcal{I} - \mathcal{K})^{-1}$ exists and is uniformly bounded on $\mathbb{X}$.

Applying $\mathcal{P}_n$ and $\mathcal{I} - \mathcal{P}_n$ to (2.4) yields

$$u_n^L - (\mathcal{P}_n\mathcal{K} + \mathcal{P}_n\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K})u_n^L = \mathcal{P}_n f + \mathcal{P}_n\mathcal{K}(\mathcal{I} - \mathcal{P}_n)f, \tag{2.7a}$$

$$u_n^H = (\mathcal{I} - \mathcal{P}_n)(\mathcal{K}u_n^L + f), \tag{2.7b}$$

where $u_n^L = \mathcal{P}_n u_n$, and $u_n^H = (\mathcal{I} - \mathcal{P}_n)u_n$.

3. Implementation of multiprojection method

For a constant $N \in \mathbb{N}$, let

$$I_h = \{x_n | x_n \in [0, T],\ 0 = x_0 < x_1 < \cdots < x_N = T\}$$

denote a mesh on the given interval $I = [0, T]$ where

$$h = \max\{h_n = x_n - x_{n-1}, n = 1, 2, \ldots, N\}, \quad h' = \min\{h_n = x_n - x_{n-1}, n = 1, 2, \ldots, N\},$$

and set

$$\sigma_0 := [x_0, x_1], \quad \sigma_n := (x_n, x_{n+1}].$$

For uniform mesh, we have $x_n = \frac{n}{N}T$ and

$$h_n = h' = h = \frac{T}{N} \quad (n = 1, 2, \dots, N).$$

We define the graded mesh with grading exponent r as

$$\begin{aligned} x_n &= \frac{T}{2}\left(\frac{2n}{N}\right)^r, \quad 0 \le n \le \frac{N}{2}, \\ x_n &= T - x_{N-n}, \quad \frac{N}{2} < n \le N, \end{aligned} \tag{3.1a}$$

or

$$x_n = T\left(\frac{n}{N}\right)^r, \quad 0 \le n \le N. \tag{3.1b}$$

In this section the solution of (2.7) is approximated by collocation in the piecewise polynomial space

$$\mathbb{P}_{m-1}^{(-1)}(I_h) = \{p : p|_{\sigma_n} \in \pi_{m-1}\,(0 \le n \le N-1)\}.$$

Assume that $\mathcal{P}_h$ is a collocation projection into $\mathbb{P}_{m-1}^{(-1)}(I_h)$ and the given mesh I_h and the given collocation parameters $\{c_i\} \subset [0,1]$ determine the set of collocation points

$$X_h = \{x_{n,i} = x_n + c_i h_n |\, 0 \le c_1 \le c_2 \le \cdots \le c_m \le 1\,(n = 0, 1, \dots, N-1)\}. \tag{3.2}$$

3.1. The exact multiprojection method

Let $u_h = u_h^L + u_h^H$ denote the desired collocation solution of (2.7) on X_h. Hence, u_h^L satisfies the following collocation equation:

$$u_h^L(x) - (\mathcal{K}\mathcal{P}_h + \mathcal{K}\mathcal{K}\mathcal{P}_h - \mathcal{K}\mathcal{P}_h\mathcal{K}\mathcal{P}_h)u_h^L(x) = f(x) + (\mathcal{K} - \mathcal{K}\mathcal{P}_h)f(x), \quad x \in X_h. \tag{3.3}$$

For $x \in [x_n, x_{n+1}]$, set $x := x_n + \upsilon h_n$, $(0 < \upsilon < 1)$. Thus, we have

$$\int_0^x \frac{K(x,t)}{(x-t)^\alpha} u_h^L(t)\mathrm{d}t = \int_0^{x_n} \frac{K(x,t)}{(x-t)^\alpha} u_h^L(t)\mathrm{d}t + \int_{x_n}^x \frac{K(x,t)}{(x-t)^\alpha} u_h^L(t)\mathrm{d}t,$$

and

$$\int_0^{x_n} \frac{K(x,t)}{(x-t)^\alpha} u_h^L(t)\mathrm{d}t = \sum_{\ell=0}^{n-1} \int_{x_\ell}^{x_{\ell+1}} \frac{K(x,t)}{(x-t)^\alpha} u_h^L(t)\mathrm{d}t.$$

The collocation solution u_h^L on the subinterval σ_ℓ employing the Lagrange base functions with respect to the collocation parameters $\{c_i\}$ has the following representation:

$$u_h^L(t) = u_h^L(x_\ell + s h_\ell) = \sum_{j=1}^m U_{\ell,j}^L \phi_j(s), \tag{3.4}$$

where

$$\phi_j(s) = \begin{cases} \prod\limits_{\substack{k=1\\k\neq j}}^{m} \frac{s-c_k}{c_j-c_k}, & 0 < s \leq 1, \\ 0, & o.w., \end{cases}$$

and $U^L_{\ell,j} = u^L_h(x_\ell + c_j h_\ell) \quad (j = 1, \ldots, m)$. On the subinterval σ_ℓ, we can write

$$\begin{aligned}\int_{x_\ell}^{x_{\ell+1}} \frac{K(x,t)}{(x-t)^\alpha} u^L_h(t)\mathrm{d}t &= h_\ell \int_0^1 \frac{K(x, x_\ell + sh_\ell)}{(x - x_\ell - sh_\ell)^\alpha} u^L_h(x_\ell + sh_\ell)\mathrm{d}s \\ &= h_\ell \sum_{j=1}^{m} U^L_{\ell,j} \int_0^1 \frac{K(x, x_\ell + sh_\ell)}{(x - x_\ell - sh_\ell)^\alpha} \phi_j(s)\mathrm{d}s.\end{aligned}$$

Set

$$b^{(\ell)}_{n,j}(x) := h_\ell \int_0^1 \frac{K(x, x_\ell + sh_\ell)}{(x - x_\ell - sh_\ell)^\alpha} \phi_j(s)\mathrm{d}s,$$

and define $\mathbf{b}^{(\ell)}_n(x) := [b^\ell_{n,1}(x), \ldots, b^\ell_{n,m}(x)]$, $\mathbf{u}^L_\ell := [U^L_{\ell,1}, \ldots, U^L_{\ell,m}]^T$. Hence, we have

$$\int_{x_\ell}^{x_{\ell+1}} \frac{K(x,t)}{(x-t)^\alpha} u^L_h(t)\mathrm{d}t = \mathbf{b}^{(\ell)}_n(x)\mathbf{u}^L_\ell.$$

For $x = x_n + \upsilon h_n$ in the subinterval σ_n, we can write

$$\begin{aligned}\int_{x_n}^{x_n+\upsilon h_n} \frac{K(x,t)}{(x-t)^\alpha} u^L_h(t)\mathrm{d}t &= h_n \int_0^\upsilon \frac{K(x, x_n + sh_n)}{(x - x_n - sh_n)^\alpha} u^L_h(x_n + sh_n)\mathrm{d}s \\ &= h_n \sum_{j=1}^{m} \int_0^\upsilon \frac{K(x, x_n + sh_n)}{(x - x_n - sh_n)^\alpha} u^L_h(x_n + sc_j h_n)\phi_j(s)\mathrm{d}s \\ &= \upsilon h_n \sum_{j=1}^{m} U^L_{n,j} \int_0^1 \frac{K(x, x_n + s\upsilon h_n)}{(x - x_n - s\upsilon h_n)^\alpha} \phi_j(s\upsilon)\mathrm{d}s \\ &= \upsilon h_n \sum_{j=1}^{m}\sum_{k=1}^{m} U^L_{n,j}\phi_j(c_k\upsilon) \int_0^1 \frac{K(x, x_n + s\upsilon h_n)}{(x - x_n - s\upsilon h_n)^\alpha} \phi_k(s)\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}\upsilon \int_0^1 \frac{K(x, x_n + s\upsilon h_n)\phi_k(s)}{(x_n - \upsilon h_n - x_n - s\upsilon h_n)^\alpha}\mathrm{d}s &= h_n^{-\alpha} \int_0^1 \frac{K(x, x_n + s\upsilon h_n)}{(\upsilon - s\upsilon)^\alpha} \phi_k(s)\upsilon\mathrm{d}s \\ &= h_n^{-\alpha} \int_0^\upsilon \frac{K(x, x_n + sh_n)}{(\upsilon - s)^\alpha} \phi_k(s/\upsilon)\mathrm{d}s.\end{aligned}$$

Since $\upsilon = (x - x_n)/h_n$, set

$$b_{n,j}(x) := h_n^{1-\alpha} \sum_{k=1}^{m} \phi_j(c_k(x - x_n)/h_n) \int_0^{(x-x_n)/h_n} K(x, x_n + sh_n) \frac{\phi_k(sh_n/(x - x_n))}{((x - x_n)/h_n - s)^\alpha}\mathrm{d}s,$$

and define $\mathbf{b}_n(x) := [b_{n,1}(x), \ldots, b_{n,m}(x)]$. We have

$$\int_{x_n}^{x_n+\upsilon h_n} \frac{K(x,t)}{(x-t)^\alpha} u^L_h(t)\mathrm{d}t = \mathbf{b}_n(x)\mathbf{u}^L_n.$$

Set $\mathbf{b}(x) := [\mathbf{b}_n^{(0)}(x), \ldots, \mathbf{b}_n^{(n-1)}(x), \mathbf{b}_n(x), \overbrace{\mathbf{0}, \ldots, \mathbf{0}}^{m(N-n-1)}]$ and $\mathbf{u}^L := \begin{bmatrix} \mathbf{u}_0^L \\ \vdots \\ \mathbf{u}_{N-1}^L \end{bmatrix}$. Hence, we have $\mathcal{K}\mathcal{P}_h u_h^L(x) = \mathbf{b}(x)\mathbf{u}^L$. Set

$$\mathbf{B}_n^{(\ell)} := \begin{bmatrix} \mathbf{b}_n^{(\ell)}(x_{n,1}) \\ \mathbf{b}_n^{(\ell)}(x_{n,2}) \\ \vdots \\ \mathbf{b}_n^{(\ell)}(x_{n,m}) \end{bmatrix}, \mathbf{B}_n := \begin{bmatrix} \mathbf{b}_n(x_{n,1}) \\ \mathbf{b}_n(x_{n,2}) \\ \vdots \\ \mathbf{b}_n(x_{n,m}) \end{bmatrix}, \mathbf{B} := \begin{bmatrix} \mathbf{B}_0 & & & & \\ \mathbf{B}_1^{(0)} & \mathbf{B}_1 & & & \\ \mathbf{B}_2^{(0)} & \mathbf{B}_2^{(1)} & \mathbf{B}_2 & & \\ \vdots & \vdots & & \ddots & \\ \mathbf{B}_{N-1}^{(0)} & \mathbf{B}_{N-1}^{(1)} & \ldots & \mathbf{B}_{N-1}^{(N-2)} & \mathbf{B}_{N-1} \end{bmatrix}.$$

Hence, $\mathcal{K}\mathcal{P}_h u_h^L(x)$ and $\mathcal{K}\mathcal{P}_h\mathcal{K}\mathcal{P}_h u_h^L(x)$, $x \in X_h$, have the matrix form $\mathbf{B}\mathbf{u}^L$ and $\mathbf{B}\mathbf{B}\mathbf{u}^L$, respectively. Using Dirichlet's formula [11] we obtain

$$\begin{aligned}\int_0^x \frac{K(x,y)}{(x-y)^\alpha} \int_0^y \frac{K(y,t)}{(y-t)^\alpha} u_h^L(t)\mathrm{d}t\mathrm{d}y &= \int_0^x \int_0^y \frac{K(x,y)K(y,t)}{[(x-y)(y-t)]^\alpha} u_h^L(t)\mathrm{d}t\mathrm{d}y \\ &= \int_0^x \int_t^x \frac{K(x,y)K(y,t)}{[(x-y)(y-t)]^\alpha}\mathrm{d}y\, u_h^L(t)\mathrm{d}t.\end{aligned}$$

