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Multiplier and approximation theorems in Smirnov classes with variable exponent

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Abstract: Let $G \subset \mathbb{C}$ be a bounded Jordan domain with a rectifiable Dini-smooth boundary Γ and let $G^- := ext\,\Gamma$. In terms of the higher order modulus of smoothness the direct and inverse problems of approximation theory in the variable exponent Smirnov classes $E^{p(\cdot)}(G)$ and $E^{p(\cdot)}(G^-)$ are investigated. Moreover, the Marcinkiewicz and Littlewood–Paley type theorems are proved. As a corollary some results on the constructive characterization problems in the generalized Lipschitz classes are presented.

Key words: Variable exponent Smirnov classes, direct and inverse theorems, Faber series, Lipschitz classes, Littlewood–Paley theorems, Marcinkiewicz theorems

1. Introduction

Let $G \subset \mathbb{C}$ be a bounded Jordan domain, bounded by a rectifiable Jordan curve Γ, and let $G^- := Ext\ \Gamma$. Let also $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$, $\mathbb{D} := Int\ \mathbb{T}$ and $\mathbb{D}^- := Ext\ \mathbb{T}$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Gamma)$ for a given Lebesgue measurable variable exponent $p(z) \geq 1$ on Γ is defined as the set of the Lebesgue measurable functions f, such that $\int_\Gamma |f(z)|^{p(z)} |dz| < \infty$. If $p^+ := ess\sup_{z\in\Gamma} p(z) < \infty$, then it becomes a Banach space, equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf\left\{\lambda > 0 : \int\limits_\Gamma |f(z)/\lambda|^{p(z)} |dz| \leq 1\right\} < \infty.$$

In the case of $p(\cdot) \equiv p$ it turns to the classical Lebesgue space $L^p(\Gamma)$.

If $\Gamma := \mathbb{T}$, then we obtain the space $L^{p(\cdot)}(\mathbb{T})$ with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf\left\{\lambda > 0 : \int\limits_0^{2\pi} \left|f(e^{it})/\lambda\right|^{p(e^{it})} dt \leq 1\right\}.$$

A function f analytic in G is said to be of the Smirnov class $E^p(G)$ if there exists a sequence of rectifiable Jordan curves (γ_n) in G, tending to the boundary Γ in the sense that γ_n eventually surrounds each compact

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subdomain of G, such that

$$\int_{\gamma_n} |f(z)|^p |dz| \le M < \infty, \quad 1 \le p < \infty.$$

Each function $f \in E^p(G)$ has [8, pp. 419–438] nontangential boundary values almost everywhere ($a.e.$) on Γ and the boundary function belongs to $L^p(\Gamma)$. The Smirnov class $E^p(G^-)$ is defined similarly. The sets

$$E^{p(\cdot)}(G) := \left\{ f \in E^1(G) : f \in L^{p(\cdot)}(\Gamma) \right\},$$

$$E^{p(\cdot)}(G^-) := \left\{ f \in E^1(G^-) : f \in L^{p(\cdot)}(\Gamma) \right\}$$

are called the variable exponent Smirnov classes of analytic functions in G and G^-, respectively. Equipped with the norm

$$\|f\|_{E^{p(\cdot)}(G)} := \|f\|_{L^{p(\cdot)}(\Gamma)}, \quad \|f\|_{E^{p(\cdot)}(G^-)} := \|f\|_{L^{p(\cdot)}(\Gamma)}$$

we make $E^{p(\cdot)}(G)$ and $E^p(G^-)$ into the Banach spaces.

Throughout this work we suppose that $f(\infty) = 0$ as soon as $f \in E^{p(\cdot)}(G^-)$.

Let $\mathcal{E}$ be the segment $[0, 2\pi]$ or a Jordan rectifiable curve Γ and let $p(\cdot) : \mathcal{E} \to \mathbb{R}^+ := [0, \infty)$ be a Lebesgue measurable function, which is defined on $\mathcal{E}$ such that

$$1 \le p_- := ess \inf_{z \in \mathcal{E}} p(z) \le ess \sup_{z \in \mathcal{E}} p(z) =: p^+ < \infty. \tag{1}$$

Definition 1 *We say that $p(\cdot) \in \mathcal{P}(\mathcal{E})$, if $p(\cdot)$ satisfies the conditions (1) and the inequality*

$$|p(z_1) - p(z_2)| \ln \left(\frac{|\mathcal{E}|}{|z_1 - z_2|} \right) \le c(p), \quad \forall z_1, z_2 \in \mathcal{E}, \; z_1 \ne z_2$$

with a positive constant $c(p)$, where $|\mathcal{E}|$ is the Lebesgue measure of $\mathcal{E}$.

If $p(\cdot) \in \mathcal{P}(\mathcal{E})$ and $p_- > 1$, Then we say that $p(\cdot) \in \mathcal{P}_0(\mathcal{E})$.

Let g be a continuous function and let

$$\omega(g, t) := \sup_{|t_1 - t_2| \le t} |g(t_1) - g(t_2)|, \qquad t > 0$$

be its modulus of continuity.

Definition 2 *Let Γ be a smooth Jordan curve and let $\theta(s)$ be the angle between the tangent and the positive real axis expressed as a function of arclength s. If θ has a modulus of continuity $\omega(\theta, s)$, satisfying the Dini-smooth condition*

$$\int_0^\delta [\omega(\theta, s)/s] \, ds < \infty, \; \delta > 0,$$

Then we say that Γ is a Dini-smooth curve.

The set of Dini-smooth curves is denoted by $\mathfrak{D}$.

