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The classification of rings with its genus of class of graphs

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Abstract: Let R be a commutative ring, I be a proper ideal of R, and $S(I) = \{a \in R : ra \in I \text{ for some } r \in R \setminus I\}$ be the set of all elements of R that are not prime to I. The total graph of R with respect to I, denoted by $T(\Gamma_I(R))$, is the simple graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x+y \in S(I)$. In this paper, we determine all isomorphic classes of commutative Artinian rings whose ideal-based total graph has genus at most two.

Key words: Commutative rings, total graph, planar, toroidal, genus

1. Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and has gained considerable attention. The research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other.

In the literature, one can find a number of different types of graphs attached to rings or other algebraic structures. The present paper deals with what is known as the total graph of a ring with respect to the ideal. The concept of the total graph of a ring, one of the most interesting concepts of the algebraic structures in graph theory, was first introduced by Anderson and Badawi in [4]. The *total graph* of a commutative ring R, denoted by $T(\Gamma(R))$, is an undirected graph with vertex set as R and the distinct vertices x and y are adjacent if and only if $x+y \in Z(R)$ where $Z(R)$ is the set of all zero divisors of R. The total graph (as in [4]) has been investigated in [3, 7, 15], and several generalizations of the total graph have been studied in [1, 5, 8–10, 16]. One such generalization is called the ideal-based total graph, introduced by Abbasi et al. in [1]. The *total graph of a commutative ring R with respect to an ideal I*, denoted by $T(\Gamma_I(R))$, is the graph whose vertices are all elements of R and two distinct vertices $x, y \in R$ are adjacent if and only if $x+y \in S(I)$, where $S(I)$ is the set of elements of R that are not prime to I; i.e. $S(I) = \{a \in R : ra \in I \text{ for some } r \in R \setminus I\}$. It is easy to check that $T(\Gamma_I(R)) \cong T(\Gamma(R))$ whenever I is a zero ideal of R. Also, if I is an ideal of a local ring R, then $S(I) = Z(R)$ and so $T(\Gamma_I(R)) \cong T(\Gamma(R))$. Note that the set $S(I)$ is not necessarily an ideal of R (i.e. not always closed under addition); $S(I)$ is a union of prime ideals of R containing I and $S(I) = I$ for any prime ideal I of R. Moreover, if I is a proper ideal of a finite ring R then $I \subseteq S(I) \subseteq Z(R)$. Some of the properties of the set $S(I)$ and the ideal-based total graph $T(\Gamma_I(R))$ have been studied in detail in [1].

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In the recent past, considerable work was carried out on characterizing rings regarding the genus of the constructed graph (see [6, 7, 9, 13, 15]). It can be recalled here that the *genus* of a graph G, denoted by $\gamma(G)$, is the smallest nonnegative integer g such that the graph G can be embedded on the surface obtained by attaching g handles to a sphere. The graphs of genus 0 and 1 are called *planar* and *toroidal* graphs, respectively. Maimani et al. [13] characterized all commutative Artinian rings whose total graphs have genus at most one and Tamizh Chelvam et al. [15] classified all commutative Artinian rings whose total graphs have genus two. In this paper, our aim is to extend some of the results of total graphs proved in [13, 15] to the more general structure called the ideal-based total graph. We classify, up to isomorphism, all commutative Artinian rings with nonzero identity whose ideal-based total graphs have genus at most two.

We now summarize some notations and concepts from graph theory that will be used throughout the paper. Throughout, R is a commutative ring with $1 \neq 0$ and $2 = 1 + 1$. We take ideal I as proper; that is, $I \neq R$. We denote $R_1 \times R_2$ for the direct product of two rings, R_1 and R_2. For $x \in R$, we denote (x) as an ideal generated by x in R. The *characteristic* of R is the least positive integer n such that $na = 0$, for all $a \in R$, and it is denoted by $char(R)$. A *complete graph* is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. A *complete bipartite graph* is a bipartite graph such that each vertex in V_1 is joined by an edge to each vertex in V_2 and is denoted by $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$. For a graph G, the *degree* of a vertex v in G, denoted as $deg(v)$, is the number of edges of G incident with v. The number $\delta(G) = min\{deg(v) : v \in V(G)\}$ is the minimum degree of G. For a nonnegative integer k, a graph is called k*-regular* if every vertex has degree k. For any graph G, the disjoint union of k copies of G is denoted as kG. Let S be a nonempty subset of the vertex set of graph G. The *subgraph induced by* S is the subgraph with the vertex set S and with any edges whose endpoints are both in the set S and is denoted by $\langle S \rangle$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* of G_1 and G_2, denoted by $G = G_1 \square G_2$, is a graph with vertex set $V_1 \times V_2$ and (x, y) is adjacent to (x', y') if $x = x'$ and y is adjacent to y' in G_2 or $y = y'$ and x is adjacent to x' in G_1. For unexplained terminology and notations in this paper, we refer the reader to [11, 12].

2. Basic properties of ideal-based total graphs

In this section, we obtain some properties of ideal-based total graphs. We state and prove some lemmas that will be used in the proof of the main results. Furthermore, for the convenience of the reader, we state without proof a few known results in the form of propositions, which will be used in the proofs of the main theorems. Let us start the section with the lemma that obtains the degree of each vertex of the ideal-based total graph of a ring.

Lemma 2.1 *Let R be a commutative ring with proper ideal I. If $S(I)$ is finite, then the following are true:*

(1) *If $2 \in S(I)$, then $deg(v) = |S(I)| - 1$ for every $v \in V(T(\Gamma_I(R)))$.*

(2) *If $2 \notin S(I)$, then $deg(v) = |S(I)| - 1$ for every $v \in S(I)$ and $deg(v) = |S(I)|$ for every vertex $v \notin S(I)$.*

Proof If x is adjacent to y, then $x + y = a$ for some $a \in S(I)$ and hence $y = a - x$. Now we have two cases: if $2x \in S(I)$, then x is adjacent to $a - x$ for every $a \in S(I) \setminus \{2x\}$. Thus, the degree of x is $|S(I)| - 1$ for every $x \in V(T(\Gamma_I(R)))$. If $2x \notin S(I)$, then x is adjacent to $a - x$ for every $a \in S(I)$. Thus, the degree of x is $|S(I)|$.