By the change of variable $y = (x-t)Y + t$,

$$\int_t^x \frac{K(x,y)K(y,t)}{[(x-y)(y-t)]^\alpha}\mathrm{d}y = \frac{1}{(x-t)^{2\alpha-1}} \int_0^1 \frac{K(x,(x-t)Y+t)K((x-t)Y+t,t)}{[Y(1-Y)]^\alpha}\mathrm{d}Y.$$

Set

$$H(x,t) := \int_0^1 \frac{K(x,(x-t)Y+t)K((x-t)Y+t,t)}{[Y(1-Y)]^\alpha}\mathrm{d}Y. \tag{3.5}$$

Hence, we have

$$\int_0^x \frac{K(x,y)}{(x-y)^\alpha} \int_0^y \frac{K(y,t)}{(y-t)^\alpha} u_h^L(t)\mathrm{d}t\mathrm{d}y = \int_0^x \frac{H(x,t)}{(x-t)^{2\alpha-1}} u_h^L(t)\mathrm{d}t. \tag{3.6}$$

By substituting $H(x,t)$ for $K(x,t)$ in the definition of entries in $\mathbf{B}$, we define lower triangular block matrix $\mathbf{B}_H$ similar to the definition of $\mathbf{B}$. We define $\mathbf{f} = [f(x_{0,1}), \ldots, f(x_{0,m}), \ldots, f(x_{N-1,1}), \ldots, f(x_{N-1,m})]^T$. Hence, $\mathcal{K}\mathcal{P}_h f(x)$, $x \in X_h$ has the matrix form $\mathbf{B}\mathbf{f}$. We also define

$$\mathbf{kf} = [\mathcal{K}f(x_{0,1}), \ldots, \mathcal{K}f(x_{0,m}), \ldots, \mathcal{K}f(x_{N-1,1}), \ldots, \mathcal{K}f(x_{N-1,m})]^T.$$

The low resolution solution u_h^L is determined by the solution of the following liner system of equations:

$$\mathbf{A}_M\mathbf{u}^L = \mathbf{R}_M, \tag{3.7}$$

where $\mathbf{A}_M = \mathbf{I} - (\mathbf{B} + \mathbf{B}_H - \mathbf{B}^2)$ and $\mathbf{R}_M = \mathbf{f} + \mathbf{kf} - \mathbf{B}\mathbf{f}$. Note that $\mathbf{I}$ denotes the identity matrix of size mN.

Theorem 3.1 *Assume that f and K in (1.1) are continuous on their respective domains $[0,T]$ and D. Then there is an $\bar{h} = \bar{h}(\alpha) > 0$ such that, for any $\alpha \in (0,1]$ and every mesh I_h with mesh diameter h less than $\bar{h}$, the linear algebraic system (3.7) has a unique solution $\mathbf{u}^L$.*

Proof The proof is similar to that of Theorem 2.2.1 of [6]. By the assumptions on the kernel factor K in (1.1), the entries of the matrices $\mathbf{B}$ and $\mathbf{B}_H$ are bounded for $\alpha \in (0,1]$. This implies that the inverse of $\mathbf{A}_M = \mathbf{I} - (\mathbf{B} + \mathbf{B}_H - \mathbf{B}^2)$ exists if $\|\mathbf{B} + \mathbf{B}_H - \mathbf{B}^2\| < 1$ for some matrix norm. Since each element of $\mathbf{B}$ and $\mathbf{B}_H$ has a mesh distance h_ℓ $(0 \le \ell \le N-1)$ multiplier, the inverse of $\mathbf{A}_M$ for any mesh with sufficiently small mesh diameter exists. In other words, there exists an $\bar{h} = \bar{h}(\alpha) > 0$ such that for every mesh I_h with $h = \max\{h_\ell : 0 \le \ell \le N-1\} < \bar{h}$, the matrix $\mathbf{A}_M$ has a bounded inverse. Hence, the assertion holds. □

Set

$$\boldsymbol{\phi}(x) := \left[\phi_1((x-x_0)/h_0), \ldots, \phi_m((x-x_0)/h_0), \ldots, \phi_1((x-x_{N-1})/h_{N-1}), \ldots, \phi_m((x-x_{N-1})/h_{N-1})\right]^T.$$

Then $u_h^H(x)$ is given by

$$u_h^H(x) = f(x) - \mathbf{f}^T\boldsymbol{\phi}(x) + \mathbf{b}(x)\mathbf{u}^L - \boldsymbol{\phi}^T(x)\mathbf{B}\mathbf{u}^L. \tag{3.8}$$

3.2. Fully discretized multiprojection method

Let $\hat{\mathcal{K}}$ and $\breve{\mathcal{K}}$ denote the discrete versions of $\mathcal{K}\mathcal{P}_h$ and $\mathcal{K}$, where the *interpolatory product Gauss-type quadrature formula* [12] on the uniform and graded meshes is employed to approximate the singular integrals, respectively. Hence, the fully discretized version of (3.3) is the following collocation equation:

$$\hat{u}_h^L(x) - (\hat{\mathcal{K}} + \breve{\mathcal{K}}\hat{\mathcal{K}} - \hat{\mathcal{K}}\hat{\mathcal{K}})\hat{u}_h^L(x) = f(x) + (\breve{\mathcal{K}} - \hat{\mathcal{K}})f(x), \quad x \in X_h, \tag{3.9}$$

where $\hat{u}_h = \hat{u}_h^L + \hat{u}_h^H$ denotes the discrete version associated with the solution u_h to (2.7). On the subinterval σ_n for $x = x_n + \upsilon h_n$ $(0 < \upsilon < 1)$, we have

$$\int_0^x \frac{K(x,t)}{(x-t)^\alpha}\hat{u}_h^L(t)\mathrm{d}t = \sum_{\ell=0}^{n-1}\int_{x_\ell}^{x_{\ell+1}} \frac{K(x,t)}{(x-t)^\alpha}\hat{u}_h^L(t)\mathrm{d}t + \int_{x_n}^{x} \frac{K(x,t)}{(x-t)^\alpha}\hat{u}_h^L(t)\mathrm{d}t.$$

The discrete collocation solution $\hat{u}_h^L$ on the subinterval σ_ℓ similar to (3.4) has the following representation:

$$\hat{u}_h^L(t) = \hat{u}_h^L(x_\ell + sh_\ell) = \sum_{j=1}^m \hat{U}_{\ell,j}^L\phi_j(s), \tag{3.10}$$

where $\hat{U}_{\ell,j}^L = \hat{u}_h^L(x_\ell + c_jh_\ell)$ $\quad (j = 1, \ldots, m)$. On the subinterval σ_ℓ, we can write

$$\begin{aligned}\int_{x_\ell}^{x_{\ell+1}} \frac{K(x,t)}{(x-t)^\alpha}\hat{u}_h^L(t)\mathrm{d}t &= h_\ell\int_0^1 \frac{K(x,x_\ell+sh_\ell)}{(x-x_\ell-sh_\ell)^\alpha}\hat{u}_h^L(x_\ell+sh_\ell)\mathrm{d}s \\ &= h_\ell\sum_{j=1}^m \hat{U}_{\ell,j}^L\int_0^1 \frac{K(x,x_\ell+c_jh_\ell)}{(x-x_\ell-sh_\ell)^\alpha}\phi_j(s)\mathrm{d}s.\end{aligned}$$

Set

$$w_{n,j}^{(\ell)}(x,\alpha) := \int_0^1 \frac{\phi_j(s)}{(x-x_\ell-sh_\ell)^\alpha}\mathrm{d}s, \quad \hat{b}_{n,j}^{(\ell)}(x) := h_\ell w_{n,j}^{(\ell)}(x,\alpha)K(x,x_\ell+c_jh_\ell),$$

and define $\hat{\mathbf{b}}_n^{(\ell)}(x) := [\hat{b}_{n,1}^{\ell}(x), \ldots, \hat{b}_{n,m}^{\ell}(x)]$, $\hat{\mathbf{u}}_\ell^L := [\hat{U}_{\ell,1}^L, \ldots, \hat{U}_{\ell,m}^L]^T$. Hence, we have

$$\int_{x_\ell}^{x_{\ell+1}} \frac{K(x,t)}{(x-t)^\alpha} \hat{u}_h^L(t)\mathrm{d}t = \hat{\mathbf{b}}_n^{(\ell)}(x)\hat{\mathbf{u}}_\ell^L.$$

For $x = x_n + \upsilon h_n$ in the subinterval σ_n, we can write

$$\begin{aligned}
\int_{x_n}^{x_n+\upsilon h_n} \frac{K(x,t)}{(x-t)^\alpha} \hat{u}_h^L(t)\mathrm{d}t = h_n \int_0^\upsilon \frac{K(x, x_n + sh_n)}{(x - x_n - sh_n)^\alpha} \hat{u}_h^L(x_n + sh_n)\mathrm{d}s \\
= & h_n \sum_{j=1}^m \int_0^\upsilon \frac{K(x, x_n + sh_n)}{(x - x_n - sh_n)^\alpha} \hat{u}_h^L(x_n + c_j h_n)\phi_j(s)\mathrm{d}s \\
= & \upsilon h_n \sum_{j=1}^m \hat{u}_{n,j}^L \int_0^1 \frac{K(x, x_n + s\upsilon h_n)}{(x - x_n - s\upsilon h_n)^\alpha} \hat{\phi}_j(s\upsilon)\mathrm{d}s \\
= & \upsilon h_n \sum_{j=1}^m \sum_{k=1}^m \hat{u}_{n,j}^L K(x, x_n + c_k \upsilon h_n)\phi_j(c_k\upsilon) \int_0^1 \frac{\phi_k(s)}{(x - x_n - s\upsilon h_n)^\alpha}\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\upsilon \int_0^1 \frac{\phi_k(s)}{(x_n - \upsilon h_n - x_n - s\upsilon h_n)^\alpha}\mathrm{d}s = h_n^{-\alpha} \int_0^1 \frac{\phi_k(s)\upsilon}{(\upsilon - s\upsilon)^\alpha}\mathrm{d}s \\
= h_n^{-\alpha} \int_0^\upsilon \frac{\phi_k(s/\upsilon)}{(\upsilon - s)^\alpha}\mathrm{d}s.
\end{aligned}$$

Since $\upsilon = (x - x_n)/h_n$, set

$$w_{n,k}(x,\alpha) := h_n^{-\alpha} \int_0^{(x-x_n)/h_n} \frac{\phi_k(sh_n/(x - x_n))}{((x - x_n)/h_n - s)^\alpha}\mathrm{d}s,$$

$$\hat{b}_{n,j}(x) := h_n \sum_{k=1}^m K(x, x_n + c_k(x - x_n))\phi_j(c_k(x - x_n)/h_n)w_{n,k}(x,\alpha),$$

and define $\hat{\mathbf{b}}_n(x) := [\hat{b}_{n,1}(x), \ldots, \hat{b}_{n,m}(x)]$. We have

$$\int_{x_n}^{x_n+\upsilon h_n} \frac{K(x,t)}{(x-t)^\alpha} \hat{u}_h^L(t)\mathrm{d}t = \hat{\mathbf{b}}_n(x)\hat{\mathbf{u}}_n^L.$$

Set $\hat{\mathbf{b}}(x) := [\hat{\mathbf{b}}_n^{(0)}(x), \ldots, \hat{\mathbf{b}}_n^{(n-1)}(x), \hat{\mathbf{b}}_n(x), \overbrace{\mathbf{0}, \ldots, \mathbf{0}}^{m(N-n-1)}]$ and $\hat{\mathbf{u}}^L := \begin{bmatrix} \hat{\mathbf{u}}_0^L \\ \vdots \\ \hat{\mathbf{u}}_{N-1}^L \end{bmatrix}$. Hence, we have $\hat{\mathcal{K}}\hat{u}_h^L(x) = \hat{\mathbf{b}}(x)\hat{\mathbf{u}}^L$.