We suppose that φ and φ_1 are the conformal mappings of G^- and G onto $\mathbb{D}^-$, respectively, and normalized by the following conditions:

$$\varphi(\infty)=\infty,\ \lim_{z\to\infty}\varphi(z)/z>0 \quad \text{and} \quad \varphi_1(0)=\infty,\ \lim_{z\to 0} z\varphi_1(z)>0.$$

Let ψ and ψ_1 be the inverse mappings of φ and φ_1, respectively. The pairs (φ,φ_1) and (ψ,ψ_1) have continuous extensions to Γ and $\mathbb{T}$, respectively. Their derivatives (φ',φ_1') and (ψ',ψ_1') have definite nontangential boundary values *a.e.* on Γ and $\mathbb{T}$, which are integrable on Γ and $\mathbb{T}$, respectively [8, p. 419–438].

For $f\in L^{p(\cdot)}(\Gamma)$, $p\in\mathcal{P}(\Gamma)$, we set $f_0:=f\circ\psi$, $p_0:=p\circ\psi$, $f_1:=f\circ\psi_1$, and $p_1:=p\circ\psi_1$. If $\Gamma\in\mathfrak{D}$, then as was proved in [17, Lemma 1], the following relations hold:

$$\begin{aligned} f_1 &\in L^{p_1(\cdot)}(\mathbb{T})\Leftrightarrow f\in L^{p(\cdot)}(\Gamma)\Leftrightarrow f_0\in L^{p_0(\cdot)}(\mathbb{T}),\\ p_0 &\in \mathcal{P}(\mathbb{T})\Leftrightarrow p\in\mathcal{P}(\Gamma)\Leftrightarrow p_1\in\mathcal{P}(\mathbb{T}).\end{aligned}$$

For $f\in L^{p(\cdot)}(\Gamma)$ we define the Cauchy type integrals

$$f_0^+(w):=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_0(\tau)}{\tau-w}d\tau,\ \ w\in\mathbb{D} \ \text{ and } \ f_1^+(w):=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_1(\tau)}{\tau-w}d\tau,\ \ w\in\mathbb{D},$$

which are analytic in $\mathbb{D}$.

Definition 3 *For $f\in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$, and $t>0$ we set*

$$\Delta_t^r f(w):=\sum_{s=0}^{r}(-1)^{r+s}\binom{r}{s}f\left(we^{ist}\right), \qquad r=1,2,3,...$$

and define the rth modulus of smoothness by

$$\Omega_r(f,\delta)_{\mathbb{T},p(\cdot)}:=\sup_{0<|h|\le\delta}\left\|\frac{1}{h}\int_0^h\Delta_t^r f(w)\,dt\right\|_{L^{p(\cdot)}(\mathbb{T})}.$$

We also define the moduli of smoothness for $f\in E^{p(\cdot)}(G)$ and $f\in E^{p(\cdot)}(G^-)$ as follows:

$$\begin{aligned}\Omega_r(f,\delta)_{G,p(\cdot)} &:= \Omega_r\left(f_0^+,\delta\right)_{\mathbb{T},p_0(\cdot)},\\ \Omega_r(f,\delta)_{G^-,p(\cdot)} &:= \Omega_r\left(f_1^+,\delta\right)_{\mathbb{T},p_1(\cdot)}, \qquad \delta>0.\end{aligned}$$

The approximation aggregates used by us are constructed via the Faber polynomials $F_k(z)$ and $\widetilde{F}_k(1/z)$ with respect to z and $1/z$, respectively. These polynomials can be defined in particular by the following series representations [17] (see also [29]):

$$\frac{\psi'(w)}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{F_k(z)}{w^{k+1}},\quad w\in\mathbb{D}^-,\quad z\in G, \tag{2}$$

$$\frac{\psi_1'(w)}{\psi_1(w)-z}=\sum_{k=1}^{\infty}-\frac{\widetilde{F}_k(1/z)}{w^{k+1}},\quad w\in\mathbb{D}^- \quad z\in G^-. \tag{3}$$

Using (2) and Cauchy's integral formulae

$$f(z)=\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int_{|w|=1}\frac{f_0(w)\psi'(w)}{\psi(w)-z}dw,\;\; z\in G,$$

which hold for every $f\in E^{p(\cdot)}(G)\subset E^1(G)$, we have

$$f(z)\sim\sum_{k=0}^{\infty}a_kF_k(z)\;\;,\; z\in G, \tag{4}$$

where

$$a_k=a_k(f):=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_0(w)}{w^{k+1}}dw,\qquad k=0,1,2,...\;. \tag{5}$$

Using (3) and the integral representation

$$f(z)=\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int_{|w|=1}\frac{f_1(w)\psi_1'(w)}{\psi_1(w)-z}dw,\;\; z\in G^-,$$

which holds for every $f\in E^{p(\cdot)}(G^-)\subset E^1(G^-)$, we also have

$$f(z)\sim\sum_{k=1}^{\infty}\widetilde{a}_k\widetilde{F}_k(1/z),\quad z\in G^-, \tag{6}$$

with the coefficients

$$\widetilde{a}_k=\widetilde{a}_k(f):=-\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_1(w)}{w^{k+1}}dw,\qquad k=1,2,...\;. \tag{7}$$

In this work, in terms of the higher order modulus of smoothness, the direct and inverse theorems of approximation theory in the variable exponent Smirnov classes $E^{p(\cdot)}(G)$ and $E^{p(\cdot)}(G^-)$ are proved. Moreover, the Marcinkiewicz and Littlewood–Paley type theorems are obtained. As a corollary some results on the constructive characterization problems in the generalized Lipschitz classes are represented.

By $c(\cdot)$, $c_1(\cdot)$, $c_2(\cdot)$, and $c(\cdot,\cdot)$, $c_1(\cdot,\cdot)$, $c_2(\cdot,\cdot)$,... we denote the constants, depending in general only on the parameters, given in the corresponding brackets and independent of n.