Hence, the statement follows from the fact that $2x \in S(I)$ if and only if either $2 \in S(I)$ or $x \in S(I)$. □

The next result is due to Abbasi et al. [1], which gives us the structure of $T(\Gamma_I(R))$ in the case that $S(I)$ is an ideal of R.

Proposition 2.2 [1, Theorem 3.5] *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R, and let $|S(I)| = \alpha$ and $|R/S(I)| = \beta$ (we allow α and β to be infinite, so then we have $\beta - 1 = (\beta - 1)/2 = \beta$).*

(1) *If $2 \in S(I)$, then $T(\Gamma_I(R))$ is the union of $\beta - 1$ disjoint K'_αs.*

(2) *If $2 \notin S(I)$, then $T(\Gamma_I(R))$ is the union of K_α and $(\beta-1)/2$ disjoint $K'_{\alpha,\alpha}$s.*

Now, Proposition 2.2 together with Proposition 3.1 gives the following result, which determines the genus of $T(\Gamma_I(R))$ whenever $S(I)$ is an ideal of R.

Proposition 2.3 *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R, and let $|S(I)| = \alpha$ and $|R/S(I)| = \beta$ (α and β may be infinite cardinals also). Then*

$$\gamma(T(\Gamma_I(R))) = \begin{cases} \beta \left\lceil \frac{(\alpha-3)(\alpha-4)}{12} \right\rceil & \text{if } 2 \in S(I), \\ \\ \left\lceil \frac{(\alpha-3)(\alpha-4)}{12} \right\rceil + (\frac{\beta-1}{2}) \left\lceil \frac{(\alpha-2)^2}{4} \right\rceil & \text{if } 2 \notin S(I). \end{cases}$$

We now state and prove the following lemma. Here, the result deals with the structure of $S(I)$ for an ideal I of a ring R where R is isomorphic to the direct product of local rings. It is worth mentioning that Artinian rings have such a decomposition to local rings.

Lemma 2.4 *For $i = 1, 2, \ldots, n$ ($n \geq 2$), let R_i be finite local rings and I_i be ideals of R_i. Let $R \cong R_1 \times \cdots \times R_n$, $I \cong I_1 \times \cdots \times I_n$, and $B = \{i \in \{1, 2, \cdots, n\} : I_i \neq R_i\}$. Then*

$$S(I) = \bigcup_{i \in B} (R_1 \times \cdots \times R_{i-1} \times Z(R_i) \times R_{i+1} \times \cdots \times R_n).$$

Proof It is enough to prove that $x = (x_1, \ldots, x_n) \in S(I)$ if and only if $x_i \in Z(R_i)$ for some $i \in B$. Let $x_i \in Z(R_i)$ for some $i \in B$. Since R_i is local, we have $S(I_i) = Z(R_i)$ and so $x_i \in S(I_i)$. Note that $I_i \neq R_i$. By the definition of $S(I)$, there exists $z_i \in R_i \setminus I_i$ such that $x_i z_i \in I_i$. Then $(0, \ldots, 0, z_i, 0, \ldots, 0) \in R \setminus I$ and $(x_1, \ldots, x_i, \ldots, x_n)(0, \ldots, 0, z_i, 0, \ldots, 0) \in I$. Thus, $(x_1, \ldots, x_i, \ldots, x_n) \in S(I)$. Conversely, assume that $(x_1, \ldots, x_n) \in S(I)$. Suppose $x_i \notin Z(R_i)$ for all $i \in B$. Thus, $x_i \notin S(I_i)$ for all $i \in B$ and $I_i = R_i$ for all $i \notin B$. Therefore, for any $r_i \in R_i$ ($1 \leq i \leq n$), $x_i r_i \in I_i$ implies that $r_i \in I_i$. Let $(r_1, \ldots, r_n)$ be an arbitrary element of R. Then $(x_1, \ldots, x_n)(r_1, \ldots, r_n) \in I$ implies that $(r_1, \ldots, r_n) \in I$, which is a contradiction to our assumption. Hence, $x_i \in Z(R_i)$ for some $i \in B$. □

We close this section by providing a necessary condition for the isomorphism relation between a total graph and ideal-based total graph.

Remark 2.5 Let $R \cong R_1 \times \cdots \times R_n$, where each R_i is a local ring, and let $I \cong I_1 \times \cdots \times I_n$ be an ideal of R. Let $B = \{i \in \{1, 2, \cdots, n\} : I_i \neq R_i\}$. If $|B| = n$, then by Lemma 2.4, $S(I) = \bigcup_{i=1}^{n} (R_1 \times \cdots \times R_{i-1} \times Z(R_i) \times R_{i+1} \times \cdots \times R_n)$, which is nothing but $Z(R)$. Thus, $T(\Gamma_I(R)) \cong T(\Gamma(R))$.

Our goal in this paper is to characterize all commutative Artinian rings whose ideal-based total graphs have genus at most two. Recall that every Artinian ring is decomposed into Artinian local rings and papers [13, 15] already characterized all commutative Artinian rings whose total graphs have genus at most two. With these facts together with Remark 2.5, to achieve our goal it is enough to look for ideals $I \cong I_1 \times \cdots \times I_n$ with $0 < |B| < n$ (if $|B| = 0$, then $I = R$, a nonproper ideal).

3. Characterization of planar graphs

In this section, we characterize all commutative rings whose ideal-based total graphs have genus zero. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. The Kuratowski theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (see [11, Theorem 9.7]).

In order to characterize the rings with planar total graphs, we need the following results, which deal with genus properties of graphs. The first one gives us the genus of complete and complete bipartite graphs.

Proposition 3.1 [13, Theorem 1.2] *The following statements hold:*

1. *For $n \geq 3$ we have $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.*
2. *For $m, n \geq 2$ we have $\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$.*

Proposition 3.2 [17, Euler formula] *If G is a finite connected graph with n vertices, m edges, and genus γ, then*

$$n - m + f = 2 - 2\gamma$$

where f is the number of faces created when G is minimally embedded on a surface of genus γ.

We also need the following proposition.

Proposition 3.3 [18, Proposition 2.1] *If G is a graph with n vertices and genus γ, then*

$$\delta(G) \leq 6 + \frac{12\gamma - 12}{n}.$$

Thus, every planar graph has a vertex v such that $deg(v) \leq 5$. As mentioned earlier, Maimani et al. [13] classified rings with planar total graph. The relevant result is stated below for ready reference.

