Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1457 – 1470
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doi:10.3906/mat-1708-6

Jakimovski–Leviatan operators of Durrmeyer type involving Appell polynomials

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Received: 03.08.2017 • **Accepted/Published Online:** 09.01.2018 • **Final Version:** 08.05.2018

Abstract: The purpose of the present paper is to establish the rate of convergence for a Lipschitz-type space and obtain the degree of approximation in terms of Lipschitz-type maximal function for the Durrmeyer type modification of Jakimovski–Leviatan operators based on Appell polynomials. We also study the rate of approximation of these operators in a weighted space of polynomial growth and for functions having a derivative of bounded variation.

Key words: Jakimovski–Leviatan–Durrmeyer-type operators, Appell polynomials, weighted modulus of continuity, bounded variation

1. Introduction

By utilizing Appell polynomials and the assumption that $p_k(y) \geq 0, \quad \forall k \geq 0$, Jakimovski and Leviatan generalized the well-known Szász–Mirakyan operators as (see [8])

$$P_n^*(f;y) = \frac{e^{-ny}}{g(1)} \sum_{k=0}^{\infty} p_k(ny) f\left(\frac{k}{n}\right), \tag{1.1}$$

having a generating function of the form

$$g(u)e^{uy} = \sum_{k=0}^{\infty} p_k(y)u^k,$$

where $g(z) = \sum_{k=0}^{\infty} a_n z^n, a_0 \neq 0$, is an analytic function in the disk $|z| < R, R > 1, g(1) \neq 0$, and the explicit form of $p_k(y)$ is given by

$$p_k(y) = \sum_{\nu=0}^{k} a_\nu \frac{y^{k-\nu}}{(k-\nu)!} \quad k = 0, 1, 2, \ldots.$$

If $g(z) = 1$, we have $p_k(y) = \frac{y^k}{k!}$ and hence we obtain from (1.1), the following Szász–Mirakyan operators

$$S_n(f;y) = e^{-ny} \sum_{k=0}^{\infty} \frac{(ny)^k}{k!} f\left(\frac{k}{n}\right).$$

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2010 *AMS Mathematics Subject Classification:* 26A15, 41A35, 26A45

Karaisa [10] introduced a Durrmeyer-type modification of the positive linear operator defined by (1.1) for $f \in C_B[0,\infty)$, the class of all continuous, bounded, and real valued functions on the interval $[0,\infty)$ as follows:

$$L_n(f;y) = \frac{e^{-ny}}{g(1)} \sum_{k=1}^{\infty} \frac{p_k(ny)}{B(n+1,k)} \int_0^{\infty} b_{n,k}(r) f(r) dr + \frac{e^{-ny}}{g(1)} a_0 f(0), \quad y \geq 0, \tag{1.2}$$

where $B(k+1,n)$ is the beta function, $b_{n,k}(r) = \frac{r^{k-1}}{(1+r)^{n+k+1}}$, and studied local approximation properties and the convergence in a weighted space of functions. A Voronovskaja-type theorem for these operators was also proved. We note that the operators given by (1.2) are defined for a bigger class of functions, e.g., $C_\gamma[0,\infty)$ than the class $C_B[0,\infty)$ considered in [10], where for a given $\gamma > 0, C_\gamma[0,\infty)$ is defined as follows:

$$C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(r)| \leq C_f(1+r^\gamma), \forall r \geq 0\}.$$

endowed with the norm $||f||_\gamma = \sup\limits_{r \in [0,\infty)} \dfrac{|f(r)|}{1+r^\gamma}$.

For every $\gamma > 0$, we can find an integer m such that $\gamma < m$ and hence using ([10], Lemma 2.2)

$$\frac{e^{-ny}}{g(1)} \sum_{k=1}^{\infty} \frac{p_k(ny)}{B(n+1,k)} \int_0^{\infty} b_{n,k}(r)(1+r^m) dr + \frac{e^{-ny}}{g(1)} a_0 \quad exists\ finitely,$$

the integral

$$\frac{e^{-ny}}{g(1)} \sum_{k=1}^{\infty} \frac{p_k(ny)}{B(n+1,k)} \int_0^{\infty} b_{n,k}(r)(1+r^\gamma) dr + \frac{e^{-ny}}{g(1)} a_0$$

also exists and is finite.

