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The Brezis–Lieb lemma in convergence vector lattices

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Abstract: Recently measure-free versions of the Brezis–Lieb lemma were proved for unbounded order convergence in vector lattices. In this article, we extend these versions to convergence vector lattices.

Key words: Vector lattice, unbounded order convergence, almost order bounded set, Brezis–Lieb lemma, convergence vector space, convergence vector lattice

1. Introduction

The Brezis–Lieb lemma [4, Theorem 2] has numerous applications mainly in calculus of variations (see for example [5, 9]). We begin with its statement. Let $j : \mathbb{C} \to \mathbb{C}$ be a continuous function with $j(0) = 0$. In addition, let j satisfy the following hypothesis: for every sufficiently small $\varepsilon > 0$, there exist two continuous, nonnegative functions φ_ε and ψ_ε such that

$$|j(a+b) - j(a)| \leq \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b) \tag{1}$$

for all $a, b \in \mathbb{C}$. The following result has been stated and proved by Brezis and Lieb in [4].

Theorem 1.1 (*Brezis–Lieb lemma, [4, Theorem 2]*). *Let (Ω, Σ, μ) be a measure space. Let the mapping j satisfy the above hypothesis, and let $f_n = f + g_n$ be a sequence of measurable functions from Ω to $\mathbb{C}$ such that:*

(i) $g_n \xrightarrow{\text{a.e.}} 0$;

(ii) $j \circ f \in L_1$;

(iii) $\int \varphi_\varepsilon \circ g_n d\mu \leq C < \infty$ *for some C independent of ε and n;*

(iv) $\int \psi_\varepsilon \circ f d\mu < \infty$ *for all $\varepsilon > 0$.*

Then, as $n \to \infty$,

$$\int |j(f+g_n) - j(g_n) - j(f)| d\mu \to 0\,. \tag{2}$$

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Recall that a subset A in a normed lattice $(X, \|\cdot\|)$ is said to be *almost order bounded* if for any $\varepsilon > 0$ there is $u_\varepsilon \in X_+$ such that $\sup_{a\in A} \big\| |a| - u_\varepsilon \wedge |a| \big\| < \varepsilon$. Motivated by the proof of Theorem 1.1 the following results were proven in [7].

Proposition 1.2 *[7, Proposition 1.2] (Brezis–Lieb lemma for mappings on L_0). Let (Ω, Σ, μ) be a measure space, $f_n = f + g_n$ be a sequence in $L_0(\mu)$ such that $g_n \xrightarrow{\text{a.e.}} 0$, and $J : L_0(\mu) \to L_0(\mu)$ be a mapping satisfying $J(0) = 0$ that preserves almost everywhere convergence and such that the sequence $\big(J(f_n) - J(g_n)\big)_{n\in\mathbb{N}}$ is almost order bounded. Then, as $n \to \infty$,*

$$\int |J(f + g_n) - \big(J(g_n) + J(f)\big)| d\mu \to 0 . \tag{3}$$

Proposition 1.3 *[7, Proposition 1.3] (Brezis–Lieb lemma for uniform integrable sequence $\big(J(f_n) - J(g_n)\big)_{n\in\mathbb{N}}$). Let (Ω, Σ, μ) be a finite measure space, $f_n = f + g_n$ be a sequence in $L_0(\mu)$ such that $g_n \xrightarrow{\text{a.e.}} 0$, and $J : L_0(\mu) \to L_0(\mu)$ be a mapping satisfying $J(0) = 0$ that preserves almost everywhere convergence and such that the sequence $\big(J(f_n) - J(g_n)\big)_{n\in\mathbb{N}}$ is uniformly integrable. Then*

$$\lim_{n\to\infty} \int |J(f + g_n) - \big(J(g_n) + J(f)\big)| d\mu = 0 . \tag{4}$$

Recall that a sequence (x_n) in a vector lattice X is *order convergent* (or o-convergent, for short) to $x \in X$ if there is a sequence (z_n) in X satisfying $z_n \downarrow 0$ and $|x_n - x| \le z_n$ for all $n \in \mathbb{N}$ (we write $x_n \xrightarrow{o} x$); see e.g. [10, Theorem 16.1]. In a vector lattice X, a sequence (x_n) is *unbounded order convergent* (or uo-convergent, for short) to $x \in X$ if $|x_n - x| \wedge y \xrightarrow{o} 0$ for all $y \in X_+$ (we write $x_n \xrightarrow{uo} x$). It is well known that if (Ω, Σ, μ) is a measure space, then in L_p spaces $(1 \le p \le \infty)$, uo-convergence of sequences is the same as the almost everywhere convergence; see e.g. [8].