Proposition 3.4 [13, Theorem 1.5] *Let R be a commutative ring such that $T(\Gamma(R))$ is a planar graph. Then the following hold:*

(a) *If R is a local ring, then R is a field or R is isomorphic to one of the 9 following rings:*

$$\mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_2[x,y]/(x,y)^2,\ \mathbb{Z}_4[x]/(2x, x^2),$$

$$\mathbb{Z}_4[x]/(2x, x^2 - 2),\ \mathbb{Z}_8,\ \mathbb{F}_4[x]/(x^2),\ \mathbb{Z}_4[x]/(x^2 + x + 1).$$

(b) *If R is not a local ring, then R is an infinite integral domain or R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$.*

We are now in a position to classify all rings and their ideals such that the ideal-based total graph is planar. The following theorem is a more general version of Proposition 3.4.

Theorem 3.5 *Let R be a commutative Artinian ring such that $T(\Gamma_I(R))$ is planar. Then the following hold:*

(1) *If R is a local ring, then R is a field or R is isomorphic to one of the 9 following rings:*

$$\mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_2[x,y]/(x,y)^2,\ \mathbb{Z}_4[x]/(2x,x^2),\ \mathbb{Z}_4[x]/(2x,x^2-2),$$

$$\mathbb{Z}_8,\ \mathbb{F}_4[x]/(x^2),\ \mathbb{Z}_4[x]/(x^2+x+1).$$

(2) *If R is nonlocal, then:*

- (a) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_3$*;*
- (b) $R \cong \mathbb{Z}_2 \times S$ *with* $I \neq \{0\} \times \{0\}$*, where* $S \cong \mathbb{F}_4$ *or* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$*;*
- (c) $R \cong \mathbb{F}_4 \times \mathbb{F}_4$ *with* $I \neq \{0\} \times \{0\}$*;*
- (d) $R \cong \mathbb{Z}_2 \times \mathbb{F}_{2^m}$ *with* $I = \mathbb{Z}_2 \times \{0\}$*, where* $2 < m \in \mathbb{Z}^+$*;*
- (e) $R \cong \mathbb{Z}_2 \times \mathbb{F}_{p^m}$ *with* $I = \mathbb{Z}_2 \times \{0\}$*, where* $2 < p$ *is prime and* $1 < m \in \mathbb{Z}^+$*;*
- (f) $R \cong \mathbb{Z}_3 \times \mathbb{F}_{2^m}$ *with* $I = \mathbb{Z}_3 \times \{0\}$*, where* $1 < m \in \mathbb{Z}^+$*;*
- (g) $R \cong S \times \mathbb{F}_4$ *with* $I = S \times \{0\}$*, where* $S \cong \mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$*;*
- (h) $R \cong S \times \mathbb{F}_{2^m}$ *with* $I = S \times \{0\}$*, where* $S \cong \mathbb{F}_4$ *or* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$ *and* $2 < m \in \mathbb{Z}^+$*;*
- (i) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *with* $I = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$*;*
- (j) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_{2^m}$ *with* $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$*, where* $1 < m \in \mathbb{Z}^+$*.*

Proof (1). If R is local, then $S(I) = Z(R)$ for every ideal I of R and so $T(\Gamma_I(R)) \cong T(\Gamma(R))$. Thus, the result follows from Proposition 3.4(a).

(2). Let R be a nonlocal ring. Let $R \cong R_1 \times \cdots \times R_n$, $n \geq 2$, where each R_i is local with $|R_1| \leq \ldots \leq |R_n|$ and let $I \cong I_1 \times \cdots \times I_n$ be an ideal of R. Let $B = \{i \in \{1,2,\cdots,n\} : I_i \neq R_i\}$. If $|B| = 0$ or $|B| = n$, then by Remark 2.5, we have that $T(\Gamma_I(R))$ is planar if and only if $T(\Gamma(R))$ is planar. Hence, by Proposition 3.4(b), $T(\Gamma_I(R))$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $I = \{0\} \times \{0\}$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ with $I = \{0\} \times \{0\}$.

Assume that $0 < |B| < n$. Let $j \in B$ be the least positive integer such that $I_j \neq R_j$. Let $A = R_1 \times \cdots \times R_{j-1} \times \{0\} \times R_{j+1} \times \cdots \times R_n$. Note that the subgraph induced by the set A in $T(\Gamma_I(R))$ is complete.

Suppose $n \geq 4$. Since $|R_i| \geq 2$ for all i, $|A| \geq 8$ and so K_8 is a subgraph of $T(\Gamma_I(R))$, which is not planar. Thus, $n \leq 3$.

Case 1. Let $n = 3$.

Case 1.1. Suppose that either $I_1 \neq R_1$ or $I_2 \neq R_2$ (that is, $j = 1$ or 2). If $|R_3| \geq 3$, then by the fact that $|R_i| \geq 2$ for $i = 1,2$, we have $|A| \geq 6$ and so K_6 is a subgraph of $T(\Gamma_I(R))$, which is not planar. Thus, $|R_3| = 2$. Since $|R_1| \leq |R_2| \leq |R_3|$, we have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Now we have the following candidates for I:

$$\{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_2, \{0\} \times \{0\} \times \mathbb{Z}_2, \{0\} \times \mathbb{Z}_2 \times \{0\}, \text{ and } \mathbb{Z}_2 \times \{0\} \times \{0\}.$$

If $I = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_2$, then $S(I) = I$ and so, by Proposition 2.3, $T(\Gamma_I(R)) = 2K_4$, a planar graph. If $I = \{0\} \times \{0\} \times \mathbb{Z}_2$ or $\{0\} \times \mathbb{Z}_2 \times \{0\}$ or $\mathbb{Z}_2 \times \{0\} \times \{0\}$, then $T(\Gamma_I(R))$ is a regular graph of degree 5 with 8 vertices and so number of edges 'm' is 20. If $T(\Gamma_I(R))$ is planar, then by the Euler formula (refer to Proposition 3.2), the number of faces 'f' in any planar embedding is 14. The average number of edges per face is $\frac{2m}{f} = 2.86$, which is less than the minimum girth value, a contradiction.

Hence, $T(\Gamma_I(R))$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with $I = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_2$.