Let Π_n be the class of algebraic polynomials of degree not exceeding n and let

$$E_n(f)_{G,p(\cdot)}:=\inf\left\{\|f-P_n\|_{L^{p(\cdot)}(\Gamma)}:\;\; P_n\in\Pi_n\right\}\;,\; n=1,2,...$$

be the best approximation number of $f\in E^{p(\cdot)}(G)$ in Π_n.

Our new results are presented as follows:

Theorem 4 *Let $\Gamma \in \mathfrak{D}$. If $f \in E^{p(\cdot)}(G)$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$, Then there is a positive constant $c(p,r)$ such that the inequality*

$$E_n(f)_{G,p(\cdot)} \leq c(p,r)\,\Omega_r(f,1/n)_{G,p(\cdot)}, \ n=1,2,3,...$$

holds.

Theorem 5 *Let $\Gamma \in \mathfrak{D}$. If $f \in E^{p(\cdot)}(G)$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$, Then there is a positive constant $c(p,r)$ such that the inequality*

$$\Omega_r(f,1/n)_{G,p(\cdot)} \leq \frac{c(p,r)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{G,p(\cdot)}\ , \ n=1,2,3,...$$

holds.

Theorems 4 and 5 in the case of $r=1$ were proved in [16] (see also [17]). In the variable exponent Lebesgue spaces $L^{p(\cdot)}([0,2\pi])$ these theorems in the case of *r=1* and $p(\cdot) \in \mathcal{P}_0([0,2\pi])$, using some other modulus of smoothness, were proved in [1–3,10,20,21]. For a wider class of the exponents $p(\cdot)$, namely when $p(\cdot) \in \mathcal{P}([0,2\pi]) \supset \mathcal{P}_0([0,2\pi])$, the direct and inverse theorems in $L^{p(\cdot)}([0,2\pi])$ in term of the first modulus of smoothness were obtained in the papers [16, 25–28].

Note that in the classical Smirnov classes the direct and inverse problems of approximation theory were investigated adequately. Detailed information about these investigations can be found in [4,5,11–13,22,30] and the references given therein. In the variable exponent Smirnov classes some approximation problems were also investigated in [14, 16, 18].

Corollary 6 *If $E_n(f)_{G,p(\cdot)} = \mathcal{O}(n^{-\alpha})$, $\alpha>0$, Then under the conditions of Theorem 5,*

$$\Omega_r(f,\delta)_{G,p(\cdot)} = \begin{cases} \mathcal{O}(\delta^\alpha), & r>\alpha \\ \mathcal{O}(\delta^r \log 1/\delta), & r=\alpha \\ \mathcal{O}(\delta^r), & r<\alpha. \end{cases}$$

If we define the generalized Lipschitz class $Lip(G,p(\cdot),\alpha)$ with $\alpha>0$ and $r:=[\alpha]+1$, where $[\alpha]$ is the integer part of α, by

$$Lip(G,p(\cdot),\alpha) := \left\{ f \in E^{p(\cdot)}(G) : \Omega_r(f,\delta)_{G,p(\cdot)} = \mathcal{O}(\delta^\alpha),\ \delta>0 \right\},$$

then from Corollary 6 we obtain:

Corollary 7 *If $E_n(f)_{G,p(\cdot)} = \mathcal{O}(n^{-\alpha})$, $\alpha>0$, Then under the conditions of Theorem 5 we have that $f \in Lip(G,p(\cdot),\alpha)$.*

On the other hand, from Theorem 4 we have:

Corollary 8 *If $f \in Lip(G,p(\cdot),\alpha)$, $\alpha>0$, Then $E_n(f)_{G,p(\cdot)} = \mathcal{O}(n^{-\alpha})$.*

Combining Corollaries 7 and 8 we obtain the following constructive characterization of the generalized Lipschitz class $Lip(G,p(\cdot),\alpha)$:

Theorem 9 *Let $\Gamma \in \mathfrak{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. Then for $\alpha > 0$ the following statements are equivalent:*

$$i) f \in Lip\left(G, p\left(\cdot\right), \alpha\right), \qquad ii) E_n\left(f\right)_{G,p(\cdot)} = \mathcal{O}\left(n^{-\alpha}\right) \qquad\qquad n = 1, 2, 3, ...$$

Let $\{\lambda_k\}_0^\infty$ be a sequence of complex numbers, satisfying for every natural numbers k and m the conditions

$$|\lambda_k| \le c, \qquad \sum_{k=2^{m-1}}^{2^m-1} |\lambda_k - \lambda_{k+1}| \le c \tag{8}$$

with some positive constant $c > 0$.

Theorem 10 *Let $\Gamma \in \mathfrak{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. If $f \in E^{p(\cdot)}(G)$ with the Faber series (4) and $\{\lambda_k\}_0^\infty$ is a sequence of complex numbers satisfying the condition (8), then there exist a function $F \in E^{p(\cdot)}(G)$ and a positive constant $c(p)$ such that*

$$F(z) \sim \sum_{k=0}^{\infty} \lambda_k a_k\left(f\right) F_k\left(z\right), \quad z \in G$$

and $\|F\|_{L^{p(\cdot)}(\Gamma)} \le c \|f\|_{L^{p(\cdot)}(\Gamma)}$.

Let

$$\Delta_k(f)(z) := \sum_{j=2^{k-1}}^{2^k-1} a_j\left(f\right) F_j\left(z\right), \; k = 1, 2, ...$$

be the lacunary partial sums of Faber series of $f \in E^{p(\cdot)}(G)$. The following Littlewood–Paley type theorem holds:

Theorem 11 *Let $\Gamma \in \mathfrak{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. If $f \in E^{p(\cdot)}(G)$, Then there exist the constants $c_1(p)$ and $c_2(p)$ such that the inequalities*

$$c_1(p) \|f\|_{L^{p(\cdot)}(\Gamma)} \le \left\| \left(\sum_{k=0}^{\infty} |\Delta_k(f)|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(\Gamma)} \le c_2(p) \|f\|_{L^{p(\cdot)}(\Gamma)}$$

hold.