Hence for every $f \in C_\gamma[0,\infty)$, $\frac{e^{-ny}}{g(1)} \sum\limits_{k=1}^{\infty} \dfrac{p_k(ny)}{B(n+1,k)} \displaystyle\int_0^{\infty} b_{n,k}(r)|f(r)| dr + \frac{e^{-ny}}{g(1)} a_0 |f(0)|$ exists finitely. Thus $L_n(f^+;y)$ and $L_n(f^-;y)$ both exist and are finite. Now, since $\frac{e^{-ny}}{g(1)} \frac{p_k(ny)}{B(n+1,k)} b_{n,k}(r) f^+(r)$ is nonegative for every $r \in [0,\infty)$, using monotone convergence theorem

$$\begin{aligned} L_n(f^+;y) &= \frac{e^{-ny}}{g(1)} \int_0^{\infty} \sum_{k=1}^{\infty} \frac{p_k(ny)}{B(n+1,k)} b_{n,k}(r) f^+(r) dr + \frac{e^{-ny}}{g(1)} a_0 f^+(0) \\ &= \frac{e^{-ny}}{g(1)} \sum_{k=1}^{\infty} \frac{p_k(ny)}{B(n+1,k)} \int_0^{\infty} b_{n,k}(r) f^+(r) dr + \frac{e^{-ny}}{g(1)} a_0 f^+(0). \end{aligned} \tag{1.3}$$

By the same argument,

$$L_n(f^-;y) = \frac{e^{-ny}}{g(1)} \sum_{k=1}^{\infty} \frac{p_k(ny)}{B(n+1,k)} \int_0^{\infty} b_{n,k}(r) f^-(r) dr + \frac{e^{-ny}}{g(1)} a_0 f^-(0). \tag{1.4}$$

Thus, using (1.3) and (1.4)

$$\begin{aligned}
L_n(f;y) &= \frac{e^{-ny}}{g(1)}\int_0^\infty \sum_{k=1}^\infty \frac{p_k(ny)}{B(n+1,k)} b_{n,k}(r)f(r)dr + \frac{e^{-ny}}{g(1)}a_0 f(0)\\
&= \frac{e^{-ny}}{g(1)}\int_0^\infty \sum_{k=1}^\infty \frac{p_k(ny)}{B(n+1,k)} b_{n,k}(r)f^+(r)dr + \frac{e^{-ny}}{g(1)}a_0 f^+(0)\\
&- \frac{e^{-ny}}{g(1)}\int_0^\infty \sum_{k=1}^\infty \frac{p_k(ny)}{B(n+1,k)} b_{n,k}(r)f^-(r)dr - \frac{e^{-ny}}{g(1)}a_0 f^-(0)\\
&= \frac{e^{-ny}}{g(1)} \sum_{k=1}^\infty \frac{p_k(ny)}{B(n+1,k)} \int_0^\infty b_{n,k}(r)(f^+(r)-f^-(r))dr + \frac{e^{-ny}}{g(1)}a_0 (f^+(0)-f^-(0))\\
&= \frac{e^{-ny}}{g(1)} \sum_{k=1}^\infty \frac{p_k(ny)}{B(n+1,k)} \int_0^\infty b_{n,k}(r)f(r)dr + \frac{e^{-ny}}{g(1)}a_0 f(0).
\end{aligned}$$

Hence the operator (1.2) can be rewritten as

$$L_n(f;y) = \int_0^\infty K_n(y,r)f(r)dr,$$

where the kernel $K_n(y,t)$ is given by

$$K_n(y,r) = \sum_{k=1}^\infty \frac{e^{-ny}}{g(1)} \frac{p_k(ny)}{B(n+1,k)} b_{n,k}(r) + \frac{e^{-ny}}{g(1)} a_0 \delta(r),$$

$\delta(r)$ being the Dirac-delta function.

In the present paper we obtain the rate of convergence in terms of the weighted modulus of continuity and the Lipschitz-type maximal function for the operators given by (1.2). Moreover, we study the rate of convergence of these operators in a weighted space and for functions having a derivative locally of bounded variation. Several researchers have contributed in this direction. For some of the related works we refer the readers to (cf.[1–3, 5, 6, 9, 11–13, 16] etc.).