Case 1.2. Suppose that $I_1 = R_1$ and $I_2 = R_2$. By the fact that $|B| < n$, we get $I_3 \neq R_3$ (that is, $j = 3$). If $|R_2| \geq 3$, then $|A| \geq 6$ and so $T(\Gamma_I(R))$ is not planar. Therefore, $|R_2| = 2$. Since $|R_1| \leq |R_2|$, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times R_3$ and $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times I_3$ for some ideal I_3 of R_3. Then, by Lemma 2.4, $S(I) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times Z(R_3)$, which is an ideal of R. Note that $\alpha = |S(I)| \geq 4$. Thus, by Proposition 2.3, $T(\Gamma_I(R))$ is planar if and only if $\alpha = 4$ and $2 \in S(I)$. Now $\alpha = 4$ implies that $S(I) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$ and so $Z(R_3) = \{0\}$. Moreover, $2 \in S(I)$ gives us $\text{char}(R_3) = 2$. Thus, $R_3 \cong \mathbb{F}_{2^m}$ where $m \in \mathbb{Z}^+$.

Hence, $T(\Gamma_I(R))$ is planar if and only if $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_{2^m}$ with $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$.

Case 2. Let $n = 2$. In this case $|B| = 1$ and so exactly one of $I_1 = R_1$ or $I_2 = R_2$.

Assume that $I_1 = R_1$. If $|R_1| \geq 5$, then $|A| \geq 5$ and so $T(\Gamma_I(R))$ is not planar. Thus, $|R_1| \leq 4$. By Lemma 2.4, we have $S(I) = R_1 \times Z(R_2)$, a prime ideal. Therefore, by Proposition 2.3, $T(\Gamma_I(R))$ is planar if and only if $\alpha \leq 4$ with $2 \in S(I)$ or $\alpha = 2$ with $2 \notin S(I)$.

Case 2.1. Let $2 \in S(I)$ with $\alpha = |S(I)| \leq 4$.

(a). Let $|R_1| = 4$. Since $\alpha \leq 4$, $Z(R_2) = \{0\}$ and so R_2 is a field. Since $2 \in I$, $char(R_2) = 2$. Thus, $R_2 \cong \mathbb{F}_{2^m}$ for some $m \in \mathbb{Z}^+$. Hence, $T(\Gamma_I(R))$ is planar if and only if $R \cong R_1 \times \mathbb{F}_{2^m}$ with $I = R_1 \times \{0\}$, where $R_1 \cong \mathbb{F}_4$, or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, where $m \in \mathbb{Z}^+$.

(b). Let $|R_1| = 3$. As seen in (a), we get $R_2 \cong \mathbb{F}_{2^m}$, where $m \in \mathbb{Z}^+$. Hence, $T(\Gamma_I(R))$ is planar if and only if $R \cong \mathbb{Z}_3 \times \mathbb{F}_{2^m}$ with $I = \mathbb{Z}_3 \times \{0\}$, where $m \in \mathbb{Z}^+$.

(c). Let $|R_1| = 2$. This implies that $R_1 \cong \mathbb{Z}_2$. Since $\alpha \leq 4$, $|Z(R_2)| \leq 2$. Since $2 \in S(I)$, we conclude that $R_2 \cong \mathbb{F}_{2^m}$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Hence, $T(\Gamma_I(R))$ is planar if and only if $R \cong \mathbb{Z}_2 \times R_2$ with $I = \mathbb{Z}_2 \times \{0\}$ or $\mathbb{Z}_2 \times Z(R_2)$, where $R_2 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, or $R \cong \mathbb{Z}_2 \times \mathbb{F}_{2^m}$ with $I = \mathbb{Z}_2 \times \{0\}$, where $m \in \mathbb{Z}^+$.

Case 2.2 Let $2 \notin S(I)$ with $\alpha = |S(I)| = 2$. This implies that $|R_1| = 2$ and $Z(R_2) = \{0\}$. Thus, R_2 is a field and $char(R_2) \neq 2$. Therefore, $R_2 \cong \mathbb{F}_{p^m}$, where $p \neq 2$. Hence, $T(\Gamma_I(R))$ is planar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}_{p^m}$ with $I = \mathbb{Z}_2 \times \{0\}$, where $m \in \mathbb{Z}^+$ and $p \neq 2$.

In a similar way one can easily check that if $I_2 = R_2$, then $T(\Gamma_I(R))$ is planar if and only if $R \cong \mathbb{Z}_2 \times R_2$ with $I = \{0\} \times R_2$, where $R_2 \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{F}_4$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ or $R \cong \mathbb{F}_4 \times R_2$ with $I = \{0\} \times R_2$, where $R_2 \cong \mathbb{F}_4$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. □

4. Characterization of toroidal graphs

In this section, we characterize all commutative Artinian rings whose ideal-based total graphs have genus one. First of all, we state the toroidal total graph result.

Proposition 4.1 [13, Theorem 1.6] *Let R be a finite commutative ring such that $T(\Gamma(R))$ is toroidal. Then the following statements hold:*

(a) If R is a local ring, then R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$.

(b) If R is not a local ring, then R is isomorphic to one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{F}_4,\ \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Now we characterize all commutative Artinian rings R and their ideal I such that the graph $T(\Gamma_I(R))$ is toroidal.

Theorem 4.2 *Let R be a commutative Artinian ring such that $T(\Gamma_I(R))$ is a toroidal graph. Then one of the following conditions hold:*

(1) *If R is local then R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$.*

(2) *If R is not local then:*

- (a) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.
- (b) $R \cong \mathbb{Z}_2 \times S$ *with* $I = \{0\} \times \{0\}$, *where* $S \cong \mathbb{F}_4$ *or* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$.
- (c) $R \cong \mathbb{Z}_3 \times S$ *with* $I = \{0\} \times S$, *where* $S \cong \mathbb{F}_4$ *or* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$.
- (d) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *with* $I = \{0\} \times \{0\} \times \{0\}$ *or* $\{0\} \times \{0\} \times \mathbb{Z}_2$ *or* $\{0\} \times \mathbb{Z}_2 \times \{0\}$ *or* $\mathbb{Z}_2 \times \{0\} \times \{0\}$.
- (e) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ *with* $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$.