Definition 1.1 *[7, page 23] A mapping $f : X \to Y$ between vector lattices is said to be σ-unbounded order continuous (in short, σuo-continuous) if $x_n \xrightarrow{uo} x$ in X implies $f(x_n) \xrightarrow{uo} f(x)$ in Y.*

In [7] we gave two variants of the Brezis–Lieb lemma in vector lattice setting by replacing $a.e.$-convergence by uo-convergence, integral functionals by strictly positive functionals, and the continuity of the scalar function j (in Theorem 1.1) by the σ-unbounded order continuity of the mapping $J : X \to Y$ between vector lattices X and Y. As standard references for basic notions on vector lattices we adopt the books [1, 10, 14] and on unbounded order convergence the paper [8]. In this article, all vector lattices are assumed to be Archimedean.

Let Y be a vector lattice and l be a strictly positive linear functional on Y. Define the following norm on Y:

$$\|y\|_l := l(|y|). \tag{5}$$

Then the $\|\cdot\|_l$-completion $(Y_l, \|\cdot\|_l)$ of $(Y, \|\cdot\|_l)$ is an AL-space, and so it is an order continuous Banach lattice. The following result is a measure-free version of Proposition 1.2.

Proposition 1.4 *[7, Proposition 2.2] (A Brezis–Lieb lemma for strictly positive linear functionals). Let X be a vector lattice and Y_l be the AL-space constructed above. Let $J : X \to Y_l$ be σuo-continuous with $J(0) = 0$, and (x_n) be a sequence in X such that:*

1. *$x_n \xrightarrow{uo} x$ in X;*
2. *the sequence $(J(x_n) - J(x_n - x))_{n\in\mathbb{N}}$ is almost order bounded in Y_l.*

Then

$$\lim_{n\to\infty} \|J(x_n) - J(x_n - x) - J(x)\|_l = 0. \tag{6}$$

Similar to Proposition 1.4 one can easily show the following result.

Proposition 1.5 *Let X and Y be vector lattices and l be a strictly positive linear functional on Y. Let $J : X \to Y$ be σuo-continuous with $J(0) = 0$, and (x_n) be a sequence in X such that:*

1. *$x_n \xrightarrow{uo} x$ in X;*
2. *the sequence $(J(x_n) - J(x_n - x))_{n\in\mathbb{N}}$ is almost order bounded in Y.*

Then

$$\lim_{n\to\infty} \|J(x_n) - J(x_n - x) - J(x)\|_l = 0. \tag{7}$$

The next result is another measure-free version of Proposition 1.2.

Proposition 1.6 *[7, Proposition 2.3] (A Brezis–Lieb lemma for σuo-continuous linear functionals). Let X, Y be vector lattices and l be a σuo-continuous functional on Y. Assume further $J : X \to Y$ is a σuo-continuous mapping with $J(0) = 0$ and (x_n) is a sequence in X such that $x_n \xrightarrow{uo} x$. Then*

$$\lim_{n\to\infty} l(J(x_n) - J(x_n - x) - J(x)) = 0\,. \tag{8}$$

2. Two variants of the Brezis–Lieb lemma in convergence vector lattices

Recently, order convergence in vector lattices has been studied from the viewpoint of convergence structures; see e.g. [2, 12, 13], and in this section we recall basic notions related to convergence structures and use them to investigate variants of the Brezis–Lieb lemma. For unexplained notions and results we refer the reader to the monograph [3].

A *filter* $\mathcal{F}$ on a set X is a nonempty collection of subsets of X that does not contain the empty set and is closed under the formation of finite intersections and supersets. A subset $\mathcal{B}$ of a filter $\mathcal{F}$ is called a *basis* of $\mathcal{F}$ and $\mathcal{F}$ the filter *generated* by $\mathcal{B}$ if each set in $\mathcal{F}$ contains a set of $\mathcal{B}$. If $\mathcal{B}$ is a collection of nonempty subsets of X that is directed downward with respect to inclusion, then $[\mathcal{B}] = \{F \subseteq X : B \subseteq F \text{ for some } B \in \mathcal{B}\}$ is the *filter generated by* $\mathcal{B}$. If $\mathcal{B} = \{x\}$ then we write $[x]$ for $[\{x\}]$. If $f : X \to Y$ is a mapping then $\{f(F) : F \in \mathcal{F}\}$ is the basis of a filter called the *image filter* of $\mathcal{F}$ under f and denoted by $f(\mathcal{F})$.