Set

$$\hat{\mathbf{B}}_n^{(\ell)} := \begin{bmatrix} \hat{\mathbf{b}}_n^{(\ell)}(x_{n,1}) \\ \hat{\mathbf{b}}_n^{(\ell)}(x_{n,2}) \\ \vdots \\ \hat{\mathbf{b}}_n^{(\ell)}(x_{n,m}) \end{bmatrix}, \hat{\mathbf{B}}_n := \begin{bmatrix} \hat{\mathbf{b}}_n(x_{n,1}) \\ \hat{\mathbf{b}}_n(x_{n,2}) \\ \vdots \\ \hat{\mathbf{b}}_n(x_{n,m}) \end{bmatrix}, \hat{\mathbf{B}} := \begin{bmatrix} \hat{\mathbf{B}}_0 & & & & \\ \hat{\mathbf{B}}_1^{(0)} & \hat{\mathbf{B}}_1 & & & \\ \hat{\mathbf{B}}_2^{(0)} & \hat{\mathbf{B}}_2^{(1)} & \hat{\mathbf{B}}_2 & & \\ \vdots & \vdots & & \ddots & \\ \hat{\mathbf{B}}_{N-1}^{(0)} & \hat{\mathbf{B}}_{N-1}^{(1)} & \ldots & \hat{\mathbf{B}}_{N-1}^{(N-2)} & \hat{\mathbf{B}}_{N-1} \end{bmatrix}.$$

Hence, $\hat{\mathcal{K}}\hat{u}_h^L(x)$ and $\hat{\mathcal{K}}\hat{\mathcal{K}}\hat{u}_h^L(x)$, $x \in X_h$, have the matrix form $\hat{\mathbf{B}}\hat{\mathbf{u}}^L$ and $\hat{\mathbf{B}}\hat{\mathbf{B}}\hat{\mathbf{u}}^L$, respectively. In the following multiple integral, using Dirichlet's formula similar to (3.6) we obtain

$$\int_0^x \frac{K(x,y)}{(x-y)^\alpha} \int_0^y \frac{K(y,t)}{(y-t)^\alpha} \hat{u}_h^L(t)\mathrm{d}t\mathrm{d}y = \int_0^x \frac{H(x,t)}{(x-t)^{2\alpha-1}} \hat{u}_h^L(t)\mathrm{d}t, \tag{3.11}$$

where $H(x,t)$ is given by (3.5). For simple $K(x,t)$, e.g., polynomial functions, $H(x,t)$ can be computed exactly. For more complex $K(x,t)$, $H(x,t)$ is approximated by the *interpolatory product Gauss-type quadrature formula* [12] employing the graded mesh defined in (3.1a) with grading exponent $r = \frac{m}{1-\alpha}$. Let $\breve{H}(x,t)$ denote the approximate value of $H(x,t)$. In the case of $\frac{1}{2} < \alpha < 1$, we will proceed in the following way. Similar to $\hat{\mathbf{B}}$, define the lower triangular block matrix $\hat{\breve{\mathbf{B}}}$ by the following entries:

$$\hat{\breve{b}}_{n,j}^{(\ell)}(x) = h_\ell w_{\ell,j}(x, 2\alpha-1)\breve{H}(x, x_\ell + c_j h_\ell),$$

$$\hat{\breve{b}}_{n,j}(x) = h_n \sum_{k=1}^{m} \breve{H}(x, x_n + c_k(x-x_n))\phi_j(c_k(x-x_n)/h_n) w_{n,k}(x, 2\alpha-1).$$

In the case of $0 < \alpha < \frac{1}{2}$, we will approximate (3.11) by the *Gauss-Legendre quadrature* [2, p. 887]. Let $\{(p_i, w_i)\}_{i=1}^{m'}$ denote the abscissas and weights of the *Gauss-Legendre quadrature* over the interval $[0,1]$. In this case, we define

$$\begin{aligned} \hat{\breve{b}}_{n,j}^{(\ell)}(x) &= h_\ell \sum_{i=1}^{m'} w_i \frac{\breve{H}(x, x_\ell + p_i h_\ell)}{(x - x_\ell - p_i h_\ell)^{2\alpha-1}} \phi_j(p_i), \\ \hat{\breve{b}}_{n,j}(x) &= h_n^{2(1-\alpha)} \sum_{k=1}^{m} \phi_j(c_k\upsilon) \sum_{i=1}^{m'} w_i \frac{\breve{H}(x, x_n + p_i\upsilon h_n)}{(\upsilon - p_i\upsilon)^{2\alpha-1}} \phi_k(p_i\upsilon), \quad (\upsilon = (x-x_n)/h_n). \end{aligned} \tag{3.12}$$

Hence, $\breve{\mathcal{K}}\hat{\mathcal{K}}\hat{u}_h^L(x)$, $x \in X_h$ has the matrix form $\hat{\breve{\mathbf{B}}}\hat{\mathbf{u}}^L$. Furthermore, $\hat{\mathcal{K}}f(x)$, $x \in X_h$ has the matrix form $\hat{\mathbf{B}}\mathbf{f}$. In order to evaluate $\breve{\mathcal{K}}f(x)$ for $x \in X_h$, we use the graded mesh $\{y_n\}_{n=0}^N$ of type (3.1b). For $x \in [y_n, y_{n+1}]$, set $x := y_n + \upsilon\breve{h}_n$ $(0 < \upsilon < 1)$, where $\breve{h}_n = y_{n+1} - y_n$. Thus, we have

$$\int_0^x \frac{K(x,t)}{(x-t)^\alpha} f(t)\mathrm{d}t = \sum_{\ell=0}^{n-1} \int_{y_\ell}^{y_{\ell+1}} \frac{K(x,t)}{(x-t)^\alpha} f(t)\mathrm{d}t + \int_{y_n}^x \frac{K(x,t)}{(x-t)^\alpha} f(t)\mathrm{d}t,$$

and

$$\int_{y_\ell}^{y_{\ell+1}} \frac{K(x,t)}{(x-t)^\alpha} f(t)\mathrm{d}t = \sum_{j=1}^{m} \breve{b}_{n,j}^{(l)}(x) f(y_\ell + c_j\breve{h}_\ell),$$

$$\int_{y_n}^x \frac{K(x,t)}{(x-t)^\alpha} f(t)\mathrm{d}t = \sum_{j=1}^{m} \breve{b}_{n,j}(x) f(y_\ell + c_j\breve{h}_\ell),$$

where

$$\breve{w}_{n,j}^{(\ell)}(x,\alpha) = \int_0^1 \frac{\phi_j(s)}{(x-y_\ell-s\breve{h}_\ell)^\alpha}\mathrm{d}s,$$

$$\breve{b}_{n,j}^{(\ell)}(x) = \breve{h}_\ell \breve{w}_{n,j}^{(\ell)}(x,\alpha)K(x,y_\ell+c_j\breve{h}_\ell),$$

$$\breve{w}_{n,k}(x,\alpha) = \breve{h}_n^{-\alpha}\int_0^{(x-y_n)/\breve{h}_n} \frac{\phi_k(s\breve{h}_n/(x-y_n))}{((x-y_n)/\breve{h}_n-s)^\alpha}\mathrm{d}s,$$

$$\breve{b}_{n,j}(x) = \breve{h}_n\sum_{k=1}^{m} K(x,y_n+c_k(x-y_n))\phi_j(c_k(x-y_n)/\breve{h}_n)\breve{w}_{n,k}(x,\alpha).$$

Using the above $\breve{b}_{n,j}^{(\ell)}(x)$ and $\breve{b}_{n,j}(x)$ entries for $x \in X_h$, define $\breve{\mathbf{B}}$ similar to the definition of $\hat{\mathbf{B}}$.

Define $\breve{\mathbf{f}} = [f(y_{0,1}),\ldots,f(y_{0,m}),\ldots,f(y_{N-1,1}),\ldots,f(y_{N-1,m})]^T$. Hence, $\breve{\mathcal{K}}f(x)$, $x \in X_h$ has the matrix form $\breve{\mathbf{B}}\breve{\mathbf{f}}$. The low resolution solution $\hat{u}_h^L$ is determined by the solution of the following linear system of equations:

$$\hat{\mathbf{A}}_M\hat{\mathbf{u}}^L = \hat{\mathbf{R}}_M, \tag{3.13}$$

where $\hat{\mathbf{A}}_M = \mathbf{I} - (\hat{\mathbf{B}} + \hat{\breve{\mathbf{B}}} - \hat{\mathbf{B}}\hat{\mathbf{B}})$ and $\hat{\mathbf{R}}_M = \mathbf{f} + \breve{\mathbf{B}}\breve{\mathbf{f}} - \hat{\mathbf{B}}\mathbf{f}$.

Theorem 3.2 *Assume that the assumptions of Theorem 3.1 hold. Then there is an $\hat{h} = \hat{h}(\alpha) > 0$ such that, for any $\alpha \in (0,1]$ and every mesh I_h with mesh diameter h less than $\hat{h}$, the linear algebraic system (3.13) has a unique solution $\hat{\mathbf{u}}^L$.*

Proof The proof proceeds along the same lines as the proof of Theorem 3.1. Since the weights $w_{n,j}^{\ell}(x,\alpha)$, $w_{n,j}(x,\alpha)$, $\breve{w}_{n,j}^{\ell}(x,\alpha)$, and $\breve{w}_{n,j}(x,\alpha)$ in the entries of $\hat{\mathbf{B}}$ and $\hat{\breve{\mathbf{B}}}$ are bounded, it follows from the continuity of K and the Neumann lemma [18, p. 26] that the matrix $\hat{\mathbf{A}}_M$ possesses a bounded inverse on any uniform mesh with mesh diameter h satisfying $h < \hat{h}$, for sufficiently small $\hat{h} > 0$ depending on α. □

After solving (3.13) for $\hat{\mathbf{u}}^L$, $\hat{u}_h^H(x)$ is given by

$$\hat{u}_h^H(x) = f(x) - \mathbf{f}^T\boldsymbol{\phi}(x) + \hat{\mathbf{b}}(x)\hat{\mathbf{u}}^L - \boldsymbol{\phi}^T(x)\hat{\mathbf{B}}\hat{\mathbf{u}}^L. \tag{3.14}$$

In the ordinary collocation (OC) method the linear system of equations $\hat{\mathbf{A}}\mathbf{y} = \mathbf{f}$ should be solved and the coefficient matrix $\hat{\mathbf{A}}$ is a lower triangular matrix defined by $\hat{\mathbf{A}} = \mathbf{I} - \hat{\mathbf{B}}$. Computational costs of different terms including multiplications/divisions involved in OC and multiprojection (MP) methods are presented in Table 1. The operation counts needed for solution of linear systems are also presented, where μ is the quotient of a division cost versus multiplication cost. Computer specification determines the value of μ and usually we have $2.5 \leq \mu \leq 3$ [3]. Let CC_{OC} and CC_{MP} denote the computational complexity of OC and MP methods, respectively. Using the information given in Table 1, we get the following comparison results of computational costs in an asymptotic sense

$$\lim_{N\to\infty} \frac{CC_{MP}(n,m,m',\mu)/CC_{OC}(n,m,m',\mu)}{5+\frac{\mu}{2}+m'+\frac{Nm}{6}} = 1, \quad (0<\alpha\leq\frac{1}{2}),$$

Table 1. Computational cost of different terms involved in OC and MP methods.