Proof (1) If R is local, then $T(\Gamma_I(R)) \cong T(\Gamma(R))$. Hence, the result follows from Proposition 4.1(a).

(2) Let R be a nonlocal ring. We may write $R \cong R_1 \times \cdots \times R_n$, $n \geq 2$, where each R_i is local with $|R_1| \leq \ldots \leq |R_n|$. Let $I \cong I_1 \times \cdots \times I_n$ be an ideal of R and $B = \{i \in \{1, 2, \cdots, n\} : I_i \neq R_i\}$. If $|B| = 0$ or $|B| = n$, then by remark 2.5, $T(\Gamma_I(R))$ is toroidal if and only if $T(\Gamma(R))$ is toroidal. Hence, by Proposition 4.1(b), $T(\Gamma_I(R))$ is toroidal if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with the zero ideal I.

Assume that $0 < |B| < n$. Let j, $1 \leq j < n$, be the least positive integer such that $I_j \neq R_j$. Let $A = R_1 \times \cdots \times R_{j-1} \times \{0\} \times R_{j+1} \times \cdots \times R_n$. Note that the subgraph $\langle A \rangle$ in $T(\Gamma_I(R))$ is complete.

If $n \geq 4$ then $|A| \geq 8$ and so K_8 is a subgraph of $T(\Gamma_I(R))$, which is not toroidal. Thus, $n \leq 3$.

Case 1. Let $n = 3$. That is, $R \cong R_1 \times R_2 \times R_3$.

Case 1.1 Suppose either $I_1 \neq R_1$ or $I_2 \neq R_2$. If $|R_3| \geq 4$, then by the fact that $|R_i| \geq 2$ for $i = 1, 2$, we have $|A| \geq 8$ and so K_8 is a subgraph of $T(\Gamma_I(R))$, which is not toroidal. Thus, $|R_3| \leq 3$. Since $|R_1| \leq |R_2| \leq |R_3|$, $|R_i| \leq 3$ for all $i = 1, 2, 3$ and we have the following candidates for R:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \text{ and } \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

(a) Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $I = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_2$, then Theorem 3.5 says that $T(\Gamma_I(R))$ is planar. If $I = \{0\} \times \{0\} \times \mathbb{Z}_2$, then by the Figure, $T(\Gamma_I(R))$ is toroidal. If $I = \{0\} \times \mathbb{Z}_2 \times \{0\}$ or $\mathbb{Z}_2 \times \{0\} \times \{0\}$, then $T(\Gamma_I(R))$ is isomorphic to the graph given in the Figure and so $T(\Gamma_I(R))$ is toroidal.

(b) Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. If $I = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_3$, then $T(\Gamma_I(R)) \cong 2K_6$ and so $\gamma(T(\Gamma_I(R))) = 2$. If $I = \{0\} \times \{0\} \times \mathbb{Z}_3$ or $\{0\} \times \mathbb{Z}_2 \times \{0\}$ or $\mathbb{Z}_2 \times \{0\} \times \{0\}$, then $T(\Gamma_I(R))$ is regular graph with degree ≥ 7. Since every toroidal graph has degree at most 6, $T(\Gamma_I(R))$ is not toroidal.

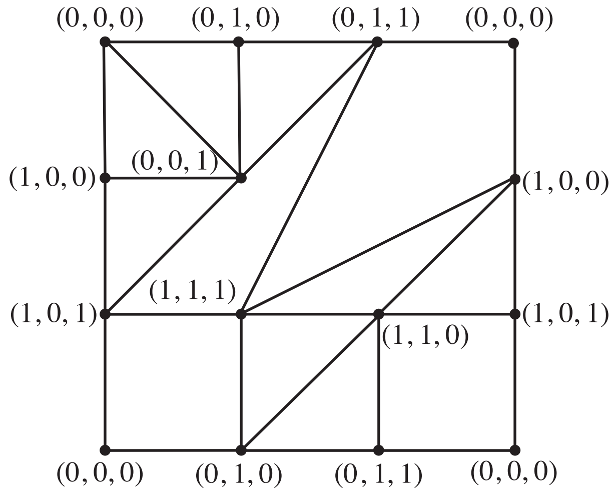

Figure. Embedding of $T(\Gamma_{\{0\}\times\{0\}\times\mathbb{Z}_2}(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2))$ in S_1.

(c) Let $R\cong\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$. If $I=\{0\}\times\mathbb{Z}_3\times\mathbb{Z}_3$, a prime ideal, then $T(\Gamma_I(R))\cong 2K_9$, which is not toroidal. If $I=\mathbb{Z}_2\times\{0\}\times\mathbb{Z}_3$ or $\mathbb{Z}_2\times\mathbb{Z}_3\times\{0\}$, then $T(\Gamma_I(R))\cong K_6\cup K_{6,6}$ and so $\gamma(T(\Gamma_I(R)))=5$. If $I=\{0\}\times\{0\}\times\mathbb{Z}_3$ or $\{0\}\times\mathbb{Z}_3\times\{0\}$ or $\mathbb{Z}_2\times\{0\}\times\{0\}$, then it is easy to check that $\delta(T(\Gamma_I(R)))\geq 9$, and so $T(\Gamma_I(R))$ is not toroidal.

(d) Let $R\cong\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$. Since $S(I)$ is a union of maximal ideals of R containing I, $|S(I)|\geq 9$ for all ideals I of R and so by Lemma 2.1 $\delta(T(\Gamma_I(R)))\geq 8$, $T(\Gamma_I(R))$ is not toroidal.

Hence, $T(\Gamma_I(R))$ is toroidal if and only if $R\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ with $I=\{0\}\times\{0\}\times\mathbb{Z}_2$ or $\{0\}\times\mathbb{Z}_2\times\{0\}$ or $\mathbb{Z}_2\times\{0\}\times\{0\}$.