2. Preliminaries

Lemma 2.1 *[10] For the linear positive operators* (1.2)*, the estimates of moments are given by*

(i) $L_n(1;y) = 1$;

(ii) $L_n(r;y) = y + \frac{1}{n}\left(\frac{g'(1)}{g(1)}\right)$;

(iii) $L_n(r^2;y) = \frac{1}{n(n-1)}\left(n^2y^2 + ny\frac{2(g'(1)+g(1))}{g(1)} + \frac{g''(1)+2g'(1)}{g(1)}\right)$;

(iv) $L_n(r^3;y) = \frac{1}{n(n-1)(n-2)}\left(n^3y^3 + n^2y^2\frac{\{3g'(1)+7g(1)\}}{g(1)} + ny\frac{\{3g''(1)+14g'(1)+5g(1)\}}{g(1)} + \frac{g'''(1)+7g''(1)+5g'(1)}{g(1)}\right)$;

(v) $L_n(r^4;y) = \frac{1}{n(n-1)(n-2)(n-3)}\Bigg(n^4y^4 + \frac{n^3y^3\{16g(1)+4g'(1)\}}{g(1)} + ny\frac{\{4g'''(1)+90g'(1)+48g''(1)+17g(1)\}}{g(1)}$

$+ \frac{17g'(1)+45g''(1)+16g'''(1)+g''''(1)}{g(1)}\Bigg).$

Following [15], the Lipschitz-type space is defined by

$$Lip_M^*(s) := \left\{f \in C[0,\infty) : |f(r)-f(y)| \leq M\frac{|r-y|^s}{(r+y)^{\frac{s}{2}}}; r \geq 0 \ \ and \ \ y \in (0,\infty)\right\},$$

where M is a positive constant and $0 < s \leq 2$.

In what follows, let $\gamma_n(y) = L_n((r-y)^2;y)$ and $||.||_{C[a,b]}$ denotes the sup norm over $[a,b]$.

Lemma 2.2 *For every $y \geq 0$ and $n \geq 2$, we have*

$$L_n(|r-y|;y) \leq \sqrt{\gamma_n(y)}.$$

Proof We have

$$L_n(|r-y|;y) = \int_0^\infty K_n(y,r)|r-y|dr.$$

Applying the Cauchy–Schwarz inequality and Lemma 2.1 we obtain

$$\begin{aligned} L_n(|r-y|;y) &\leq \left(\int_0^\infty K_n(y,r)dr\right)^{\frac{1}{2}}\left(\int_0^\infty K_n(y,r)(r-y)^2dr\right)^{\frac{1}{2}} \\ &= \sqrt{L_n((r-y)^2;y)} = \sqrt{\gamma_n(y)}. \end{aligned}$$

This completes the proof. □

Let the space $C_2^*[0,\infty)$ be defined as

$$C_2^*[0,\infty) := \{f \in C_2[0,\infty) : lim_{y\to\infty}\frac{|f(y)|}{1+y^2} \ \ exists\}.$$

Following [17] for $f \in C_2^*[0,\infty)$, the weighted modulus of continuity is given by

$$\Omega(f,\delta) = \sup_{y\geq 0}\sup_{0<|h|\leq\delta}\frac{|f(y+h)-f(y)|}{1+(y+h)^2}.$$

The weighted modulus of continuity $\Omega(f,\delta)$ satisfies the following properties:

Lemma 2.3 *[17] If $f \in C_2^*[0,\infty)$, then*

1. *$\Omega(f,\delta)$ is a monotone increasing function of δ;*

2. *$lim_{\delta\to 0^+}\Omega(f,\delta) = 0$;*

3. *for any $\lambda \in [0,\infty), \Omega(f,\lambda\delta) \leq (1+\lambda)\Omega(f,\delta)$.*

Using Lemma 2.1, for any $\lambda > 3$ and sufficiently large n we have

$$\begin{aligned}\gamma_n(y) &= \frac{y^2}{n-1} + \frac{2y}{n}\left\{1 + \frac{1}{n-1}\left(\frac{g'(1)}{g(1)} + 1\right)\right\} + \frac{1}{n(n-1)}\left(\frac{g''(1) + 2g'(1)}{g(1)}\right) \\ &\leq \frac{\lambda(1+y^2)}{n}, \end{aligned} \tag{2.1}$$

for every $y \in [0, \infty)$.

Lemma 2.4 *For a fixed* $y \in (0, \infty)$, $\lambda > 3$, *and* n *sufficiently large, we have*

$$\lambda_n(y; q) := \int_0^q K_n(y, r)dr < \frac{\lambda(1+y^2)}{n(y-q)^2}, \quad 0 \leq q < y,$$

and

$$1 - \lambda_n(y, z) := \int_z^\infty K_n(y, r)dr < \frac{\lambda(1+y^2)}{n(y-z)^2}, \quad y < z < \infty.$$

Proof We may write

$$\lambda_n(y; q) \leq \int_0^q K_n(y; r)\frac{(r-y)^2}{(y-q)^2}dr \leq \frac{\gamma_n(y)}{(y-q)^2} < \frac{\lambda(1+y^2)}{n(y-q)^2}.$$

Similarly, we can prove the other inequality. □

3. Main results

In the following result we obtain the rate of convergence of the operators L_n for functions in a Lipschitz-type space.