	Multiplications	Divisions	Computational cost
$\hat{\mathbf{B}}$	$N(N-1)m^2$ $+\frac{1}{2}Nm(m+1)(2m+1)$	0	$N(N-1)m^2$ $+\frac{1}{2}Nm(m+1)(2m+1)$
$\hat{\hat{\mathbf{B}}}^{\dagger}$	$\frac{1}{2}N(N-1)m^2(2m'+1)$ $+\frac{1}{2}m(m+1)((2m'+1)m+1)$	$\frac{1}{2}Nm^2(m+1)m'$ $+\frac{1}{2}n(N-1)m^2$	$\frac{1}{2}N(N-1)m^2(2m'+1)$ $\mu\big(\frac{1}{2}m(m+1)((2m'+1)m+1)$ $+\frac{1}{2}Nm^2(m+1)m'\big)$
$\hat{\hat{\mathbf{B}}}^{\ddagger}$	$N(N-1)m^2$ $+\frac{1}{2}Nm(m+1)(2m+1)$	0	$N(N-1)m^2$ $+\frac{1}{2}Nm(m+1)(2m+1)$
$\hat{\mathbf{B}}\hat{\mathbf{B}}$	$\frac{1}{2}(\frac{1}{6}Nm(Nm+1)(2Nm+1)$ $+\frac{1}{2}Nm(Nm+1))$	0	$\frac{1}{2}(\frac{1}{6}Nm(Nm+1)(2Nm+1)$ $+\frac{1}{2}Nm(Nm+1))$
$\breve{\mathbf{B}}$	$N(N-1)m^2$ $+\frac{1}{2}Nm(m+1)(2m+1)$	0	$N(N-1)m^2$ $+\frac{1}{2}Nm(m+1)(2m+1)$
$\breve{\mathbf{B}}\breve{\mathbf{f}}$	$\frac{1}{2}Nm(Nm+1)$	0	$\frac{1}{2}Nm(Nm+1)$
$\hat{\mathbf{B}}\mathbf{f}$	$\frac{1}{2}Nm(Nm+1)$	0	$\frac{1}{2}Nm(Nm+1)$
$\hat{\mathbf{A}}\mathbf{y}=\mathbf{f}$	$\frac{1}{2}Nm(Nm-1)$	Nm	$\frac{1}{2}Nm(Nm-1)+\mu Nm$
$\hat{\mathbf{A}}_M\hat{\mathbf{u}}^L=\hat{\mathbf{R}}_M$	$\frac{1}{2}Nm(Nm-1)$	Nm	$\frac{1}{2}Nm(Nm-1)+\mu Nm$

† $0<\alpha<\frac{1}{2}$.

‡ $\frac{1}{2}<\alpha<1$.

$$\lim_{N\to\infty}\frac{CC_{MP}(n,m,m',\mu)/CC_{OC}(n,m,m',\mu)}{\frac{11}{2}+\frac{Nm}{6}}=1,\quad (\frac{1}{2}<\alpha<1).$$

In the case of $0<\alpha\leq\frac{1}{2}$, for $\mu=2.7$ we get asymptotic result $\frac{127}{20}+m'+\frac{Nm}{6}$. As implied by these limits, the computational cost underlying the MP method is significantly more than that of the OC method. Let us note, however, that the term $\frac{Nm}{6}$ appearing in these limits is related to the matrix-matrix multiplication $\hat{\mathbf{B}}\hat{\mathbf{B}}$ in the MP method, which can be performed by parallel computation to reduce the computational time.

4. Local error estimate

In this section we estimate the decay-rate of $\|\hat{u}_h-y\|_\infty$ at the collocation points as $h\to 0$. In this regard, for $x=x_n+\upsilon h_n$, we define

$$\begin{aligned}
(Q_n^{(\ell)}\hat{u}_h)(x;\alpha) &:= \int_0^1\frac{K(x,x_\ell+sh_\ell)}{(x-x_\ell-sh_\ell)^\alpha}\hat{u}_h(x_\ell+sh_\ell)\mathrm{d}s,\\
(Q_n\hat{u}_h)(x;\alpha) &:= \int_0^\upsilon\frac{K(x,x_n+sh_n)}{(x-x_n-sh_n)^\alpha}\hat{u}_h(x_n+sh_n)\mathrm{d}s,\\
&= \upsilon\int_0^1\frac{K(x,x_n+s\upsilon h_n)}{(x-x_n-s\upsilon h_n)^\alpha}\hat{u}_h(x_n+s\upsilon h_n)\mathrm{d}s,
\end{aligned}$$

$$(Q\hat{u}_h)(x;\alpha) := \sum_{\ell=0}^{n-1}(Q_n^{(\ell)}\hat{u}_h)(x;\alpha) + (Q_n\hat{u}_h)(x;\alpha),$$

$$(\hat{Q}_n^{(\ell)}\hat{u}_h)(x;\alpha) := \sum_{j=1}^{m} w_{n,j}^{(\ell)}(x,\alpha)K(x,x_\ell + c_j h_\ell)\hat{u}_h(x_\ell + c_j h_\ell),$$

$$(\hat{Q}_n\hat{u}_h)(x;\alpha) := \sum_{j=1}^{m}\sum_{k=1}^{m} K(x,x_n + c_k \upsilon h_n)\phi_j(c_k\upsilon)w_{n,k}(x,\alpha)\hat{u}_h(x_n + c_j h_n),$$

$$(\hat{Q}\hat{u}_h)(x;\alpha) := \sum_{\ell=0}^{n-1}(\hat{Q}_n^{(\ell)}\hat{u}_h)(x;\alpha) + (\hat{Q}_n\hat{u}_h)(x;\alpha),$$

$$(\breve{Q}_n^{(\ell)}\hat{u}_h)(x;\alpha) := \sum_{j=1}^{m} \breve{w}_{n,j}^{(\ell)}(x,\alpha)K(x,x_\ell + c_j \breve{h}_\ell)\hat{u}_h(x_\ell + c_j \breve{h}_\ell),$$

$$(\breve{Q}_n\hat{u}_h)(x;\alpha) := \sum_{j=1}^{m}\sum_{k=1}^{m} K(x,x_n + c_k \upsilon \breve{h}_n)\phi_j(c_k\upsilon)\breve{w}_{n,k}(x,\alpha)\hat{u}_h(x_n + c_j \breve{h}_n),$$

$$(\breve{Q}\hat{u}_h)(x;\alpha) := \sum_{\ell=0}^{n-1}(\breve{Q}_n^{(\ell)}\hat{u}_h)(x;\alpha) + (\breve{Q}_n\hat{u}_h)(x;\alpha),$$

and introduce the interpolatory product quadrature errors at $x_{n,j} \in X_h$ $(n = 0,1,\dots,N-1,\ j = 1,\dots,m)$,

$$\hat{E}_n^{(\ell)}(x_{n,j};\alpha) := (Q_n^{(\ell)}\hat{u}_h)(x_{n,j};\alpha) - (\hat{Q}_n^{(\ell)}\hat{u}_h)(x_{n,j};\alpha),$$

$$\hat{E}_n(x_{n,j};\alpha) := (Q_n\hat{u}_h)(x_{n,j};\alpha) - (\hat{Q}_n\hat{u}_h)(x_{n,j};\alpha),$$

$$\breve{E}_n^{(\ell)}(x_{n,j};\alpha) := (Q_n^{(\ell)}\hat{u}_h)(x_{n,j};\alpha) - (\breve{Q}_n^{(\ell)}\hat{u}_h)(x_{n,j};\alpha),$$

$$\breve{E}_n(x_{n,j};\alpha) := (Q_n\hat{u}_h)(x_{n,j};\alpha) - (\breve{Q}_n\hat{u}_h)(x_{n,j};\alpha),$$

where $\ell < n$. In the following, we present some results about the optimal order of the product quadrature rules on the uniform and graded meshes.

Theorem 4.1 *([9]). Assume that $u \in C^d(I)$ for some $d \geq m$ and the parameters $\{c_i\}$ with $0 \leq c_1 < c_2 < \cdots < c_m \leq 1$ satisfy the following orthogonality condition:*

$$k = \min\left\{\varrho \in \mathbb{N} \cup \{0\} :\ J_\varrho = \int_0^1 s^\varrho \prod_{i=1}^{m}(s - c_i)\mathrm{d}s \neq 0 \quad (k \leq m)\right\}. \tag{4.1}$$

Then, for any $\alpha \in (0,1)$ on the uniform mesh I_h,

$$|\ (Qu)(x_n;\alpha) - (\hat{Q}u)(x_n;\alpha)\ | \leq C(\alpha)\begin{cases} h^m & \text{if} \quad k = 0, \\ h^{m+1-\alpha} & \text{if} \quad k > 0, \end{cases}$$

where $x_n = nh$ $(n = 0,1,\dots,N)$.

Theorem 4.2 *([20]). Suppose that k is defined as (4.1) in Theorem 4.1 and assume that $u \in C^{m+k}(I)$ in the case of $k > 0$; otherwise, $u \in C^{m+1}(I)$. Then for any $\alpha \in (0,1)$ on the graded mesh $I_{\check{h}}$ characterized by*

$$x_n = T(\frac{n}{N})^{\rho}, \quad \textit{with} \quad \rho = \frac{m+k+1}{m+1-\alpha},$$

we have

$$\mid (Qu)(x_n;\alpha) - (\check{Q}u)(x_n;\alpha) \mid = O(n^{-(m+k)}) \quad (n = 1, \ldots, N).$$

Remark 4.1 *([20]). Since $\frac{m}{1-\alpha} > \rho$, and the assertion of Theorem 4.2 remains true for graded mesh with grading exponent larger than ρ, the error estimate obtained in Theorem 4.2* holds *for graded mesh with $r = \frac{m}{1-\alpha}$ grading exponent.*

Lemma 4.1 *([6, P. 381]). Let $I_{\check{h}}$ denote the graded mesh (3.1b) on the interval $I = [0,T]$, with grading exponent $r \geq 1$. Then for $\{c_i\}$ $(0 \leq c_1 < \cdots < c_m \leq 1)$, we have*

$$\int_0^1 \left(\frac{x_{n,i} - x_\ell}{\check{h}_\ell} - s \right)^{-\alpha} s^\eta \mathrm{d}s \leq \rho(\alpha)(n-\ell)^{-\alpha} \quad (i = 1, \ldots, m),$$

where $1 \leq \ell \leq n \leq N-1$, $\eta \in \mathbb{N} \cup \{0\}$, and $\rho(\alpha) = 2^\alpha/(1-\alpha)$.

Theorem 4.3 *([15]). Let the nonnegative sequence $\{z_n\}$ satisfy the following* discrete Gronwall inequality*:*

$$z_n \leq \mu_n + Mh^{1-\alpha} \sum_{\ell=0}^{n-1} (n-\ell)^{-\alpha} z_\ell \quad (n = 0, 1, \ldots, N),$$

with $M > 0$, $\alpha \in (0,1)$, and nonnegative and nondecreasing sequence μ_n. Then the elements of $\{z_n\}$ are bounded by

$$z_n \leq E_{1-\alpha}(M\Gamma(1-\alpha)(nh)^{1-\alpha})\mu_n,$$

where E_β denotes the Mittag-Leffler function [16].