Case 1.2. Suppose that $I_1=R_1$ and $I_2=R_2$. By the fact that $|B|<n$, we get $I_3\neq R_3$. If $|R_2|\geq 4$, then $|A|\geq 8$ and so $T(\Gamma_I(R))$ is not toroidal. Therefore, $|R_2|\leq 3$. Thus, $R\cong R_1\times R_2\times R_3$ with $R_1,R_2\cong\mathbb{Z}_2$ or $\mathbb{Z}_3$. Since $I_1=R_1$ and $I_2=R_2$, by Lemma 2.4, we get $S(I)=R_1\times R_2\times Z(R_3)$ for every ideal I of R. Note that $S(I)$ is an ideal of R. Thus, applying Proposition 2.3 with $\beta\neq 1$, we have that $T(\Gamma_I(R))$ is toroidal if and only if $\alpha=3,4$ and $\beta=3$ with $2\notin S(I)$. Here $\alpha=|S(I)|\leq 4$ and the only possible ring is $\mathbb{Z}_2\times\mathbb{Z}_2\times R_3$. Since $\beta=|\frac{R}{S(I)}|=3$, we get $R_3\cong\mathbb{Z}_3$. Thus, $R\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$ with $I=\mathbb{Z}_2\times\mathbb{Z}_2\times\{0\}$.

Hence, $T(\Gamma_I(R))$ is toroidal if and only if $R\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$ with $I=\mathbb{Z}_2\times\mathbb{Z}_2\times\{0\}$.

Case 2. Let $n=2$. In this case $|B|=1$ and so exactly one of $I_1=R_1$ or $I_2=R_2$. That is, by Lemma 2.4, $S(I)=R_1\times Z(R_2)$ or $Z(R_1)\times R_2$, which is an ideal of R. Now applying Proposition 2.3 with $\beta\neq 1$, we get that $T(\Gamma_I(R))$ is toroidal if and only if $\alpha=3,4$ and $\beta=3$ with $2\notin S(I)$. Here $\alpha=\beta=3$ implies that $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$ with $S(I)=\mathbb{Z}_3\times\{0\}$ or $\{0\}\times\mathbb{Z}_3$. If $\alpha=4$ and $\beta=3$, then $R\cong\mathbb{Z}_3\times R_2$ with $S(I)=\{0\}\times R_2$, where $R_2\cong\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{F}_4$.

Hence, $T(\Gamma_I(R))$ is toroidal if and only if $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$ with $I=\mathbb{Z}_3\times\{0\}$ or $\{0\}\times\mathbb{Z}_3$ or $R\cong\mathbb{Z}_3\times R_2$ with $I=\{0\}\times R_2$, where $R_2\cong\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{F}_4$. □

5. Characterization of genus two graphs

In the last section of this paper, we characterize all commutative Artinian rings whose ideal-based total graphs have genus two. As mentioned in the introduction, Tamizh Chelvam et al. [15] provided all finite commutative rings with genus two total graphs. The relevant result is given below.

Proposition 5.1 [15, Theorem 4.3] *Let R be a finite commutative ring. Then $\gamma(T(\Gamma(R))) = 2$ if and only if R is isomorphic to either $\mathbb{Z}_{10}$ or $\mathbb{Z}_3 \times \mathbb{F}_4$.*

We now state the following proposition, which is needed for the proof of Theorem 5.4. In the sequel, $G_1 \Box G_2$ denotes the Cartesian product of graphs G_1 and G_2.

Proposition 5.2 [14, Theorem 1] *For all integers $n \not\equiv 5$ or $9 (mod\, 12)$ with $n \geq 2$, $\gamma(K_n \Box K_2) = \left\lceil \frac{(n-2)(n-3)}{6} \right\rceil$. If $n = 5$, then $\gamma(K_n \Box K_2) = 2$.*

We also need the following remark.

Remark 5.3 If G is a graph with $\gamma(G) = 2$, then by Proposition 3.3, we have $\delta(G) \leq 6$ whenever $|V(G)| \geq 13$ and $\delta(G) \leq 7$ whenever $7 \leq |V(G)| \leq 12$.

We are now in a position to prove the genus two classification result. This result is a more general form of Proposition 5.1.

Theorem 5.4 *Let R be a commutative ring. Then $\gamma(T(\Gamma_I(R))) = 2$ if and only if one of the following holds:*

(1) *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$ with $I = \{0\} \times \{0\}$ or $\{0\} \times \mathbb{Z}_5$.*

(2) *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_7$ with $I = \{0\} \times \mathbb{Z}_7$.*

(3) *$R \cong \mathbb{Z}_3 \times \mathbb{F}_4$ with $I = \{0\} \times \{0\}$.*

(4) *$R \cong \mathbb{Z}_3 \times S$ with $I = \mathbb{Z}_3 \times \{0\}$ or $\mathbb{Z}_3 \times Z(S)$, where $S \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

(5) *$R \cong S \times \mathbb{Z}_5$ with $I = S \times \{0\}$, where $S \cong \mathbb{Z}_3$ or $\mathbb{F}_4$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

(6) *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ with $I = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_3$.*

Proof If R is local, then $T(\Gamma_I(R)) \cong T(\Gamma(R))$ and by Proposition 5.1, there is no ring R with $\gamma(T(\Gamma_I(R))) = 2$. Thus, R is a nonlocal ring.

Assume that $\gamma(T(\Gamma_I(R))) = 2$. Let $R \cong R_1 \times \cdots \times R_n$, $n \geq 2$, where each R_i is local with $|R_1| \leq \ldots \leq |R_n|$, and let $I \cong I_1 \times \cdots \times I_n$ be an ideal of R. Let $B = \{i \in \{1, 2, \ldots, n\} : I_i \neq R_i\}$. If $|B| = 0$ or $|B| = n$, then by Remark 2.5, $\gamma(T(\Gamma_I(R))) = 2$ if and only if $\gamma(T(\Gamma(R))) = 2$ if and only if, by Proposition 5.1, $R \cong \mathbb{Z}_3 \times \mathbb{F}_4$ with $I = \{0\} \times \{0\}$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$ with $I = \{0\} \times \{0\}$.

Assume that $0 < |B| < n$. Let $j \in B$ be the least positive integer such that $I_j \neq R_j$. Let $A = R_1 \times \cdots \times R_{j-1} \times \{0\} \times R_{j+1} \times \cdots \times R_n$. Note that the subgraph $\langle A \rangle$ in $T(\Gamma_I(R))$ is complete.

If $n \geq 5$, then $|A| \geq 16$ and so $\gamma(T(\Gamma_I(R))) > 2$; thus, $n \leq 4$.

Case 1. Let $n = 4$.