Theorem 3.1 *Let* $0 < s \leq 2$ *and* $f \in Lip_M^*(s)$. *Then for all* $y > 0$ *and* $n \geq 2$, *we have*

$$|L_n(f; y) - f(y)| \leq M\left(\frac{\gamma_n(y)}{y}\right)^{\frac{s}{2}}.$$

Proof First we prove the result for s=2,

$$\begin{aligned}|L_n(f; y) - f(y)| &\leq \int_0^\infty K_n(y, r)|f(r) - f(y)|dr \\ &\leq M\int_0^\infty K_n(y, r)\frac{|r-y|^2}{r+y}dr \\ &\leq \frac{M}{y}\gamma_n(y)\end{aligned}$$

Now for $(0 < s < 2)$, by our hypothesis we have

$$\begin{aligned} |L_n(f;y) - f(y)| &\leq \int_0^\infty K_n(y,r)|f(r) - f(y)|dr \\ &\leq M \int_0^\infty K_n(y,r) \frac{|r-y|^s}{(r+y)^{\frac{s}{2}}} dr \\ &\leq \frac{M}{(y)^{\frac{s}{2}}} \int_0^\infty K_n(y,r)|r-y|^s dr. \end{aligned}$$

Applying Hölder's inequality by taking $p = \frac{2}{s}$ and $q = \frac{2}{2-s}$ and Lemma 2.1, we have

$$\begin{aligned} |L_n(f;y) - f(y)| &\leq \frac{M}{(y)^{\frac{s}{2}}} \left(\int_0^\infty K_n(y,r)(|r-y|^s)^p dr \right)^{\frac{1}{p}} \left(\int_0^\infty K_n(y,r) dr \right)^{\frac{1}{q}} \\ &\leq \frac{M}{(y)^{\frac{s}{2}}} \left(\int_0^\infty K_n(y,r)|r-y|^{sp} dr \right)^{\frac{1}{p}} \\ &= M \left(\frac{\gamma_n(y)}{y} \right)^{\frac{s}{2}}, \end{aligned}$$

which completes the proof. □

In the next theorem we obtain the rate of convergence of the operators L_n for functions in the weighted space $C_2[0,\infty)$.

Theorem 3.2 *Let $f \in C_2[0,\infty)$ and ω_{a+1} be its modulus of continuity on the finite interval $[0, a+1], a > 0$. Then for every $n \geq 2$*

$$||L_n(f) - f||_{C[0,a]} \leq 4M_f(1+a^2)\eta_n + 2\omega_{a+1}(f, \sqrt{\eta_n}),$$

where $\eta_n = max_{y\in[0,a]}\gamma_n(y)$.

Proof From [7] for $y \in [0,a]$ *and* $r \in [a+1,\infty)$, we have $r - y > 1$; hence it follows that

$$\begin{aligned} |f(r) - f(y)| &\leq M_f(2 + r^2 + y^2) \\ &= M_f(2 + (r-y+y)^2 + y^2) \\ &\leq M_f(r-y)^2(2 + 2y^2 + 2y) \\ &\leq 4M_f(1+y^2)(r-y)^2. \end{aligned}$$

And for $r \in [0, a+1]$ and $y \in [0,a]$

$$\begin{aligned} |f(r) - f(y)| &\leq \omega_f\left(\frac{|r-y|}{\delta}\right) \\ &\left(1 + \frac{|r-y|}{\delta}\right)\omega_f(\delta). \end{aligned}$$

Thus

$$|f(r)-f(y)| \leq 4M_f(1+y^2)(r-y)^2 + \left(1+\frac{|r-y|}{\delta}\right)\omega_{a+1}(f;\delta), \tag{3.1}$$

for any $\delta > 0$. Hence applying the operator $L_n(.;y)$ and using Lemma 2.2 and the Cauchy–Schwarz inequality we get

$$\begin{aligned}|L_n(f;y)-f(y)| &\leq 4M_f(1+y^2)L_n((r-y)^2;y) + \left(1+\frac{1}{\delta}L_n(|r-y|;y)\right)\omega_{a+1}(f,\delta)\\ &\leq 4M_f(1+y^2)\eta_n + \omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\eta_n}\right).\end{aligned}$$