The following theorem expresses the main result of this section on the error estimate order for the solution of the fully discretized equation at the collocation points.

Theorem 4.4 *Assume that $f \in C^d(I)$, $K \in C^d(D)$ with $d \geq m$, and the collocation parameters $\{c_i\}_{i=1}^m$ satisfy the orthogonality condition (4.1) in Theorem 4.1. Then the estimate*

$$\|y - \hat{u}_h\|_\infty \leq C(\alpha) \begin{cases} h^m & \textit{if} \quad k = 0, \\ h^{m+1-\alpha} & \textit{if} \quad k > 0 \end{cases}$$

holds at collocation points X_h characterized by uniform mesh I_h, where y is the exact solution of (1.1) and $\hat{u}_h$ in $\mathbb{P}_{m-1}^{(-1)}(I_h)$ denotes the discretized collocation solution obtained using the interpolatory product quadrature formula in (3.10) and (3.14).

Proof Letting u_h denote the exact collocation solution of (2.5), then for any $x_{n,i} \in X_h$ we can write

$$| \hat{u}_h(x_{n,i}) - y(x_{n,i}) | \leq | \hat{u}_h(x_{n,i}) - u_h(x_{n,i}) | + | u_h(x_{n,i}) - y(x_{n,i}) | .$$

By (2.2), (2.5), and (2.6) we have

$$\begin{aligned} u_h(x_{n,i}) - y(x_{n,i}) &= ((\mathcal{I} - \mathcal{K}_h^M)^{-1} - (\mathcal{I} - \mathcal{K})^{-1}) f(x_{n,i}) \\ &= (\mathcal{I} - \mathcal{K}_h^M)^{-1} (\mathcal{K}_h^M - \mathcal{K}) y(x_{n,i}) \\ &= (\mathcal{I} - \mathcal{K}_h^M)^{-1} (\mathcal{I} - \mathcal{P}_h) \mathcal{K} (\mathcal{I} - \mathcal{P}_h) y(x_{n,i}), \\ | u_h(x_{n,i}) - y(x_{n,i}) | &\leq \|(\mathcal{I} - \mathcal{K}_h^M)^{-1}\| \, | (\mathcal{I} - \mathcal{P}_h) \mathcal{K} (\mathcal{I} - \mathcal{P}_h) y(x_{n,i}) | . \end{aligned}$$

Hence, $u_h(x_{n,i}) - y(x_{n,i}) = 0$. By the definition of $\hat{u}_h$ and u_h,

$$\hat{u}_h(x_{n,i}) - u_h(x_{n,i}) = \hat{u}_h^L(x_{n,i}) + \hat{u}_h^H(x_{n,i}) - u_h^L(x_{n,i}) - u_h^H(x_{n,i}).$$

From (2.7b) and (3.14) it follows that $u_h^H(x_{n,i}) = 0$, and $\hat{u}_h^H(x_{n,i}) = 0$, respectively. Thus,

$$\hat{u}_h(x_{n,i}) - u_h(x_{n,i}) = \hat{u}_h^L(x_{n,i}) - u_h^L(x_{n,i}).$$

By (3.7) and (3.13),

$$\begin{aligned} \mathbf{u}^L - \hat{\mathbf{u}}^L =& (\mathbf{B} + \mathbf{B}_H - \mathbf{B}^2)(\mathbf{u}^L - \hat{\mathbf{u}}^L) \\ & + [(\mathbf{B} + \mathbf{B}_H - \mathbf{B}^2) - (\hat{\mathbf{B}} + \hat{\breve{\mathbf{B}}} - \hat{\mathbf{B}}^2)]\hat{\mathbf{u}}^L \\ & + \mathbf{kf} - \breve{\mathbf{B}}\mathbf{f} - (\mathbf{B} - \hat{\mathbf{B}})\mathbf{f}. \end{aligned} \tag{4.2}$$

Let $Z_{n,i} = u_h^L(x_{n,i}) - \hat{u}_h^L(x_{n,i})$, and $\mathbf{z}_n = (Z_{n,1}, \ldots, Z_{n,m})^T$. Then $(\mathbf{B} + \mathbf{B}_H - \mathbf{B}^2)(\mathbf{u}^L - \hat{\mathbf{u}}^L)$ on σ_n can be written as

$$\begin{aligned} &\sum_{\ell=0}^{n-1} h^{1-\alpha} \mathbf{B}_n^{(\ell)} \mathbf{z}_\ell + h^{1-\alpha} \mathbf{B}_n \mathbf{z}_n + \sum_{\ell=0}^{n-1} h^{1-\alpha} \mathbf{B}_{H_n}^{(\ell)} \mathbf{z}_\ell + h^{1-\alpha} \mathbf{B}_{Hn} \mathbf{z}_n + h^{2(1-\alpha)} \left(\mathbf{B}_n^\ell \mathbf{B}_\ell + \sum_{k=\ell+1}^{n-1} \mathbf{B}_n^k \mathbf{B}_k^\ell \right) \mathbf{z}_\ell + h^{2(1-\alpha)} \mathbf{B}_n^2 \mathbf{z}_n \\ &= \sum_{\ell=0}^{n-1} \left[h^{1-\alpha} \left(\mathbf{B}_n^\ell - \mathbf{B}_{H_n}^{(\ell)} \right) + h^{2(1-\alpha)} \left(\mathbf{B}_n^\ell \mathbf{B}_\ell + \sum_{k=\ell+1}^{n-1} \mathbf{B}_n^k \mathbf{B}_k^\ell \right) \right] \mathbf{z}_\ell \\ &\quad + \left[h^{1-\alpha} (\mathbf{B}_n + \mathbf{B}_{H_n}) + h^{2(1-\alpha)} \mathbf{B}_n^2 \right] \mathbf{z}_n, \end{aligned}$$

and (4.2) on σ_n has the following form:

$$\begin{aligned}
\mathbf{z}_n =& \sum_{\ell=0}^{n-1}\left[h^{1-\alpha}\left(\mathbf{B}_n^{(\ell)}-\mathbf{B}_{H_n}^{(\ell)}\right)+h^{2(1-\alpha)}\left(\mathbf{B}_n^{(\ell)}\mathbf{B}_\ell+\sum_{k=\ell+1}^{n-1}\mathbf{B}_n^{(k)}\mathbf{B}_k^{(\ell)}\right)\right]\mathbf{z}_\ell+\left[h^{1-\alpha}(\mathbf{B}_n+\mathbf{B}_{H_n})+h^{2(1-\alpha)}\mathbf{B}_n^2\right]\mathbf{z}_n\\
&+\sum_{\ell=0}^{n-1}\left[h^{1-\alpha}\Big(\mathbf{B}_n^{(\ell)}-\hat{\mathbf{B}}_n^{(\ell)}+\mathbf{B}_{H_n}^{(\ell)}-\hat{\mathbf{B}}_{\breve{H}_n}^{(\ell)}\Big)\right.\\
&\left.+h^{2(1-\alpha)}\left(\mathbf{B}_n^{(\ell)}\mathbf{B}_{(\ell)}-\hat{\mathbf{B}}_n^{\ell}\hat{\mathbf{B}}_\ell+\sum_{k=\ell+1}^{n-1}\left(\mathbf{B}_n^{(k)}\mathbf{B}_k^{(\ell)}-\hat{\mathbf{B}}_n^{(k)}\hat{\mathbf{B}}_k^{(\ell)}\right)\right)\right]\hat{\mathbf{u}}_\ell\\
&+\left[h^{1-\alpha}(\mathbf{B}_n-\hat{\mathbf{B}}_n+\mathbf{B}_{H_n}-\hat{\mathbf{B}}_{\breve{H}_n})+h^{2(1-\alpha)}(\mathbf{B}_n^2-\hat{\mathbf{B}}_n^2)\right]\hat{\mathbf{u}}_n\\
&+\sum_{\ell=0}^{n-1}(\mathbf{k}\mathbf{f}_\ell-\breve{\mathbf{B}}_n^{(\ell)}\mathbf{f}_\ell)+\mathbf{k}\mathbf{f}_n-\breve{\mathbf{B}}_n\mathbf{f}_n-\sum_{\ell=0}^{n-1}(\mathbf{B}_n^{(\ell)}-\breve{\mathbf{B}}_n^{(\ell)})\mathbf{f}_\ell-(\mathbf{B}_n-\breve{\mathbf{B}}_n)\mathbf{f}_n.
\end{aligned}$$

Let

$$\begin{aligned}
\hat{\mathbf{E}}_n^{(\ell)} &= (\hat{E}_n^{(\ell)}(x_{n,1};\alpha),\dots,\hat{E}_n^{(\ell)}(x_{n,m};\alpha))^T,\\
\hat{\mathbf{E}}_n &= (\hat{E}_n(x_{n,1};\alpha),\dots,\hat{E}_n(x_{n,m};\alpha))^T,\\
\breve{\mathbf{E}}_n^{(\ell)} &= (\breve{E}_n^{(\ell)}(x_{n,1};\alpha),\dots,\breve{E}_n^{(\ell)}(x_{n,m};\alpha))^T,\\
\breve{\mathbf{E}}_n &= (\breve{E}_n(x_{n,1};\alpha),\dots,\breve{E}_n(x_{n,m};\alpha))^T,
\end{aligned}$$

where $\ell < n$. Then we have

$$\begin{aligned}
&\left[\mathcal{I}_m-\Big(h^{1-\alpha}(\mathbf{B}_n+\mathbf{B}_{H_n})+h^{2(1-\alpha)}\mathbf{B}_n^2\Big)\right]\mathbf{z}_n=\sum_{\ell=0}^{n-1}\left[h^{1-\alpha}\left(\mathbf{B}_n^{(\ell)}-\mathbf{B}_{H_n}^{(\ell)}\right)+h^{2(1-\alpha)}\left(\mathbf{B}_n^{(\ell)}\mathbf{B}_\ell+\sum_{k=\ell+1}^{n-1}\mathbf{B}_n^{(k)}\mathbf{B}_k^{(\ell)}\right)\right]\mathbf{z}_\ell\\
&\quad+\sum_{\ell=0}^{n-1}\left[h^{1-\alpha}\left(2\hat{\mathbf{E}}_n^{(\ell)}+\breve{\mathbf{E}}_n^{(\ell)}\right)+h^{2(1-\alpha)}\left(\mathbf{B}_n^{(\ell)}\hat{\mathbf{E}}_\ell+\hat{\mathbf{E}}_n^{(\ell)}+\sum_{k=\ell+1}^{n-1}\left(\mathbf{B}_n^{(k)}\hat{\mathbf{E}}_k^{(\ell)}-\hat{\mathbf{E}}_n^{(k)}\right)\right)\right]\\
&\quad+\left[h^{1-\alpha}(\hat{\mathbf{E}}_n+\hat{\mathbf{E}}_n^{(\ell)}+\breve{\mathbf{E}}_n^{(\ell)})+h^{2(1-\alpha)}(\mathbf{B}_n+\hat{\mathbf{B}}_n)\hat{\mathbf{E}}_n\right]+\sum_{\ell=0}^{n-1}\breve{\mathbf{E}}_n^{(\ell)}+\breve{\mathbf{E}}_n-\sum_{\ell=0}^{n-1}\hat{\mathbf{E}}_n^{(\ell)}-\hat{\mathbf{E}}_n.
\end{aligned}$$