The results, similar to Theorems 10 and 11 in the classical Lebesgue spaces $L^p([0, 2\pi])$, were first proved by Littlewood and Paley in [24]. They play an important role in the various problems of approximation theory, especially for the improvements of the direct and inverse theorems. In the classical Smirnov classes, *Theorems 10 and 11* were obtained in [9].

All of the results formulated above can be formulated also in the classes $E^{p(\cdot)}(G^-)$.

Let

$$E_n\left(f\right)_{G^-,p(\cdot)} := \inf\left\{ \|f - P_n^*\|_{L^{p(\cdot)}(\Gamma)} : P_n^* \in \Pi_n^* \right\}, \; n = 1, 2, ...$$

be the best approximation number of $f \in E^{p(\cdot)}(G^-)$ in the class Π_n^*, of the algebraic polynomials with respect to $1/z$, of degree not exceeding n.

Theorem 12 *Let* $\Gamma \in \mathfrak{D}$. *If* $f \in E^{p(\cdot)}(G^-)$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$, *Then there exists a positive constant* $c(p,r)$ *such that the inequality*

$$E_n(f)_{G^-,p(\cdot)} \leq c(p,r)\,\Omega_r(f,1/n)_{G^-,p(\cdot)},\ n = 1,2,3,...$$

holds.

Theorem 13 *Let* $\Gamma \in \mathfrak{D}$. *If* $f \in E^{p(\cdot)}(G^-)$ *with* $p(\cdot) \in \mathcal{P}_0(\Gamma)$, *Then there exists a positive constant* $c(p,r)$ *such that the inequality*

$$\Omega_r(f,1/n)_{G^-,p(\cdot)} \leq \frac{c(p,r)}{n^r} \sum_{k=1}^{n} (k+1)^{r-1} E_k(f)_{G^-,p(\cdot)},\ n = 1,2,3,...$$

holds.

Corollary 14 *If* $E_n(f)_{G^-,p(\cdot)} = \mathcal{O}(n^{-\alpha})$ *for some* $\alpha > 0$, *Then under the conditions of Theorem 13*

$$\Omega_r(f,\delta)_{G^-,p(\cdot)} = \begin{cases} \mathcal{O}(\delta^\alpha), & r > \alpha \\ \mathcal{O}(\delta^r \log(1/\delta)), & r = \alpha \\ \mathcal{O}(\delta^r), & r < \alpha. \end{cases}$$

Similarly, if we define the generalized Lipschitz class $Lip(G^-, p(\cdot), \alpha)$ *with* $\alpha > 0$ *and* $r := [\alpha] + 1$ *by*

$$Lip\left(G^-, p(\cdot), \alpha\right) := \left\{ f \in E^{p(\cdot)}\left(G^-\right) : \Omega_r(f,\delta)_{G^-,p(\cdot)} = \mathcal{O}(\delta^\alpha),\ \delta > 0 \right\},$$

then we have:

Corollary 15 *If* $E_n(f)_{G^-,p(\cdot)} = \mathcal{O}(n^{-\alpha})$, $\alpha > 0$, *Then under the conditions of Theorem 13 we have* $f \in Lip(G^-, p(\cdot), \alpha)$.

On the other hand, Theorem 12 implies:

Corollary 16 *If* $f \in Lip(G^-, p(\cdot), \alpha)$ *for some* $\alpha > 0$, *Then* $E_n(f)_{G^-,p(\cdot)} = \mathcal{O}(n^{-\alpha})$.

Hence, by means of Corollaries 15 and 16, we can formulate the following theorem, which gives a constructive characterization of the generalized Lipschitz class $Lip(G^-, p(\cdot), \alpha)$:

Theorem 17 *Let* $\Gamma \in \mathfrak{D}$, $p(\cdot) \in \mathcal{P}_0(\Gamma)$ *and let* $\alpha > 0$. *The following statements are equivalent:*

$$i)\ f \in Lip\left(G^-, p(\cdot), \alpha\right), \quad ii)\ E_n(f)_{G^-,p(\cdot)} = \mathcal{O}\left(n^{-\alpha}\right), \quad n = 1,2,3,...$$

Theorem 18 *Let* $\Gamma \in \mathfrak{D}$ *and* $p(\cdot) \in \mathcal{P}_0(\Gamma)$. *If* $f \in E^{p(\cdot)}(G^-)$ *with the Faber series* (6) *and* $\{\lambda_k\}_0^\infty$ *is a sequence of complex numbers, satisfying the condition* (8), *then there exists a function* $F \in E^{p(\cdot)}(G^-)$ *such that*

$$F(z) \sim \sum_{k=1}^{\infty} \lambda_k \widetilde{a}_k(f)\, \widetilde{F}_k(1/z), \quad z \in G^-$$

and $\|F\|_{L^{p(\cdot)}(\Gamma)} \leq c\,\|f\|_{L^{p(\cdot)}(\Gamma)}$.

Denoting the lacunary partial sums of Faber series of $f \in E^{p(\cdot)}(G^-)$ by

$$\widetilde{\Delta}_k(f)(z) := \sum_{j=2^{k-1}}^{2^k-1} \widetilde{a}_j(f)\, \widetilde{F}_j(1/z),$$

we have the following Littlewood–Paley type theorem in the classes $E^{p(\cdot)}(G^-)$:

Theorem 19 *Let $\Gamma \in \mathfrak{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. If $f \in E^{p(\cdot)}(G^-)$, Then there exist the positive constants $c_3(p)$ and $c_4(p)$ such that the inequalities*

$$c_3(p)\,\|f\|_{L^{p(\cdot)}(\Gamma)} \le \left\| \left(\sum_{k=1}^{\infty} \left| \widetilde{\Delta}_k(f) \right|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(\Gamma)} \le c_4(p)\,\|f\|_{L^{p(\cdot)}(\Gamma)}$$

hold.