Case 1.1. Suppose that at least one $I_i \neq R_i$ for $i = 1, 2, 3$. If $|R_4| \geq 3$, then by the fact that $|R_i| \geq 2$ for $i = 1, 2, 3$, we have $|A| \geq 12$ and so $\gamma(T(\Gamma_I(R))) > 2$. Thus, $|R_4| = 2$, and therefore $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Then the cardinality of each maximal ideal of R is 8. Since $S(I)$ is a union of maximal ideal containing I, $|S(I)| \geq 8$ and so $\delta(T(\Gamma_I(R))) = |S(I)| - 1 \geq 7$. Moreover, since $V(T(\Gamma_I(R))) = |R| = 16$, by Remark 5.3, $\gamma(T(\Gamma_I(R))) > 2$.

Case 1.2. Suppose that $I_i = R_i$ for $i = 1, 2, 3$. By the fact that $|B| < n$, we get $I_4 \neq R_4$. If $|R_3| \geq 3$, then $|A| \geq 12$ and so $\gamma(T(\Gamma_I(R))) > 2$; hence, $|R_3| = 2$ which means $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times R_4$ and $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times I_4$. Then $S(I) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times Z(R_4)$ for all ideal I of R. Note that here $S(I)$ is an ideal of R and $\alpha = |S(I)| \geq 8$. Thus, by Proposition 2.3, $\gamma(T(\Gamma_I(R))) = 2$ if and only if $\alpha = 8$ and $|\frac{R}{S(I)}| = \beta = 1$ with $2 \in S(I)$, which is not possible because β cannot be 1.

Case 2. Let $n = 3$.

Case 2.1. Suppose that either $I_1 \neq R_1$ or $I_2 \neq R_2$. If $|R_3| \geq 5$, then $|A| \geq 10$ and so $\gamma(T(\Gamma_I(R))) > 2$. Therefore, $|R_3| \leq 4$. If $|R_3| = 4$, then any maximal ideal of R containing I has cardinality greater than or equal to 8. Thus, $|S(I)| \geq 8$ and so $\delta(T(\Gamma_I(R))) \geq 7$, which contradicts the fact that $\delta(T(\Gamma_I(R))) \leq 6$ whenever $|V(T(\Gamma_I(R)))| \geq 13$ (see Remark 5.3). Hence, $|R_3| \leq 3$, which means that we have the following candidates for R (up to isomorphism):

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Again, if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then the cardinality of any maximal ideal of R is greater than or equal to 9, a contradiction. Furthermore, Theorems 3.5 and 4.2 show that $\gamma(T(\Gamma_I(R))) \leq 1$ whenever $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(a) Let us take $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. If $I = \{0\} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then $|S(I)| = 9$ and so $\delta(T(\Gamma_I(R))) = 8$, a contradiction. If $I = \mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_3$, then $T(\Gamma_I(R)) \cong K_6 \cup K_{6,6}$ and so $\gamma(T(\Gamma_I(R))) = 5$. If $I = \{0\} \times \{0\} \times \mathbb{Z}_3$ or $\{0\} \times \mathbb{Z}_3 \times \{0\}$, then $T(\Gamma_I(R))$ is a regular graph of degree 11, a contradiction. If $I = \mathbb{Z}_2 \times \{0\} \times \{0\}$, then $|S(I)| = 10$ and so $\delta(T(\Gamma_I(R))) \geq 9$, again a contradiction.

(b) Consider the ring $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

(b1) If $I = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_3$, then $T(\Gamma_I(R)) \cong 2K_6$ and so $\gamma(T(\Gamma_I(R))) = 2$.

(b2) If $I = \{0\} \times \{0\} \times \mathbb{Z}_3$, then $T(\Gamma_I(R))$ is regular graph of degree 8, a contradiction to Remark 5.3.

(b3) Let $I = \{0\} \times \mathbb{Z}_2 \times \{0\}$. Here $S(I) = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (0,0,2), (0,1,2), (1,0,0), (1,1,0)\}$. Let H_1 be the spanning subgraph of $T(\Gamma_I(R))$ in which two distinct vertices $x, y \in R$ are adjacent if $x + y \in S(I) \setminus \{(1,1,0)\}$. One can easily check that $H_1 = K_6 \square K_2$ and, by Proposition 5.2, $\gamma(H_1) = 2$. By the Euler formula (Proposition 3.2), the cellular embedding of H_1 in S_2 has 22 faces. Let f be the number of faces when H_1 is embedded in S_2 and f_i be the number of i-gons in the embedding of H_1 in S_2. We know that

$$f = f_3 + f_4 + f_5 + \cdots$$

and

$$2E = 3f_3 + 4f_4 + 5f_5 + \cdots$$

Consequently,

$$2E - 3f = f_4 + 2f_5 + 3f_6 + \cdots$$

so

$$6 = f_4 + 2f_5 + 3f_6 + \cdots \tag{1}$$

Let us call the six edges (x, y) such that $x + y = (1,0,0)$ in H_1 as blue edges. The other edges we call red edges. Then each vertex of $H_1 = K_6 \square K_2$ is incident with one blue and 5 red edges. If we cut the polyhedron P along each edge, then P breaks apart into f faces, each a polygon. Each polygon has a certain number of sides. The total number of sides of P is $2E$. We know that in P each edge represents two sides. In the graph H_1 each closed way w containing a blue edge has length ≥ 4 and contains an even number of blue edges. At least half of the edges of w are red. Thus, each polygon of P that has a blue side is not a triangle and at least half of its sides are red sides. Since there are exactly 12 blue sides, there are at least 12 red sides not belonging to triangular faces in P. Thus, there are 60 red sides in P, but at least 12 of them do not belong to a triangle. This means that no more than 48 red sides may form triangular faces. Note that each triangle uses three of them. Therefore, $3f_3 \leq 48$ implies that $f_3 \leq 16$. Since $f = 22$, we get

$$f_4 + f_5 + f_6 + \cdots \geq 6. \tag{2}$$

Thus, Equations (1) and (2) give us $f_4 = 6$, $f_5 = f_6 = \cdots = 0$.