Set

$$D_0(\alpha):=\left\|\left[\mathcal{I}_m-\Big(h^{1-\alpha}(\mathbf{B}_n+\mathbf{B}_{H_n})+h^{2(1-\alpha)}\mathbf{B}_n^2\Big)\right]^{-1}\right\|_1,$$

and

$$\boldsymbol{\epsilon}_n:=\sum_{\ell=0}^{n-1}\left[h^{1-\alpha}\left(2\hat{\mathbf{E}}_n^{(\ell)}+\breve{\mathbf{E}}_n^{(\ell)}\right)+h^{2(1-\alpha)}\left(\mathbf{B}_n^{(\ell)}\hat{\mathbf{E}}_\ell+\hat{\mathbf{E}}_n^{(\ell)}+\sum_{k=\ell+1}^{n-1}\left(\mathbf{B}_n^{(k)}\hat{\mathbf{E}}_k^{(\ell)}-\hat{\mathbf{E}}_n^{(k)}\right)\right)\right] \tag{4.3}$$

$$+\left[h^{1-\alpha}(\hat{\mathbf{E}}_n+\hat{\mathbf{E}}_n^{(\ell)}+\breve{\mathbf{E}}_n^{(\ell)})+h^{2(1-\alpha)}(\mathbf{B}_n+\hat{\mathbf{B}}_n)\hat{\mathbf{E}}_n\right]+\sum_{\ell=0}^{n-1}\breve{\mathbf{E}}_n^{(\ell)}+\breve{\mathbf{E}}_n-\sum_{\ell=0}^{n-1}\hat{\mathbf{E}}_n^{(\ell)}-\hat{\mathbf{E}}_n. \tag{4.4}$$

Consequently, we have

$$\begin{aligned}\|\mathbf{z}_n\|_1 \leq & D_0(\alpha)\sum_{\ell=0}^{n-1}\left[h^{1-\alpha}\left\|\mathbf{B}_n^{(\ell)}-\mathbf{B}_{H_n}^{(\ell)}\right\|_1+h^{2(1-\alpha)}\left\|\mathbf{B}_n^{(\ell)}\mathbf{B}_\ell+\sum_{k=\ell+1}^{n-1}\mathbf{B}_n^{(k)}\mathbf{B}_k^{(\ell)}\right\|_1\right]\|\mathbf{z}_\ell\|_1 \\ & + D_0(\alpha)\|\boldsymbol{\epsilon}_n\|_1.\end{aligned}$$

By Lemma 4.1, it is evident that

$$\begin{aligned}\|\mathbf{z}_n\|_1 \leq & \gamma_0(\alpha)\sum_{\ell=0}^{n-1}\Bigg[2h^{1-\alpha}(n-\ell)^{-\alpha} \\ & + h^{2(1-\alpha)}\Big((n-\ell)^{-\alpha}\Gamma(1-\alpha)+\sum_{k=\ell+1}^{n-1}(n-k)^{-\alpha}(k-\ell)^{-\alpha}\Big)\Bigg]\mathbf{z}_\ell + D_0(\alpha)\|\boldsymbol{\epsilon}_n\|_1 \\ \leq & \gamma_1(\alpha)h^{1-\alpha}\sum_{\ell=0}^{n-1}(n-\ell)^{-\alpha}\|\mathbf{z}_\ell\|_1 + D_0(\alpha)\|\boldsymbol{\epsilon}_n\|_1.\end{aligned}$$

Also, Theorem 4.3 implies

$$\|\mathbf{z}_n\|_1 \leq E_{1-\alpha}\left(\gamma_1(\alpha)\Gamma(1-\alpha)(nh)^{1-\alpha}\right)D_0(\alpha)\|\boldsymbol{\epsilon}_n\|_1.$$

We have $nh \leq T \quad (n=0,1,\ldots,N-1)$. The orders of components that belong to $\|\boldsymbol{\epsilon}_n\|_1$ in (4.3) are governed by the results of Theorem 4.1 or Theorem 4.2. Hence,

$$\|y-\hat{u}_h\|_\infty \leq C(\alpha)\begin{cases} h^m & \text{if} \quad k=0, \\ h^{m+1-\alpha} & \text{if} \quad k>0, \end{cases}$$

and the assertion of the theorem holds. □

5. Numerical examples

In the following examples, the approximate solution $\hat{u}_h$ is obtained in the space of piecewise constant functions $(m=1)$ with respect to the uniform partition $\{x_n : x_n = \frac{n}{N} \quad (n=0,1,\ldots,N)\}$, with $h=1/N$. The collocation points are given by

$$x_{n,i} = \frac{2n-1}{2N}, \quad i=1, \quad n=1,\ldots,N.$$

We have used four point Gauss–Legendre quadrature in (3.12), i.e. $m'=4$. Instead of the graded mesh defined by (3.1b), the following mesh points,

$$\left\{0,\left(\frac{1}{N}\right)^r\right\}\cup\left\{\left(1-\left(\frac{1}{N}\right)^r\right)\frac{i-1}{N-1}\right\}_{i=2}^{N},$$

are used. In order to validate the error estimate results for the convergence order (see, for instance [19]), the following O_c as the computational order of convergence (COC) is defined and reported in tables:

$$O_c = \frac{\ln\left(\|\hat{e}_{h'}\|_\infty/\|\hat{e}_h\|_\infty\right)}{\ln(2)}, \quad h=2h',$$

where $\hat{e}_h(x) = y(x) - \hat{u}_h(x)$, and $\|\hat{e}_h\|_\infty = \max_{x\in[0,T]} |\hat{e}_h(x)|$. All calculations are carried out with the Mathematica 9.0 software package.

Example 5.1 *Consider*

$$y(x) + \int_0^x \frac{1}{\sqrt{x-t}} y(t)\mathrm{d}t = 1, \quad x \in [0,1],$$

with the exact solution

$$y(x) = \exp(\pi x)\ \mathrm{erfc}(\sqrt{\pi x}),$$

where erfc *is the complementary error function [2]. In Figures 1(a)–1(d), we present the absolute errors and superposition of exact and approximate solutions obtained by OC and MP methods with* $N = 512$*. In Table 2 and Figure 2, we present the infinity norm of the OC and MP method solution errors and the corresponding order of convergence. Values of* O_c *indicate that the MP method has better convergence order than the OC method.*

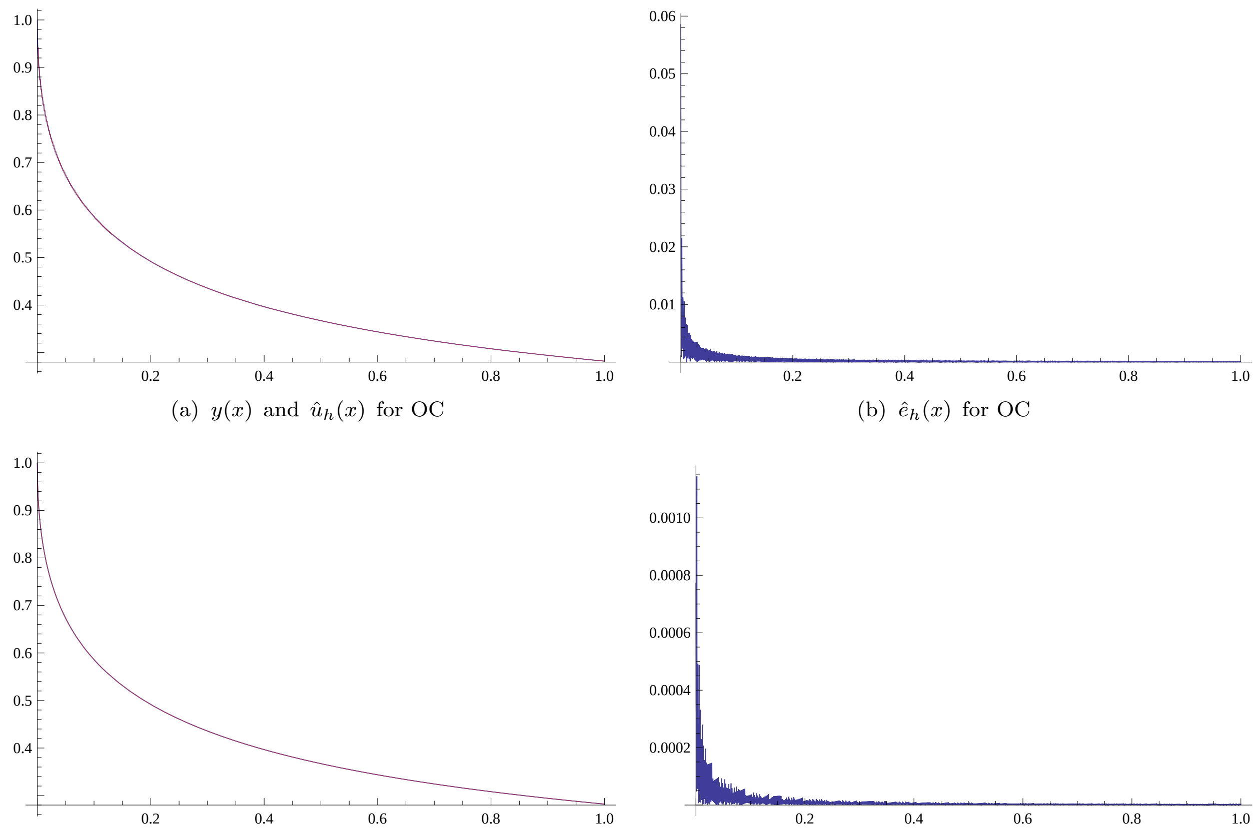
Table 2. Numerical results of Example 5.1 for ordinary collocation and multiprojection methods.

	OC		MP	
h	$\|\hat{e}_h\|_\infty$	O_c	$\|\hat{e}_h\|_\infty$	O_c
1/2	$4.997E-1$	$-$	$1.025E-1$	$-$
1/4	$4.139E-1$	0.271	$7.089E-2$	0.531
1/8	$3.330E-1$	0.313	$4.552E-2$	0.639
1/16	$2.609E-1$	0.352	$2.752E-2$	0.725
1/32	$1.997E-1$	0.385	$1.563E-2$	0.816
1/64	$1.499E-1$	0.413	$8.514E-3$	0.876
1/128	$1.108E-1$	0.436	$4.591E-3$	0.891
1/256	$8.092E-2$	0.453	$2.202E-3$	1.059
1/512	$5.853E-2$	0.467	$1.144E-3$	0.944

Example 5.2 *Let us consider the following singular integral equation:*

$$y(x) + \int_0^x \frac{K(x,t)}{(x-t)^\alpha} y(t)\mathrm{d}t = f(x), \quad x \in [0,1], \tag{5.1}$$

where $K(x,t) = x + t$*, and the exact solution is* $y(x) = x^{1-\alpha}$*. We obtained the approximate solutions* $\hat{u}_h$ *for different values of* α*. In Figures 3(a)–3(f) and Figures 4(a)–4(f), we present the absolute errors of the OC and the MP methods with* $N = 512$ *for* $\alpha = 0.1, 0.2, 0.4$ *and* $\alpha = 0.6, 0.8, 0.9$*, respectively. In Tables 3 and 4 and Figure 5, we present the infinity norm of the OC and MP method solution errors and the corresponding order of convergence of the methods. For the OC and MP methods, the convergence orders are approximately* $1-\alpha$ *and* $2-\alpha$*, respectively. Hence, we can see that the numerical results agree with the theoretical estimate given in the previous section.*



(a) $y(x)$ and $\hat{u}_h(x)$ for OC (b) $\hat{e}_h(x)$ for OC

(c) $y(x)$ and $\hat{u}_h(x)$ for MP (d) $\hat{e}_h(x)$ for MP

Figure 1. Exact and approximate solutions and absolute errors of Example 5.1 for ordinary collocation and multiprojection methods with $N = 512$.