Definition 2.1 *Let X be a set. A mapping λ from X into the power set of the set of all filters on X is called a convergence structure on X and (X, λ) a convergence space if the following hold for all $x \in X$:*

1. $[x] \in \lambda(x)$*;*

2. for all filters $\mathcal{F}, \mathcal{G} \in \lambda(x)$ *the intersection* $\mathcal{F} \cap \mathcal{G} \in \lambda(x)$*;*

3. if $\mathcal{F} \in \lambda(x)$*, then* $\mathcal{G} \in \lambda(x)$ *for all filters* $\mathcal{G}$ *on* X *for which* $\mathcal{G} \supseteq \mathcal{F}$*.*

Let (X, λ) be a convergence space. A filter $\mathcal{F}$ on X *converges to* x in the space X if $\mathcal{F} \in \lambda(x)$. We write $\mathcal{F} \to x$ in (X, λ) or $\mathcal{F} \to x$ in X. The element x is called the *limit* of $\mathcal{F}$.

Every topological space is a convergence space; see e.g. [3, Examples 1.1.2(i)], but the converse is not necessarily true; see e.g. [3, Examples 1.1.2(iii)]. Throughout this article, $\mathbb{R}$ refers to the convergence space induced by its standard topology. That is, a filter $\mathcal{F} \to x$ in $\mathbb{R}$ iff $\mathcal{F} \supseteq \mathcal{U}_x$, where $\mathcal{U}_x$ is the neighbourhood filter of x (the collection of all topological neighbourhoods of x).

Let X and Y be convergence spaces. A mapping $f : X \to Y$ is called *continuous at a point* $x \in X$ if $f(\mathcal{F}) \to f(x)$ in Y whenever $\mathcal{F} \to x$ in X. The mapping f is called *continuous* if it is continuous at every point of X.

If $\mathcal{F}$ is a filter on X_1 and $\mathcal{G}$ is a filter on X_2 then the *product filter* $\mathcal{F} \times \mathcal{G}$ is the filter on $X_1 \times X_2$ generated by $\{F \times G : F \in \mathcal{F},\ G \in \mathcal{G}\}$. Let X_1 and X_2 be convergence spaces; then a filter $\mathcal{F}$ on $X_1 \times X_2$ converges to (x_1, x_2) iff $p_i(\mathcal{F})$ converges to $p_i((x_1, x_2))$, for each $i \in \{1, 2\}$, where p_i is the projection of $X_1 \times X_2$ onto X_i.

Given a convergence space (X, λ), for a sequence (x_n) in X the *Fréchet filter* of (x_n) denoted by $<(x_n)>$ is the filter generated by $\{A_k\}_{k \in \mathbb{N}}$, where $A_k = \{x_n : n \geq k\}$. A sequence (x_n) *converges* to a point $x \in X$ if the Fréchet filter $<(x_n)>$ converges to x in X. In this case we write $x_n \xrightarrow{\lambda} x$.

Let X be a real vector space. A convergence structure λ on X is called a *vector space convergence structure* and (X, λ) a *convergence vector space* if addition and scalar multiplication are continuous.

Definition 2.2 *Let* X *be a vector lattice. A vector space convergence structure* λ *on* X *is called a vector lattice convergence structure and* (X, λ) *a convergence vector lattice whenever lattice operations are continuous.*

It is obvious by definition that every convergence vector lattice is a convergence vector space. The converse is not necessarily true. Let $X = (L_1[0, 1], \tau_w)$, where τ_w denotes the weak topology on $L_1[0, 1]$. Then X is a linear topological vector space and so it is a convergence vector space, but X is not a convergence vector lattice.

Definition 2.3 *Let* (X, λ) *be a convergence vector lattice. A sequence* (x_n) *in* X *is unbounded convergent to* $x \in X$ *if* $|x_n - x| \wedge y \xrightarrow{\lambda} 0$ *for all* $y \in X_+$*. We write* $x_n \xrightarrow{u\lambda} x$ *and say* x_n $u\lambda$*-converges to* x*.*

Given a locally solid vector lattice (X, τ) and a sequence (x_n) in X, then X is a convergence vector lattice and x_n $u\lambda$-converges to $x \in X$ iff x_n $u\tau$-converges to x; see [6]. The following notion is motivated by Definition 1.1.