2. Auxiliary results

Let $\mathcal{T}_n$ be the class of the trigonometric polynomials of degree not exceeding n. The *best approximation number* of $f \in L^{p(\cdot)}(\mathbb{T})$ in $\mathcal{T}_n$ is defined by

$$E_n(f)_{p(\cdot)} := \inf\left\{ \|f - T_n\|_{L^{p(\cdot)}(\mathbb{T})} : T_n \in \mathcal{T}_n \right\}, \quad n = 0, 1, 2, \dots .$$

We will use the following direct and inverse results proved in [19]:

Theorem A *[19] Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, and let $r \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$ such that for every $n \in \mathbb{N}$ The inequality*

$$E_n(f)_{p(\cdot)} \le c(p,r)\, \Omega_r(f, 1/n)_{\mathbb{T},p(\cdot)}$$

holds.

Theorem B *[19] Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, and let $r \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$ such that for every $n \in \mathbb{N}$ The inequality*

$$\Omega_r(f, 1/n)_{\mathbb{T},p(\cdot)} \le \frac{c(p,r)}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{p(\cdot)}$$

holds.

Let

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{\Gamma \setminus \{\zeta \in \Gamma :\, |\zeta - z| < \varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

be the Cauchy singular integral of $f \in L^{p(\cdot)}(\Gamma)$. By Privalov's theorem, the Cauchy type integrals

$$f^+(z) := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{[\psi'(w)]}{\psi(w) - z} f_0(w)\, dw, \quad z \in G$$

$$f^{-}(z) := \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{[\psi'(w)]}{\psi(w) - z}f_1(w)\,dw, \qquad z \in G^{-}$$

have the nontangential inside and outside limits f^{+} and f^{-}, respectively *a.e.* on Γ. Furthermore, the formulas

$$f^{+}(z) = S_{\Gamma}(f)(z) + \frac{1}{2}f(z) \quad \text{and} \quad f^{-}(z) = S_{\Gamma}(f)(z) - \frac{1}{2}f(z) \tag{9}$$

are valid *a.e.* on Γ, which implies that

$$f(z) = f^{+}(z) - f^{-}(z) \tag{10}$$

a.e. on Γ.

The following theorem is a special case of the more general result on the boundedness of Cauchy's singular operator $S_{\Gamma}(f)$ in $L^{p(\cdot)}(\Gamma)$, proved in [23, p. 59, Theorem 2.45].

Theorem C *Let $\Gamma \in \mathfrak{D}$ and $p \in \mathcal{P}_0(\Gamma)$. Then the Cauchy singular operator $S_{\Gamma}(f)$ is bounded in $L^{p(\cdot)}(\Gamma)$.*

Lemma 20 *[16] Let $\Gamma \in \mathfrak{D}$. If $f \in L^{p(\cdot)}(\Gamma)$, $p \in \mathcal{P}_0(\Gamma)$, Then $f^{+} \in E^{p(\cdot)}(G)$ and $f^{-} \in E^{p(\cdot)}(G^{-})$.*

Now we consider the operators

$$T(f)(z) := \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(w)\,\psi'(w)}{\psi(w) - z}dw, \qquad f \in E^{p_0(\cdot)}(\mathbb{D}),\ \ z \in G,$$

$$\widetilde{T}(f)(z) := \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(w)\,\psi_1'(w)}{\psi_1(w) - z}dw, \qquad f \in E^{p_1(\cdot)}(\mathbb{D}),\ \ z \in G^{-},$$

defined on the classes $E^{p_0(\cdot)}(\mathbb{D})$ and $E^{p_1(\cdot)}(\mathbb{D})$, respectively.

The following lemma holds.

Lemma 21 *Let $\Gamma \in \mathfrak{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. Then:*

i) The operator $T : E^{p_0(\cdot)}(\mathbb{D}) \to E^{p(\cdot)}(G)$ is linear, bounded, one-to-one, and onto. Moreover, $T\left(f_0^{+}\right) = f$ for every $f \in E^{p(\cdot)}(G)$,

ii) The operator $\widetilde{T} : E^{p_1(\cdot)}(\mathbb{D}) \to E^{p(\cdot)}(G^{-})$ is linear, bounded, one-to-one, and onto. Moreover, $\widetilde{T}\left(f_1^{+}\right) = f$ for every $f \in E^{p(\cdot)}(G^{-})$.

Note that *assertion i)* was proved in [16]. *Assertion ii)* can be proved using the same techniques.

Lemma 22 *Let $\Gamma \in \mathfrak{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. Then there exist the positive constants $c_i(p)$, i=5,6,7,8, such that the following assertions hold:*

$$i) \text{if } f \in E^{p(\cdot)}(G), \text{ Then } E_n\left(f_0^{+}\right)_{p_0(\cdot)} \le c_5(p)E_n(f)_{G,p(\cdot)} \le c_6(p)E_n\left(f_0^{+}\right)_{p_0(\cdot)};$$

$$ii) \text{if } f \in E^{p(\cdot)}\left(G^{-}\right), \text{ Then } E_n\left(f_1^{+}\right)_{p_1(\cdot)} \le c_7(p)E_n(f)_{G^{-},p(\cdot)} \le c_8(p)E_n\left(f_1^{+}\right)_{p_1(\cdot)}.$$