Choosing $\delta = \sqrt{\eta_n}$, the desired result follows. □

For $f \in C_B[0,\infty)$, the Lipschitz-type maximal function of order α given by Lenze [14] is defined as follows:

$$\tilde{\omega}_\alpha(f,y) = \sup_{r\neq y, r\in[0,\infty)} \frac{|f(r)-f(y)|}{|r-y|^\alpha}, \quad y\in[0,\infty) \ \ \textit{and} \ \ \alpha\in(0,1].$$

In the next result we obtain an estimate of the error for a Lipschitz-type maximal function.

Theorem 3.3 *Let $f \in \tilde{C}_B[0,\infty)$ and $0<\alpha\leq 1$. Then for all $y\in[0,\infty)$, we get*

$$|L_n(f;y)-f(y)| \leq \tilde{\omega}_\alpha(f,y)\gamma_n^{\alpha/2}(y).$$

Proof By the definition of $\tilde{\omega}_\alpha(f,y)$, we have

$$|f(r)-f(y)| \leq \tilde{\omega}_\alpha(f,y)|r-y|^\alpha.$$

Thus

$$|L_n(f;y)-f(y)| \leq L_n(|f(r)-f(y)|,y) \leq \tilde{\omega}_\alpha(f,y)L_n(|r-y|^\alpha,y)$$

Applying Hölder's inequality with $p=\frac{2}{\alpha}$ and $\frac{1}{q}=1-\frac{1}{p}$, we have

$$\begin{aligned}|L_n(f;y)-f(y)| &\leq \tilde{\omega}_\alpha(f,y)L_n((r-y)^2;y)^{\frac{\alpha}{2}}\\ &\leq \tilde{\omega}_\alpha(f,y)\gamma_n^{\alpha/2}(y).\end{aligned}$$

Thus, the proof is complete. □

The following result is a Korovkin-type theorem in the weighted space $C_2[0,\infty)$. This type of result has been discussed in [4] for locally integrable functions.

Theorem 3.4 *For each $f\in C_2[0,\infty)$ and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{y\in[0,\infty)} \frac{|L_n(f;y)-f(y)|}{(1+y^2)^{1+\alpha}} = 0.$$

Proof Let $y_0 \in [0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}
\sup_{y\in[0,\infty)} \frac{|L_n(f;y)-f(y)|}{(1+y^2)^{1+\alpha}} &\leq \sup_{y\leq y_0} \frac{|L_n(f;y)-f(y)|}{(1+y^2)^{1+\alpha}} + \sup_{y>y_0} \frac{|L_n(f;y)-f(y)|}{(1+y^2)^{1+\alpha}} \\
&\leq ||L_n(f)-f||_{C[0,y_0]} + ||f||_2 \sup_{y>y_0} \frac{|L_n(1+r^2;y)|}{(1+y^2)^{1+\alpha}} \\
&\quad + \sup_{y>y_0} \frac{|f(y)|}{(1+y^2)^{1+\alpha}} \\
&= I_1 + I_2 + I_3, say. \qquad (3.2)
\end{aligned}$$

Since $|f(y)| \leq ||f||_2(1+y^2)$, we have $sup_{y>y_0} \frac{|f(y)|}{(1+y^2)^{1+\alpha}} \leq \frac{||f||_2}{(1+y_0^2)^\alpha}$.

Let $\epsilon > 0$ be arbitrary. We can choose y_0 to be so large that

$$\frac{||f||_2}{(1+y_0^2)^\alpha} < \frac{\epsilon}{6}. \qquad (3.3)$$

From Lemma 2.1, there exists $n_1(y) \in \mathbb{N}$ such that

$$\begin{aligned}
||f||_2 \frac{L_n(1+r^2;y)}{(1+y^2)^{1+\alpha}} &\leq \frac{||f||_2}{(1+y^2)^{1+\alpha}} \left(1+y^2+\frac{\epsilon}{3||f||_2}\right), \forall \ n \geq n_1(y) \\
&< \frac{||f||_2}{(1+y_0^2)^\alpha} + \frac{\epsilon}{3}, \ \forall n \geq n_1(y) \ \ and \ \ y > y_0.
\end{aligned}$$