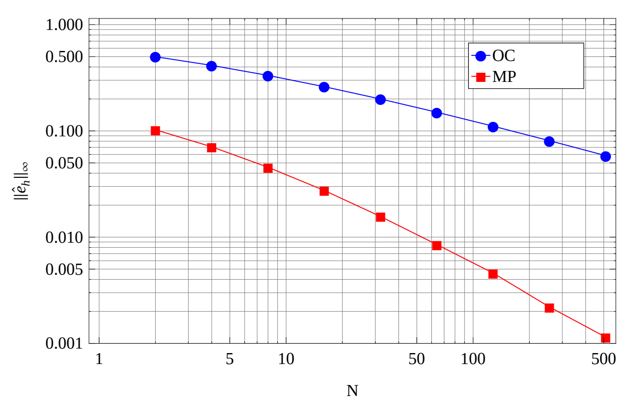
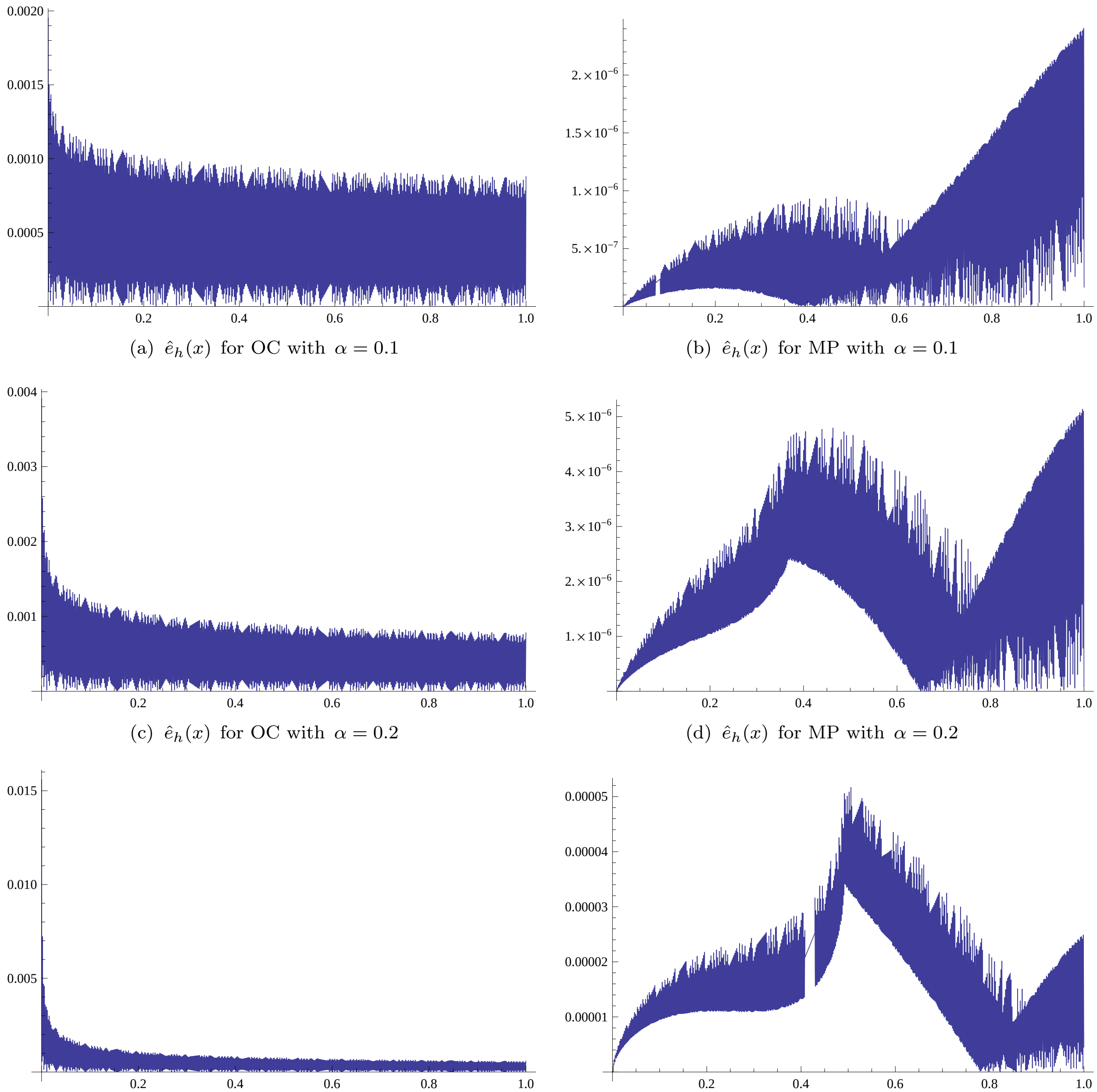


Figure 2. Distribution of $\|\hat{e}_h\|_\infty$ versus N for Example 5.1, both in logarithmic scale.

Example 5.3 *([13]). Consider the second kind of linear Abel–Volterra integral equation,*

$$y(x) + \int_0^x \frac{1}{\sqrt{x-t}} y(t)\mathrm{d}t = f(x),$$



(a) $\hat{e}_h(x)$ for OC with $\alpha = 0.1$

(b) $\hat{e}_h(x)$ for MP with $\alpha = 0.1$

(c) $\hat{e}_h(x)$ for OC with $\alpha = 0.2$

(d) $\hat{e}_h(x)$ for MP with $\alpha = 0.2$

(e) $\hat{e}_h(x)$ for OC with $\alpha = 0.4$

(f) $\hat{e}_h(x)$ for MP with $\alpha = 0.4$

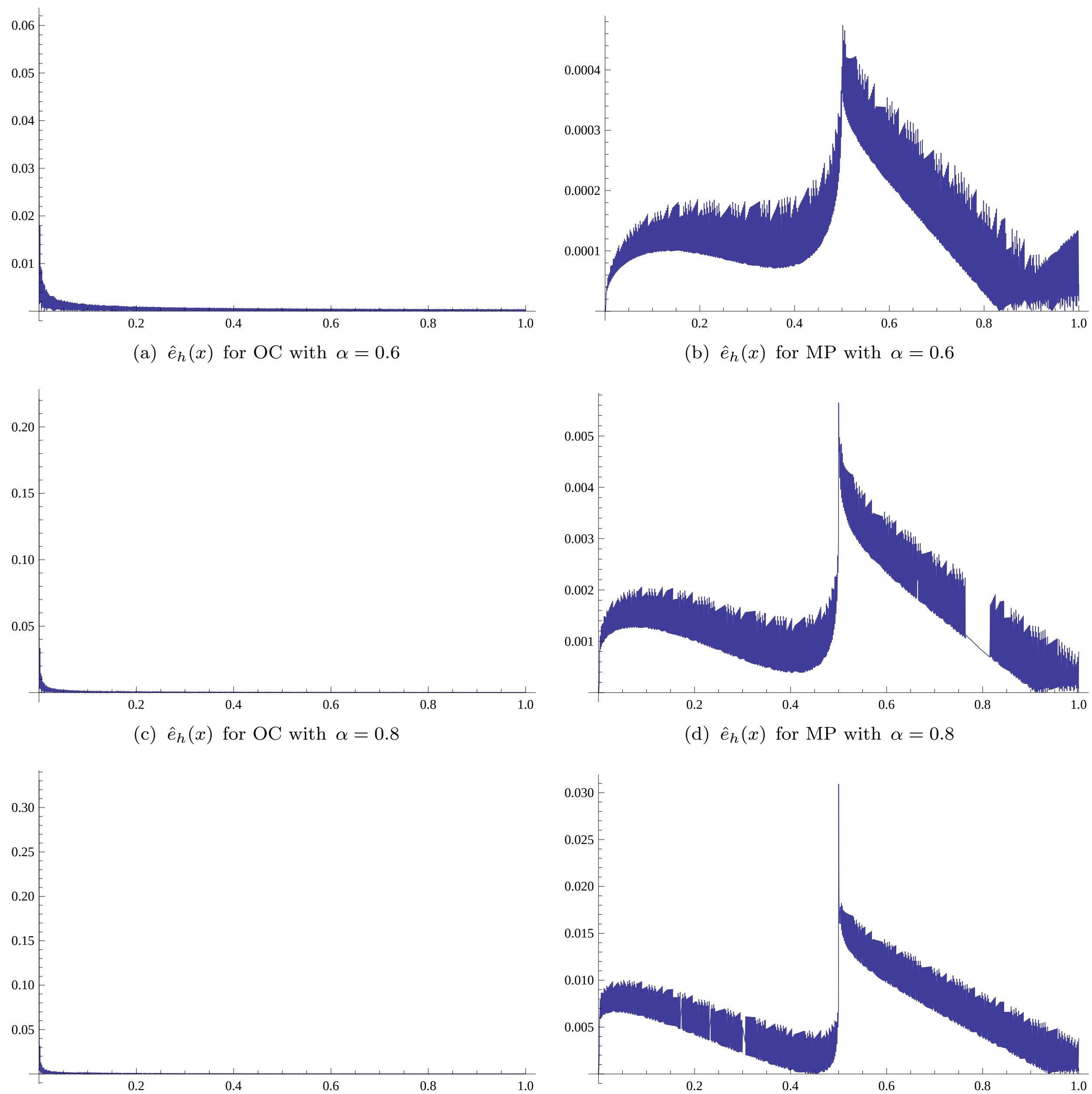
Figure 3. Absolute errors for ordinary collocation and multiprojection methods with $N = 512$ and $\alpha = 0.1, 0.2, 0.4$ for Example 5.2.

with the exact solution

$$y(x) = \frac{\sin(x)}{\sqrt{x}},$$

and

$$f(x) = \frac{\sin(x)}{\sqrt{x}} + \pi \frac{x}{2} \mathrm{J}_0(\frac{x}{2}),$$



(a) $\hat{e}_h(x)$ for OC with $\alpha = 0.6$

(b) $\hat{e}_h(x)$ for MP with $\alpha = 0.6$

(c) $\hat{e}_h(x)$ for OC with $\alpha = 0.8$

(d) $\hat{e}_h(x)$ for MP with $\alpha = 0.8$

(e) $\hat{e}_h(x)$ for OC with $\alpha = 0.9$

(f) $\hat{e}_h(x)$ for MP with $\alpha = 0.9$

Figure 4. Absolute errors for ordinary collocation and multiprojection methods with $N = 512$ and $\alpha = 0.6, 0.8, 0.9$ for Example 5.2.

where $\mathrm{J}_0(x)$ *is the Bessel function. In Figures 6(a)–6(d), we present the absolute errors and superposition of exact and approximate solutions obtained by the OC and MP methods with* $N = 512$*. In Table 5 and Figure 7, we present the infinity norm of the OC and MP method solution errors and the corresponding order of convergence. Values of* O_c *indicate that the MP method has better convergence order than the OC method.*

Table 3. Numerical results of ordinary collocation method for Example 5.2.