Proof Assertion $i)$ was proved in [16]. We will prove $ii)$. Since the operator $\widetilde{T}:E^{p_1(\cdot)}(\mathbb{D})\to E^{p(\cdot)}(G^-)$ is linear, bounded, one-to-one, and onto, it has the inverse operator $\widetilde{T}^{-1}:E^{p(\cdot)}(G^-)\to E^{p_1(\cdot)}(\mathbb{D})$, which is also linear, bounded, one-to-one, and onto. If $f\in E^{p(\cdot)}(G^-)$, then $\widetilde{T}^{-1}(f)=f_1^+\in E^{p_1(\cdot)}(\mathbb{D})$. If P_n^* is the polynomial of the best approximation to f in $E^{p(\cdot)}(G^-)$ with respect to $1/z$ and degree not exceeding n, then $\widetilde{T}^{-1}(P_n^*)$ is a polynomial with respect to w and degree not exceeding n. Therefore, denoting $c_7(p):=\left\|\widetilde{T}^{-1}\right\|$, we get

$$\begin{aligned} E_n\left(f_1^+\right)_{p_1(\cdot)} &\le \left\|f_1^+-\widetilde{T}^{-1}(P_n^*)\right\|_{L^{p_1(\cdot)}(\mathbb{T})} \\ &\le \left\|\widetilde{T}^{-1}(f)-\widetilde{T}^{-1}(P_n^*)\right\|_{L^{p_1(\cdot)}(\mathbb{T})} \\ &\le \left\|\widetilde{T}^{-1}\right\|\left\|f-P_n^*\right\|_{L^{p(\cdot)}(\Gamma)}=c_7(p)E_n(f)_{G^-,p(\cdot)}. \end{aligned}$$

On the other hand, by Lemma 21 we have that $\widetilde{T}\left(f_1^+\right)=f\in E^{p(\cdot)}(G^-)$ and then by boundedness of $\widetilde{T}$

$$\begin{aligned} E_n(f)_{G^-,p(\cdot)} &\le \left\|f-\widetilde{T}(P_n^*)\right\|_{L^{p(\cdot)}(\Gamma)} \\ &\le \left\|\widetilde{T}\left(f_1^+\right)-\widetilde{T}(P_n^*)\right\|_{L^{p(\cdot)}(\Gamma)} \\ &\le \left\|\widetilde{T}\right\|\left\|f_1^+-P_n^*\right\|_{L^{p_1(\cdot)}(\mathbb{T})}=c_8(p)E_n\left(f_1^+\right)_{p_1(\cdot)}. \end{aligned}$$

□

Lemma 23 *Let $p(\cdot)\in\mathcal{P}_0(\Gamma)$ and let $\{\lambda_k\}_0^\infty$ be a sequence of complex numbers satisfying the condition (8). If $g\in E^{p(\cdot)}(\mathbb{D})$ has the Taylor series*

$$g(w)=\sum_{k=0}^{\infty}\beta_k(g)\,w^k \quad , \quad w\in\mathbb{D}$$

Then there exists a function $g^*\in E^{p(\cdot)}(\mathbb{D})$ that has the Taylor series

$$g^*(w)=\sum_{k=0}^{\infty}\lambda_k\beta_k(g)\,w^k \quad , \quad w\in\mathbb{D}$$

and $\|g^*\|_{L^{p(\cdot)}(\mathbb{T})}\le c(p)\|g\|_{L^{p(\cdot)}(\mathbb{T})}$.

Proof Let $g\in E^{p(\cdot)}(\mathbb{D})$ and $c_k(g)$ $(k=...,-1,0,1,...)$ be the Fourier coefficients of the boundary function of g. Then (*Theorem 3.4* in [7, p. 38]) we have

$$c_k(g)=\begin{cases}\beta_k(g), & k\ge 0\\ 0, & k<0.\end{cases}$$

By the Marcinkiewicz type theorem [23, p. 120, Theorem 2.103], there is a function $h\in L^{p(\cdot)}(\mathbb{T})$ with the Fourier coefficients $c_k(h)=\lambda_k c_k(g)$ such that $\|h\|_{L^{p(\cdot)}(\mathbb{T})}\le c(p)\|g\|_{L^{p(\cdot)}(\mathbb{T})}$, for some positive constant $c(p)$.

Since $g^{*}:=h^{+}\in E^{p(\cdot)}(\mathbb{D})$, for the Taylor coefficients $\beta_k(g^{*})$, $k=0,1,2,...,$ of g^{*}, by (10) we have

$$\beta_k(g^{*})=\beta_k(h^{+})=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{h^{+}(w)}{w^{k+1}}dw=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{h(w)}{w^{k+1}}dw+\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{h^{-}(w)}{w^{k+1}}dw$$

$$=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{h(w)}{w^{k+1}}dw=c_k(h)=\lambda_k c_k(g)=\lambda_k\beta_k(g)$$

and

$$\|g^{*}\|_{L^{p(\cdot)}(\mathbb{T})}\leq\left\|h^{+}\right\|_{L^{p(\cdot)}(\mathbb{T})}\leq c_9(p)\|h\|_{L^{p(\cdot)}(\mathbb{T})}\leq c_{10}(p)\|g\|_{L^{p(\cdot)}(\mathbb{T})}.$$

□

3. Proofs of main results

Proof of Theorem 4 If $f\in E^{p(\cdot)}(G)$, then $f_0^{+}=T^{-1}(f)\in E^{p_0(\cdot)}(\mathbb{D})$. Hence, applying the second inequality of assertion $i)$ in Lemma 22 and Theorem A, we have

$$\begin{aligned}E_n(f)_{G,p(\cdot)} &\leq c_6(p)E_n\left(f_0^{+}\right)_{p_0(\cdot)}\\ &\leq c_6(p)c(p,r)\,\Omega_r\left(f_0^{+},1/n\right)_{\mathbb{T},p_0(\cdot)}=c_1(p,r)\,\Omega_r(f,1/n)_{G,p(\cdot)}.\end{aligned}$$