Hence,

$$||f||_2 \sup_{y>y_0} \frac{L_n(1+t^2;y)}{(1+y^2)^{1+\alpha}} < \frac{||f||_2}{(1+y_0^2)^\alpha} + \frac{\epsilon}{3}, \ \forall n \geq n_1(y). \qquad (3.4)$$

Using Theorem 3.2, we see that there exists $n_2 \in \mathbb{N}$ such that

$$||L_n(f)-f||_{C[0,y_0]} < \frac{\epsilon}{3}, \forall n \geq n_2. \qquad (3.5)$$

Let $n_0 = max(n_1(y), n_2)$. Then combining (3.2) − (3.5) we obtain the required result. □

In the next result we present an estimate of the rate of convergence in terms of the weighted modulus of continuity for functions in $C_2^*[0,\infty)$.

Theorem 3.5 *If $f \in C_2^*[0,\infty)$ and $y \in [0,\infty)$, then there exists $n_0 \in \mathbb{N}$ such that*

$$|L_n(f;y)-f(y)| \leq K(1+y^{2+\mu})\Omega(f,\delta_n), \quad for \quad all \quad n \geq n_0,$$

where $\mu \geq 1$, $\delta_n = \sqrt{\frac{\lambda}{n}}$, $\lambda > 3$ and K is a positive constant independent of f and n.

Proof From the definition of $\Omega(f,\delta)$ and applying Lemma 2.1, we have

$$\begin{aligned} |f(r)-f(y)| &\leq (1+(y+|r-y|)^2)\left(1+\frac{|r-y|}{\delta}\right)\Omega(f,\delta) \\ &\leq (1+(2y+r)^2)\left(1+\frac{|r-y|}{\delta}\right)\Omega(f,\delta) \\ &= \phi_y(r)\left(1+\frac{\psi_y(r)}{\delta}\right)\Omega(f,\delta), \end{aligned} \tag{3.6}$$

where $\phi_y(r)=1+(2y+r)^2$ and $\psi_y(r)=|r-y|$. Thus, from (1.1)

$$|L_n(f;y)-f(y)| \leq \left(L_n(\phi_y;y)+\frac{L_n(\phi_y\psi_y;y)}{\delta}\right)\Omega(f,\delta).$$

From (3.6), applying the Cauchy–Schwarz inequality we are led to

$$|L_n(f;y)-f(y)| \leq \left(L_n(\phi_y;y)+\frac{1}{\delta}\sqrt{L_n(\phi_y^2;y)}\sqrt{(L_n\psi_y^2;y)}\right)\Omega(f,\delta). \tag{3.7}$$

From Lemma 2.1, it follows that there exists a positive constant K_1 such that

$$L_n(\phi_y;y) \leq K_1(1+y^2). \tag{3.8}$$

Hence by a similar reasoning, there exists a positive constant K_2 such that

$$\sqrt{L_n(\phi_y^2;y)} \leq K_2(1+y^2). \tag{3.9}$$

Further, for any $\lambda>3$ and $y\in[0,\infty)$ there exists a positive integer n_0 such that

$$L_n(\psi_y^2;y) \quad \leq \frac{\lambda(1+y^2)}{n}, \forall \;\; n \geq n_0.$$

Combining (3.7)–(3.9), we have

$$|L_n(f;y)-f(y)| \leq (1+y^2)\left(K_1+\frac{1}{\delta}K_2\sqrt{\frac{\lambda(1+y^2)}{n}}\right)\Omega(f,\delta), \forall n \geq n_0.$$

Choosing $\delta=\delta_n=\sqrt{\frac{\lambda}{n}}$, $\forall y\in[0,\infty)$ we get

$$\begin{aligned} |L_n(f;y)-f(y)| &\leq (1+y^2)(K_1+K_2\sqrt{y^2+y})\Omega(f,\delta_n), \\ &\leq K_3(1+y^{2+\mu})\Omega(f,\delta_n), \forall n \geq n_0. \end{aligned}$$

Thus, the proof is completed. □

Lastly, we obtain the rate of convergence of the operators defined by (1.2) for functions with derivatives of bounded variation. Let $DBV[0,\infty)$ be the class of all functions f in $C_2[0,\infty)$ having a derivative of bounded variation on every finite subinterval of $[0,\infty)$. Then f can be represented as

$$f(y)=\int_0^y g(t)dt+f(0),$$

where $g(t)$ is a function locally of bounded variation.