	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.4$	
h	$\|\hat{e}_h\|_\infty$	O_c	$\|\hat{e}_h\|_\infty$	O_c	$\|\hat{e}_h\|_\infty$	O_c
OC						
1/2	$2.754E-1$	–	$3.159E-1$	–	$4.198E-1$	–
1/4	$1.520E-1$	0.857	$1.869E-1$	0.757	$2.832E-1$	0.567
1/8	$8.220E-2$	0.887	$1.083E-1$	0.786	$1.885E-1$	0.587
1/16	$4.415E-2$	0.896	$6.242E-2$	0.795	$1.247E-1$	0.595
1/32	$2.367E-2$	0.899	$3.588E-2$	0.798	$8.240E-2$	0.598
1/64	$1.269E-2$	0.899	$2.061E-2$	0.799	$5.437E-2$	0.599
1/128	$6.800E-3$	0.899	$1.184E-2$	0.799	$3.587E-2$	0.600
1/256	$3.644E-3$	0.900	$6.800E-3$	0.800	$2.365E-2$	0.600
1/512	$1.953E-3$	0.900	$3.905E-3$	0.800	$1.560E-2$	0.600
MP						
1/2	$7.554E-2$	–	$9.896E-2$	–	$1.804E-1$	–
1/4	$2.525E-2$	1.580	$3.467E-2$	1.512	$7.348E-2$	1.295
1/8	$7.171E-3$	1.816	$1.034E-2$	1.745	$2.921E-2$	1.330
1/16	$1.924E-3$	1.897	$2.932E-3$	1.818	$1.058E-2$	1.464
1/32	$5.071E-4$	1.923	$8.230E-4$	1.833	$3.850E-3$	1.458
1/64	$1.329E-4$	1.931	$2.308E-4$	1.833	$1.315E-3$	1.549
1/128	$3.483E-5$	1.932	$6.485E-5$	1.831	$4.571E-4$	1.525
1/256	$9.146E-6$	1.929	$1.820E-5$	1.832	$1.564E-4$	1.546
1/512	$2.405E-6$	1.926	$5.138E-6$	1.825	$5.168E-5$	1.598

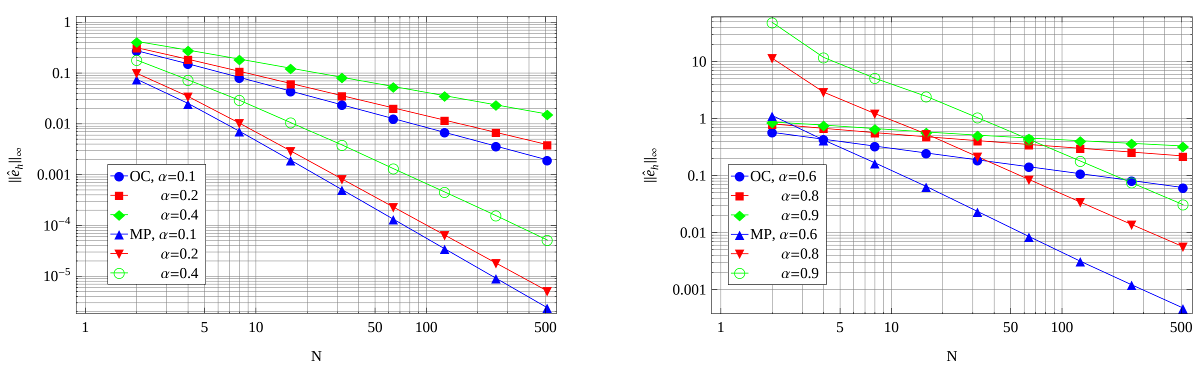


Figure 5. Distribution of $\|\hat{e}_h\|_\infty$ versus N for Example 5.2, both in logarithmic scale.

6. Conclusion

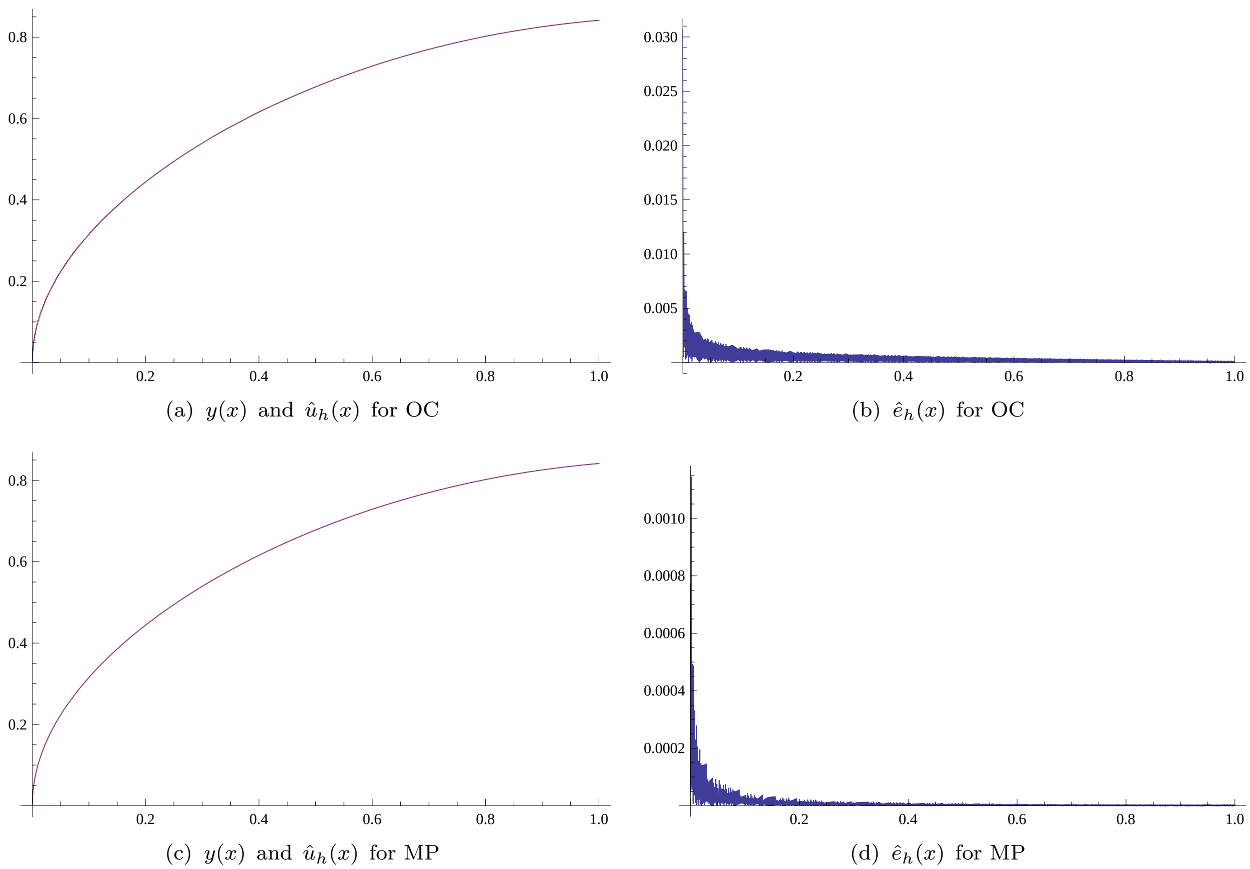
In this paper, we have extended the multiprojection method and applied it to the collocation version for solving weakly singular Volterra integral equations. We have also implemented the fully discretized multiprojection method and indicated that the fully discretized version has a superconvergence property, which the conventional collocation method lacks. Presented numerical examples confirm the error analysis results.

Table 4. Numerical results of multiprojection method for Example 5.2.

	$\alpha = 0.6$		$\alpha = 0.8$		$\alpha = 0.9$	
h	$\|\hat{e}_h\|_\infty$	O_c	$\|\hat{e}_h\|_\infty$	O_c	$\|\hat{e}_h\|_\infty$	O_c
OC						
$1/2$	$5.738E-1$	$-$	$8.073E-1$	$-$	$8.733E-1$	$-$
$1/4$	$4.345E-1$	0.401	$6.752E-1$	0.257	$7.693E-1$	0.182
$1/8$	$3.290E-1$	0.401	$5.668E-1$	0.252	$6.693E-1$	0.200
$1/16$	$2.491E-1$	0.401	$4.802E-1$	0.239	$5.825E-1$	0.200
$1/32$	$1.886E-1$	0.401	$4.100E-1$	0.228	$5.124E-1$	0.184
$1/64$	$1.427E-1$	0.402	$4.100E-1$	0.222	$4.561E-1$	0.168
$1/128$	$1.079E-1$	0.402	$3.014E-1$	0.221	$4.089E-1$	0.157
$1/256$	$8.163E-2$	0.403	$2.584E-1$	0.222	$3.678E-1$	0.153
$1/512$	$6.166E-2$	0.404	$2.211E-1$	0.224	$3.307E-1$	0.153
MP						
$1/2$	$1.132E-0$	$-$	$1.146E+1$	$-$	$4.888E+1$	$-$
$1/4$	$4.222E-1$	1.423	$2.916E-0$	1.975	$1.187E+1$	2.040
$1/8$	$1.672E-1$	1.335	$1.225E-0$	1.251	$5.101E-0$	1.219
$1/16$	$6.443E-2$	1.376	$5.378E-1$	1.187	$2.459E-0$	1.052
$1/32$	$2.344E-2$	1.458	$2.143E-1$	1.327	$1.036E-0$	1.247
$1/64$	$8.523E-3$	1.459	$8.549E-2$	1.326	$4.329E-1$	1.258
$1/128$	$3.142E-3$	1.439	$3.429E-2$	1.318	$1.801E-1$	1.265
$1/256$	$1.211E-3$	1.375	$1.385E-2$	1.307	$7.471E-2$	1.269
$1/512$	$4.740E-4$	1.353	$5.648E-3$	1.294	$3.091E-2$	1.273

Table 5. Numerical results of Example 5.3 for OC and MP methods.

	OC		MP	
h	$\|\hat{e}_h\|_\infty$	O_c	$\|\hat{e}_h\|_\infty$	O_c
$1/2$	$4.423E-1$	$-$	$2.177E-1$	$-$
$1/4$	$3.212E-1$	0.461	$2.400E-1$	-0.14
$1/8$	$2.318E-1$	0.470	$6.214E-2$	1.949
$1/16$	$1.666E-1$	0.475	$2.492E-2$	1.318
$1/32$	$1.194E-1$	0.480	$1.132E-2$	1.138
$1/64$	$8.539E-2$	0.484	$5.360E-3$	1.078
$1/128$	$6.086E-2$	0.488	$2.602E-3$	1.042
$1/256$	$4.328E-2$	0.491	$1.267E-3$	1.037
$1/512$	$3.071E-2$	0.494	$6.263E-4$	1.017



(a) $y(x)$ and $\hat{u}_h(x)$ for OC (b) $\hat{e}_h(x)$ for OC

(c) $y(x)$ and $\hat{u}_h(x)$ for MP (d) $\hat{e}_h(x)$ for MP

Figure 6. Exact and approximate solutions and absolute errors of Example 5.3 for OC and MP methods with $N = 512$.

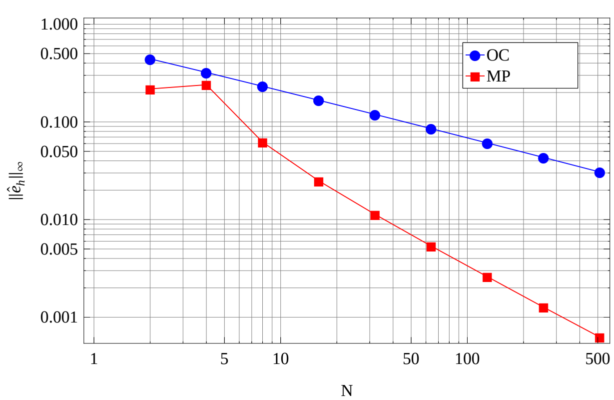


Figure 7. Distribution of $\|\hat{e}_h\|_\infty$ versus N for Example 5.3, both in logarithmic scale.

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