Consider the edges in $E(T(\Gamma_I(R))) - E(H_1)$. These edges are precisely the pairs of vertices $x, y \in R$ satisfying $x + y = (1,1,0)$. Note that the vertex $(0,0,0)$ is adjacent to $(1,1,0)$ in $T(\Gamma_I(R))$. Suppose $(0,0,0)$ and $(1,1,0)$ are not in the same region of a cellular embedding of H_1 in S_2. Then the edge joining $(0,0,0)$ and $(1,1,0)$ must cross an edge in an extension to an embedding of $T(\Gamma_I(R))$ in S_2 and so $\gamma(T(\Gamma_I(R))) \geq 3$. Suppose $(0,0,0)$ and $(1,1,0)$ are in the same region of a cellular embedding of H_1 in S_2. Since $(0,0,0)$ (or $(1,1,0)$) is adjacent to only one vertex from the set $\{1\} \times \mathbb{Z}_2 \times \mathbb{Z}_3$ (or $\{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_3$) in H_1, the region containing $(0,0,0)$ and $(1,1,0)$ is the rectangle formed by the vertices $\{(0,0,0),(1,0,0),(1,1,0),(0,1,0)\}$. Then the edge joining $(0,0,0)$ and $(1,1,0)$ can be drawn in the cellular embedding of $T(\Gamma_I(R))$ in S_2. If $(1,0,0)$ and $(0,1,0)$ are not vertices of any other 4-gons, then the edge joining $(1,0,0)$ and $(0,1,0)$ must cross an edge in embedding of $T(\Gamma_I(R))$ in S_2 and so $\gamma(T(\Gamma_I(R))) \geq 3$. If $(1,0,0)$ and $(0,1,0)$ are vertices of another rectangle face, then the rectangle must be formed by the same set of vertices $\{(0,0,0),(1,0,0),(1,1,0),(0,1,0)\}$. Note that $K_6 \cup K_6$ is a subgraph of H_1 and so the embedding of $K_6 \cup K_6$ is obtained by embedding each copy of K_6 in a torus (by Proposition 3.1). Since every vertex in one copy of K_6 in H_1 is adjacent to a vertex in the other copy of K_6, the cellular embedding of H_1 in S_2 has 6 distinct rectangle faces. Since $f_4 = 6$, the rectangle formed by $\{(0,0,0),(1,0,0),(1,1,0),(0,1,0)\}$ cannot occur more than once, a contradiction.

(b4) If $I = \mathbb{Z}_2 \times \{0\} \times \{0\}$, then it is clear that $T(\Gamma_I(R)) \cong T(\Gamma_{\{0\}\times\mathbb{Z}_2\times\{0\}}(R))$ and so $\gamma(T(\Gamma_I(R))) \geq 3$.

Hence, $\gamma(T(\Gamma_I(R))) = 2$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ with $I = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_3$.

Case 2.2. Suppose that $I_1 = R_1$ and $I_2 = R_2$. Then $I_3 \neq R_3$ and $S(I) = R_1 \times R_2 \times Z(R_3)$, an ideal of R. If $|R_2| \geq 5$, then $|A| \geq 10$ and so $\gamma(T(\Gamma_I(R))) > 2$. Thus, $|R_2| \leq 4$. Let $\alpha = |S(I)|$ and $\beta = |R/S(I)|$. Now, applying Proposition 2.3 with $S(I)$ an ideal and $\beta \neq 1$, we get $\gamma(T(\Gamma_I(R))) = 2$ if and only if $5 \leq \alpha \leq 7$ and $\beta = 2$ with $2 \in S(I)$ or $\alpha = 3,4$ and $\beta = 5$ with $2 \notin S(I)$. It is easy to verify that there is no ring with ideal $S(I)$ of the form $R_1 \times R_2 \times Z(R_3)$ that satisfies the condition $5 \leq \alpha \leq 7$ and $\beta = 2$ with $2 \in S(I)$. Thus, $\alpha = 3,4$ and $\beta = 5$ with $2 \notin S(I)$. In this category exactly one type of ring exists, namely $\mathbb{Z}_2 \times \mathbb{Z}_2 \times R_3$. Therefore, α must be 4 and so $Z(R_3) = \{0\}$. That is, R_3 is a field. Since $\beta = 5$, this gives $R_3 \cong \mathbb{Z}_5$.

Hence, $\gamma(T(\Gamma_I(R))) = 2$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$ with $I = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$.

Case 3. Let $n = 2$. As seen in Case 2 of Theorem 4.2, $S(I) = R_1 \times Z(R_2)$ or $Z(R_1) \times R_2$, an ideal of R. Now applying Proposition 2.3 with $\beta \neq 1$, $\gamma(T(\Gamma_I(R))) = 2$ if and only if $5 \leq \alpha \leq 7$ and $\beta = 2$ with $2 \in S(I)$ or $\alpha = 3,4$ and $\beta = 5$ with $2 \notin S(I)$.

(a) Consider $\beta = 2$, $5 \leq \alpha \leq 7$ with $2 \in S(I)$. If $\alpha = 5$ and $\beta = 2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$ with $S(I) = \{0\} \times \mathbb{Z}_5$. If $\alpha = 6$ and $\beta = 2$, then $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ with $S(I) = \mathbb{Z}_3 \times 2\mathbb{Z}_4$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$ with $S(I) = \mathbb{Z}_3 \times \{0, 2 + (x^2)\}$. Also, if $\alpha = 7$ and $\beta = 2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_7$ with $S(I) = \{0\} \times \mathbb{Z}_7$.

(b) Consider $\beta = 5$, $3 \leq \alpha \leq 4$ with $2 \notin S(I)$. If $\alpha = 3$ and $\beta = 5$, then $R \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ with $S(I) = \mathbb{Z}_3 \times \{0\}$. If $\alpha = 4$ and $\beta = 5$, then $R \cong R_1 \times \mathbb{Z}_5$ with $S(I) = R_1 \times \{0\}$ where $R_1 \cong \mathbb{F}_4$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

Hence, $\gamma(T(\Gamma_I(R))) = 2$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$ with $I = \{0\} \times \mathbb{Z}_5$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_7$ with $I = \{0\} \times \mathbb{Z}_7$ or $R \cong \mathbb{Z}_3 \times S$ with $I = \mathbb{Z}_3 \times \{0\}$ or $\mathbb{Z}_3 \times Z(S)$, where $S \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ with $I = \mathbb{Z}_3 \times \{0\}$ or $R \cong S \times \mathbb{Z}_5$ with $I = S \times \{0\}$, where $S \cong \mathbb{Z}_4$ or $Z_2[x]/(x^2)$ or $\mathbb{F}_4$. □

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