Theorem 3.6 *Let $f\in DBV[0,\infty)$. Then, for every $y\in(0,\infty)$ and sufficiently large n,*

$$\begin{aligned}|L_n(f;y)-f(y)| \leq\ & |g'(1)|\left(\frac{|f'(y+)+f'(y-)|}{2ng(1)}\right)+\left|\frac{f'(y+)+f'(y-)}{2}\right|\sqrt{\frac{\lambda(1+y^2)}{n}}\\ &+\frac{y}{\sqrt{n}}\bigvee_{y-\frac{y}{\sqrt{n}}}^{y+\frac{y}{\sqrt{n}}}(f'_y)+\frac{\lambda(1+y^2)}{ny}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{y-\frac{y}{k}}^{y+\frac{y}{k}}(f'_y)\\ &+4\left(M_f+\frac{M_f+|f(y)|}{y^2}\right)\frac{\lambda(1+y^2)}{n}\\ &+|f'(y+)|\sqrt{\frac{\lambda(1+y^2)}{n}}+\frac{\lambda(1+y^2)}{ny^2}|f(2y)-f(y)-yf'(y+)|,\end{aligned}$$

where

$$f'_y(r)=\begin{cases} f'(r)-f'(y+), & y<r<\infty\\ 0, & r=y\\ f'(r)-f'(y-), & 0\le r<y,\end{cases} \tag{3.10}$$

and $\bigvee_c^d(f'_y)$ is the total variation of f'_y on $[c,d]\subset(0,\infty)$.

Proof From (3.10), we may write

$$\begin{aligned}f'(r) =\ & \frac{1}{2}(f'(y+)+f'(y-))+f'_y(r)+\frac{(f'(y+)-f'(y-))}{2}sgn(r-y)\\ &+\delta_y(r)\left(f'(r)-\frac{1}{2}(f'(y+)+f'(y-))\right),\end{aligned} \tag{3.11}$$

where

$$\delta_y(r)=\begin{cases} 1, & y=r\\ 0, & y\neq r.\end{cases}$$

Clearly, using the definition of $\delta_y(r)$

$$\int_0^\infty K_n(y,r)\delta_y(r)\left(f'(r)-\frac{f'(y+)+f'(y-)}{2}\right)dr=0.$$

Hence, using (3.11) we find

$$\begin{aligned}|L_n(f;y)-f(y)| &= \left|\int_0^{\infty} K_n(y,r)\Big(\int_y^r f'(u)du\Big)dr\right| \\ &\leq \left|\frac{f'(y+)+f'(y-)}{2}\right||L_n(r-y,y)| + \left|\frac{f'(y+)-f'(y-)}{2}\right|(L_n|r-y|,y)) \\ &\quad +|E_1(n,y)+|E_2(n,y)|, \end{aligned} \tag{3.12}$$

where

$$E_1(n,y)=\int_0^y \left(\int_y^r f'_y(u)du\right)K_n(y,r)dr$$

and

$$E_2(n,y)=\int_y^{\infty} \left(\int_y^r f'_y(u)du\right)K_n(y,r)dr.$$

To find an estimate of $E_1(n,y)$, by using the definition of $\lambda_n(y,r)$ we have

$$E_1(n,y)=\int_0^y \left(\int_y^r f'_y(u)du\right)\frac{\partial}{\partial r}\lambda_n(y,r)dr.$$

Applying the integration by parts, we get

$$\begin{aligned}|E_1(n,y)| &\leq \int_0^y |f'_y(r)|\lambda_n(y,r)dt \\ &\leq \int_0^{y-\frac{y}{\sqrt{n}}} |f'_y(r)|\lambda_n(y,r)dr + \int_{y-\frac{y}{\sqrt{n}}}^{y} |f'_y(r)|\lambda_n(y,r)dr \\ &= J_1+J_2, say. \end{aligned} \tag{3.13}$$

Since $f'_y(y)=0$ and $\lambda_n(y,r)\leq 1$, we have

$$\begin{aligned}J_2 &= \int_{y-\frac{y}{\sqrt{n}}}^{y} |f'_y(r)|\lambda_n(y,r)dr \\ &\leq \int_{y-\frac{y}{\sqrt{n}}}^{y} \bigvee_r^y (f'_y)dr \\ &\leq \bigvee_{y-\frac{y}{\sqrt{n}}}^{y} (f'_y)\int_{y-\frac{y}{\sqrt{n}}}^{y} dr = \frac{y}{\sqrt{n}}\bigvee_{y-\frac{y}{\sqrt{n}}}^{y} (f'_y). \end{aligned} \tag{3.14}$$