□

Proof of Theorem 5 If $f\in E^{p(\cdot)}(G)$, then by Lemma 21 we have that $f_0^{+}\in E^{p_0(\cdot)}(\mathbb{D})$. Applying Theorem B for the boundary values of f_0^{+} and the first inequality of assertion $i)$ in Lemma 22, we obtain the desired inequality:

$$\begin{aligned}\Omega_r(f,1/n)_{G,p(\cdot)} &= \Omega_r\left(f_0^{+},1/n\right)_{\mathbb{T},p_0(\cdot)}\leq\frac{c(p,r)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k\left(f_0^{+}\right)_{p_0(\cdot)}\\ &\leq \frac{c(p,r)\,c_5(p)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_n(f)_{G,p(\cdot)}\\ &\leq \frac{c_2(p,r)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_n(f)_{G,p(\cdot)}\end{aligned}$$

□

Proof of Theorem 10 Let $f\in E^{p(\cdot)}(G)$. Then by (5) and (10) we have

$$a_k(f)=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0(w)}{w^{k+1}}dw=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^{+}(w)}{w^{k+1}}dw-\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^{-}(w)}{w^{k+1}}dw$$

$$=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^{+}(w)}{w^{k+1}}dw=\beta_k\left(f_0^{+}\right),\qquad k=0,1,2,\ldots.$$

This means that the Faber coefficients of f are also the Taylor coefficients of f_0^+ at the origin; that is,

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f)\, w^k, \quad w \in \mathbb{D}.$$

Since $f \in E^{p(\cdot)}(G)$ we have $f_0^+ \in E^{p_0(\cdot)}(\mathbb{D})$ and then by Lemma 23 there is a function $F_0 \in E^{p_0(\cdot)}(\mathbb{D})$ with the Taylor coefficients $\beta_k(F_0) = \lambda_k \beta_k\left(f_0^+\right) = \lambda_k a_k(f)$, $k = 0, 1, 2, \ldots$, such that

$$\|F_0\|_{L^{p_0(\cdot)}(\mathbb{T})} \leq c(p)\left\|f_0^+\right\|_{L^{p_0(\cdot)}(\mathbb{T})}.$$

At the same time, by Lemma 21, $F := T(F_0) \in E^{p(\cdot)}(G)$ and has the Faber coefficients $\beta_k(F_0) = \lambda_k a_k(f)$, $k = 0, 1, 2, \ldots$. Hence,

$$F(z) = T(F_0)(z) \sim \sum_{k=0}^{\infty} \lambda_k a_k(f)\, F_k(z), \quad z \in G.$$

Now using the boundedness of T, (9), and *Theorem C*, we have

$$\begin{aligned}
\|F\|_{L^{p(\cdot)}(\Gamma)} &= \|T(F_0)\|_{L^{p(\cdot)}(\Gamma)} \leq \|T\| \, \|F_0\|_{L^{p_0(\cdot)}(\mathbb{T})} \\
&\leq c(p)\left\|f_0^+\right\|_{L^{p_0(\cdot)}(\mathbb{T})} \leq c_{11}(p)\|f_0\|_{L^{p_0(\cdot)}(\mathbb{T})} \leq c_{12}(p)\|f\|_{L^{p(\cdot)}(\Gamma)}.
\end{aligned}$$

□

Let ω be a weight on Γ, i.e. an almost everywhere nonnegative integrable function on Γ, and let $B_r(z) := \{t : |t - z| < r,\ z \in \Gamma\}$, $r > 0$. ω is said to satisfy Muckenhoupt's $A_p(\Gamma)$, $1 < p < \infty$, condition if

$$\sup_{z\in\Gamma}\sup_{r>0}\left(\frac{1}{r}\int\limits_{\Gamma\cap B_r(z)} \omega(z)\,|dz|\right)\left(\frac{1}{r}\int\limits_{\Gamma\cap B_r(z)} \omega(z)^{1-p'}\,|dz|\right)^{p-1} < \infty,\ \frac{1}{p}+\frac{1}{p'} = 1.$$

Let also $\mathcal{M}f$ be the maximal function, defined as

$$\mathcal{M}f(z) := \sup_{\gamma \ni z}\left(\frac{1}{|\gamma|}\int\limits_{\gamma} |f(z)|\,|dz|\right),\ f \in L(\Gamma),$$

where the supremum is taken over all rectifiable arcs $\gamma \subset \Gamma$ that contain z and $|\gamma|$ is a Lebesgue measure of γ.

Proof of Theorem 11 Let $\omega \in A_p(\Gamma)$, $\Gamma \in \mathfrak{D}$, and $1 < p < \infty$. In [9, Theorem 3] it was proved that if $f \in E^p(G,\omega)$, then there exist the positive constants $c_{13}(p)$ and $c_{14}(p)$ such that

$$c_{13}(p)\|f\|_{L^p(\Gamma,\omega)} \leq \left\|\left(\sum_{k=0}^{\infty}|\Delta_k(f)|^2\right)^{1/2}\right\|_{L^p(\Gamma,\omega)} \leq c_{14}(p)\|f\|_{L^p(\Gamma,\omega)}. \tag{11}$$