Definition 2.4 *Let* X *and* Y *be two convergence vector lattices. A mapping* $f : X \to Y$ *is called* σu*-continuous if* $f(x_n) \xrightarrow{u\lambda} f(x)$ *in* Y *whenever* $x_n \xrightarrow{u\lambda} x$ *in* X*.*

The next result should be compared with Proposition 3.7 in [8].

Lemma 2.1 *Let (X,λ) be a convergence vector lattice and $\|\cdot\| : X \to \mathbb{R}$ be a continuous lattice norm. If a sequence (x_n) is almost order bounded in $(X,\|\cdot\|)$ and $x_n \xrightarrow{u\lambda} x$, then $x_n \xrightarrow{\|\cdot\|} x$.*

Proof It is easy to see that the sequence $(|x_n - x|)_{n\in\mathbb{N}}$ is almost order bounded. Since $x_n \xrightarrow{u\lambda} x$ then $|x_n - x| \wedge y \xrightarrow{\lambda} 0$ for all $y \in X_+$. The continuity of the lattice norm assures that $|x_n - x| \wedge y \xrightarrow{\|\cdot\|} 0$ for all $y \in X_+$. The remaining part of the proof is the same as in [8, Proposition 3.7]. □

The following two results generalize Propositions 1.5 and 1.6, respectively.

Theorem 2.2 *Let X and Y be convergence vector lattices and $l : Y \to \mathbb{R}$ be a continuous strictly positive linear functional. Suppose $J : X \to Y$ is σu-continuous mapping with $J(0) = 0$ and (x_n) is a sequence in X such that:*

1. *$x_n \xrightarrow{u\lambda} x$ in X;*
2. *the sequence $(J(x_n) - J(x_n - x))_{n\in\mathbb{N}}$ is almost order bounded in $(Y, \|\cdot\|_l)$.*

Then

$$\lim_{n\to\infty} \|J(x_n) - J(x_n - x) - J(x)\|_l = 0. \tag{9}$$

Proof Since $x_n \xrightarrow{u\lambda} x$ and J is σu-continuous, then $J(x_n) \xrightarrow{u\lambda} J(x)$ and $J(x_n - x) \xrightarrow{u\lambda} J(0) = 0$. Thus, $J(x_n) - J(x_n - x) \xrightarrow{u\lambda} J(x)$. Since $l : Y \to \mathbb{R}$ is a continuous linear functional then the norm $\|\cdot\|_l$ given by (5) is continuous on Y. Thus it follows from Lemma 2.1 that $\|J(x_n) - J(x_n - x) - J(x)\|_l \to 0$ as $n \to \infty$. □

Theorem 2.3 *Let X and Y be convergence vector lattices. Supposing $l : Y \to \mathbb{R}$ is a σu-continuous linear functional, $J : X \to Y$ is σu-continuous mapping with $J(0) = 0$ and $x_n \xrightarrow{u\lambda} x$ in X. Then*

$$\lim_{n\to\infty} l(J(x_n) - J(x_n - x) - J(x)) = 0. \tag{10}$$

Proof Since $x_n \xrightarrow{u\lambda} x$ and J is σu-continuous with $J(0) = 0$, then $J(x_n) - J(x_n - x) - J(x) \xrightarrow{u\lambda} 0$. The conclusion now follows from the σu-continuity of the linear functional l. □

Finally, we show that Propositions 1.5 and 1.6 follow from Theorems 2.2 and 2.3.

Let X be a vector lattice and $x \in X$. The following relation

$$\mathcal{F} \in \lambda_o(x) \Leftrightarrow \big\{[u_n, v_n] : n \in \mathbb{N}\big\} \subseteq \mathcal{F}, \tag{11}$$

with $(u_n), (v_n) \subseteq X$ increasing and decreasing to x, respectively, defines a vector lattice convergence structure on X and so (X, λ_o) is a convergence vector lattice; see [12, Corollary 12, Theorem 14, and Proposition 15]. Moreover, a sequence (x_n) in X converges to $x \in X$ iff (x_n) order converges to x; see e.g. [12, Corollary 12], [2, Theorem 16(iii)], or [11, Definition II.1.7].

Let X and Y be vector lattices equipped with the vector lattice convergence structure λ_o given in (11). Then $x_n \xrightarrow{u\lambda} x$ in (X, λ_o) iff $x_n \xrightarrow{uo} x$ in X and a mapping $f : X \to Y$ is σu-continuous iff it is σuo-continuous (in the sense of Definition 1.1).

Therefore, if we equip the vector lattices X and Y in Theorems 2.2 and 2.3 with the vector lattice convergence structure λ_o, then we obtain Propositions 1.5 and 1.6, respectively.

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