Applying Lemma 2.4 and substituting $r = y - \frac{y}{u}$,

$$\begin{aligned}
J_1 &\leq \frac{\lambda(1+y^2)}{n}\int_0^{y-\frac{y}{\sqrt{n}}} |f'_y(r) - f'_y(y)|\frac{dr}{(y-r)^2} \\
&\leq \frac{\lambda(1+y^2)}{n}\int_0^{y-\frac{y}{\sqrt{n}}} \bigvee_r^y (f'_y)\frac{dr}{(y-r)^2} \\
&= \frac{\lambda(1+y^2)}{ny}\int_1^{\sqrt{n}} \bigvee_{y-\frac{y}{u}}^{y} (f'_y)du \\
&\leq \frac{\lambda(1+y^2)}{ny}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{y-\frac{y}{k}}^{y} (f'_y). \qquad (3.15)
\end{aligned}$$

Collecting the estimates (3.13)–(3.15), we obtain

$$|E_1(n,y)| \leq \frac{y}{\sqrt{n}}\bigvee_{y-\frac{y}{\sqrt{n}}}^{y} (f'_y) + \frac{\lambda(1+y^2)}{ny}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{y-\frac{y}{k}}^{y} (f'_y). \qquad (3.16)$$

In order to determine an estimate of $E_2(n,y)$, using (3.10), Lemma 2.4, and integration by parts

$$\begin{aligned}
|E_2(n,y)| \;\; &\leq \;\; \left|\int_{2y}^{\infty}\left(\int_y^r f'_y(u)du\right)K_n(y,r)dr\right| \\
&+\left|\int_y^{2y}\left(\int_y^r f'_y(u)du\right)\frac{\partial}{\partial r}(1-\lambda_n(y,r))dr\right| \\
&\leq \left|\int_{2y}^{\infty}(f(r)-f(y))K_n(y,r)dr\right| + |f'(y+)|\left|\int_{2y}^{\infty}(r-y)K_n(y,r)dr\right| \\
&+\left|\int_y^{2y} f'_y(u)du\right||1-\lambda_n(y,2y)| + \left|\int_y^{2y} f'_y(r)(1-\lambda_n(y,r))dr\right|.
\end{aligned}$$

Since $|f(r)| \leq M(1+r^2)$, for every $r > 0$, applying the Cauchy–Schwarz inequality and estimating the last term on the right side of the above inequality in a manner similar to the estimate of $E_1(n,y)$, we get

$$\begin{aligned}
|E_2(n,y)| \leq & M_f\int_{2y}^{\infty}(1+r^2)K_n(y,r)dr + |f(y)|\int_{2y}^{\infty}K_n(y,r)dr + |f'(y+)|\sqrt{\frac{\lambda(1+y^2)}{n}} \\
&+ \frac{\lambda(1+y^2)}{ny^2}|f(2y)-f(y)-yf'(y+)| + \frac{y}{\sqrt{n}}\bigvee_{y}^{y+\frac{y}{\sqrt{n}}}(f'_y) \\
&+ \frac{\lambda(1+y^2)}{ny}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{y}^{y+\frac{y}{k}} f'_y. \qquad (3.17)
\end{aligned}$$

Finally, since $r \leq 2(r-y)$ and $y \leq r-y$ when $r \geq 2y$, using equation (2.1)

$$M_f \int_{2y}^{\infty} (1+r^2) K_n(y,r)dr + |f(y)| \int_{2y}^{\infty} K_n(y,r)dr$$

$$\leq (M_f + |f(y)|) \int_{2y}^{\infty} K_n(y,r)dr + 4M_f \int_{2y}^{\infty} (r-y)^2 K_n(y,r)dr$$

$$\leq \left(\frac{M_f + |f(y)|}{y^2} \right) \int_{2y}^{\infty} (r-y)^2 K_n(y,r)dr + 4M_f \int_{2y}^{\infty} (r-y)^2 K_n(y,r)dr$$

$$\leq 4\left(M_f + \frac{M_f + |f(y)|}{y^2} \right) \frac{\lambda(1+y^2)}{n}. \tag{3.18}$$

Combining (3.12) and (3.16)–(3.18) and using (2.1), we get the required result. This completes the proof of the theorem. □

Acknowledgment

The authors are extremely thankful to the reviewers for a very careful reading of the manuscript and making valuable comments and suggestions leading to a better presentation of the paper. The first author is thankful to the "Ministry of Human Resource Development", New Delhi, India, for financial support to carry out the above work.

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