Considering the operator $\mathcal{A} : f \to \left(\sum\limits_{k=0}^{\infty}|\Delta_k(f)|^2\right)^{1/2}$, we see that by the second inequality of (11), it is bounded in $L^p(\Gamma,\omega)$. On the other hand, the maximal operator $\mathcal{M}$ is bounded (see [23, p. 50, Theorem 2.29]) in $L^{p(\cdot)}(\Gamma)$, $p(\cdot)\in\mathcal{P}_0(\Gamma)$. Hence, all of the conditions of Corollary 5.32 of extrapolation Theorem 5.28, proved in [6, pp. 209–211], are fulfilled. Then

$$\left\|\left(\sum_{k=0}^{\infty}\left|\Delta_k(f)\right|^2\right)^{1/2}\right\|_{L^{p(\cdot)}(\Gamma)}\leq c_{15}(p)\left\|f\right\|_{L^{p(\cdot)}(\Gamma)},$$

and therefore, the second inequality of Theorem 11 is proved. The proof of the first inequality goes similarly. □

Proof of Theorem 12 If $f\in E^{p(\cdot)}(G^-)$, then by assertion $ii)$ of Lemma 21, we get $f_1^+\in E^{p_1(\cdot)}(\mathbb{D})$. Applying the second inequality of assertion $ii)$ in *Lemma 22* and Theorem A for f_1^+ we have that

$$\begin{aligned}E_n(f)_{G^-,p(\cdot)} &\leq c(p)\,E_n\left(f_1^+\right)_{p_1(\cdot)}\\ &\leq c(p,r)\,\Omega_r\left(f_1^+,1/n\right)_{\mathbb{T},p_1(\cdot)}=c(p,r)\,\Omega_r(f,1/n)_{G^-,p(\cdot)}.\end{aligned}$$

□

Proof of Theorem 13 If $f\in E^{p(\cdot)}(G^-)$, then by assertion ii) of Lemma 21 we have $f_1^+\in E^{p_1(\cdot)}(\mathbb{D})$. Hence, applying Theorem B for the boundary values of f_1^+ and the first inequality of assertion $ii)$ in Lemma 22, we get

$$\begin{aligned}\Omega_r(f,1/n)_{G^-,p(\cdot)} &= \Omega_r\left(f_1^+,1/n\right)_{\mathbb{T},p_1(\cdot)}\leq\frac{c(p,r)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k\left(f_1^+\right)_{p_1(\cdot)}\\ &\leq\frac{c(p,r)\,c_7(p)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_n(f)_{G^-,p(\cdot)}\\ &\leq\frac{c_3(p,r)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_n(f)_{G^-,p(\cdot)}.\end{aligned}$$

□

Proof of Theorem 18 Let $f\in E^{p(\cdot)}(G^-)$. By (7) and (10)

$$\widetilde{a}_k(f)=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_1(w)}{w^{k+1}}dw=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_1^+(w)}{w^{k+1}}dw-\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_1^-(w)}{w^{k+1}}dw$$

$$=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f_1^+(w)}{w^{k+1}}dw=\beta_k\left(f_1^+\right),$$

where $\beta_k\left(f_1^+\right)$, $k=0,1,2,...$, are the Taylor coefficients of $f_1^+\in E^{p_1(\cdot)}(\mathbb{D})$. This means that the Faber coefficients $\widetilde{a}_k(f)$ of f are the Taylor coefficients of f_1^+ at the origin; that is,

$$f_1^+(w)=\sum_{k=0}^{\infty}\widetilde{a}_k(f)\,w^k,\ \ w\in\mathbb{D}.$$

Since $f \in E^{p(\cdot)}(G^-)$ by assertion $ii)$ of Lemma 21, we have $f_1^+ \in E^{p_1(\cdot)}(\mathbb{D})$ and then by Lemma 23 there is a function $F_1 \in E^{p_1(\cdot)}(\mathbb{D})$ with the Taylor coefficients $\beta_k(F_1) = \lambda_k \beta_k(f_1^+) = \lambda_k \widetilde{a}_k(f)$, $k = 0, 1, 2, ...$, such that

$$\|F_1\|_{L^{p_1(\cdot)}(\mathbb{T})} \leq c(p) \left\|f_1^+\right\|_{L^{p_1(\cdot)}(\mathbb{T})}.$$

Since $\widetilde{T}(F_1) \in E^{p(\cdot)}(G^-)$, its Faber coefficients are $\beta_k(F_1) = \lambda_k \widetilde{a_k}(f)$ and hence

$$\widetilde{T}(F_1)(z) \sim \sum_{k=0}^{\infty} \lambda_k \widetilde{a}_k(f) \widetilde{F}_k(1/z), \quad z \in G^-.$$

Now denoting $F := \widetilde{T}(F_1)$, using the boundedness of $\widetilde{T}$ in $E^{p_1(\cdot)}(\mathbb{D})$ and the relation (9), and applying Theorem C, we have

$$\begin{aligned} \|F\|_{L^{p(\cdot)}(\Gamma)} &= \left\|\widetilde{T}(F_1)\right\|_{L^{p(\cdot)}(\Gamma)} \leq \left\|\widetilde{T}\right\| \|F_1\|_{L^{p_1(\cdot)}(\mathbb{T})} \\ &\leq c_{16}(p) \left\|f_1^+\right\|_{L^{p_1(\cdot)}(\mathbb{T})} \leq c_{17}(p) \|f_1\|_{L^{p_1(\cdot)}(\mathbb{T})} \leq c(p) \|f\|_{L^{p(\cdot)}(\Gamma)}. \end{aligned}$$

□

Proof of Theorem 19 The proof can be realized similarly to the proof of Theorem 11. We just need to apply the relation

$$c \|f\|_{L^p(\Gamma,\omega)} \leq \left\| \left(\sum_{k=1}^{\infty} \left| \widetilde{\Delta}_k(f) \right|^2 \right)^{1/2} \right\|_{L^{p(}(\Gamma,\omega)} \leq c \|f\|_{L^{p(}(\Gamma,\omega)},$$

proved in [9, Theorem 4], instead of (11). □

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