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Convergence analysis and numerical solution of the Benjamin–Bona–Mahony equation by Lie–Trotter splitting

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Abstract: In this paper, an operator splitting method is used to analyze nonlinear Benjamin–Bona–Mahony-type equations. We split the equation into an unbounded linear part and a bounded nonlinear part and then Lie–Trotter splitting is applied to the equation. The local error bounds are obtained by using the approach based on the differential theory of operators in a Banach space and the quadrature error estimates via Lie commutator bounds. The global error estimate is obtained via Lady Windermere's fan argument. Finally, to confirm the expected convergence order, numerical examples are studied.

Key words: Lie–Trotter splitting, convergence analysis, Benjamin–Bona–Mahony equation

1. Introduction

Nonlinear phenomena play a fundamental role in applied mathematics and physics. Here we study the initial value problems of nonlinear Benjamin–Bona–Mahony (BBM)-type equations in the form

$$u_t = (1-\partial_x^2)^{-1}K(\partial_x)u + \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(u^2), \quad u|_{t=t_0} = u_0, \tag{1}$$

where $x \in \mathbb{R}$, $0 \le t \le T$, and K is a polynomial of degree $d \ge 2$ satisfying $\mathrm{Re}\big(K(i\xi)\big) \le 0$ for all $\xi \in \mathbb{R}$. Moreover, the equation corresponds to the generalized BBM equation when $d = 2$, and to the KdV-BBM equation when $d = 3$; see [3, 5, 8, 18].

In 1972, Benjamin et al. improved the Benjamin–Bona–Mahony equation as an alternative to the Korteweg-de-Vries equation for modeling the unidirectional propagation of weakly long dispersive waves [4]. Many researchers have introduced various numerical methods to solve the BBM equation. Al-Khaled et al. [1] implemented Adomian's decomposition method for obtaining numerical solutions of the BBM equation. Tari and Ganji [19] have applied variational iteration and homotopy perturbation methods in order to derive approximate explicit solutions for the BBM equation. El-Wakil et al. [9] used the exp-function method to obtain generalized solitary solutions and periodic solutions. Dutykh et al. [8] used the finite volume method to solve unidirectional dispersive waves. Furthermore, finite element method and spectral method solution techniques can be found in [2, 7, 15].

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Lie–Trotter and Strang splitting are commonly used techniques that give advantages when you are interested in the solutions of complicated nonlinear problems. The idea is based on splitting equations into two parts such as linear and nonlinear and then solving each one with suitable techniques in time. Lie–Trotter is a two-step method; the other one is three-step. Both have advantages and disadvantages. When you are working with Lie–Trotter you spend less time and easily construct error bounds but have less accuracy. Strang splitting takes a long time and is hard work but gives better accuracy. During the 2000s many papers focused on the convergence analysis of these methods with different nonlinear equations [6, 10–14, 16, 17, 21]. In [11], a convergence analysis for Lie–Trotter and Strang splitting in time of the KdV equation is given. They use solutions of the KdV equation remaining bounded in a H^s space and this guarantees the existence of a uniform choice of time step Δt that prevents the solution from any Burgers' step from blowing up. In [12], Burgers-type equations are studied. Here the local errors are obtained as quadrature errors via Lie commutators for Strang splitting. In [13, 14], they identify the local error of quadrature errors estimated via bounds of the Lie commutator for operator splitting for linear evolution and for nonlinear Scrödinger equations, respectively. Similar convergence analyses are studied for operator splitting methods for various equations such as BBM-type equations, Burgers–Huxley equation, Airy equation, viscous Burgers equation, KdV equation, and Fisher's equation in [6, 10, 16, 21].

In this paper, we employ Lie–Trotter splitting to Eq. (1) in time. Firstly, Eq. (1) is split into two subequations with an unbounded linear and a bounded nonlinear operator, respectively, i.e.

$$u_t = (1-\partial_x^2)^{-1}K(\partial_x)u \quad \text{and} \quad w_t = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(w^2).$$

Then with the operators

$$Au = (1-\partial_x^2)^{-1}K(\partial_x)u \quad \text{and} \quad B(w) = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(w^2),$$

the Lie–Trotter solution at time $t = n\Delta t$, as $\Delta t \to 0$, is $u_{n+1} = \Theta^{\Delta t}(u_n) = \Phi_A^{\Delta t} \circ \Phi_B^{\Delta t}(u_n)$, where $\Phi_A^{\Delta t}$ and $\Phi_B^{\Delta t}$ are the exact solution operators and $\Theta^{\Delta t}$ is Lie–Trotter splitting solution operator.

In the present paper, we provide an error analysis for Lie–Trotter splitting in time for Benjamin–Bona–Mahony-type equations. A similar approach to [10] is followed. They study error analysis for Strang splitting for BBM-type equations, but here the error bounds for the Lie–Trotter method for BBM-type equations are constructed, which are more effective and require less computational time. We assume that the initial data and solutions of Eq. (1) are bounded in the Sobolev spaces (H^s) for a fixed time T, i.e.

$$\|u_0\|_{H^{s+d-2}} \le \alpha, \quad \|u(t)\|_{H^{s+d-2}} \le \beta \tag{2}$$

for $0 \le t \le T$, $d \ge 2$, where α and β are any constants and s is any positive integer.

2. Regularity analysis

In this section, we start with the introduction of Lemma 2.1 and Lemma 2.2, which have smoothing effects on nonlinear terms of Eq. (1). The proof of Lemma 2.1 in L^2 norm is given in [18] and the proof of Lemma 2.2 in H^s Sobolev norm is given in [10]. During this study, Lemma 2.2 is used while constructing stability and local error bounds for Lie–Trotter splitting for BBM equations. In Lemma 2.3 the boundedness of nonlinear flow is proved in the local bases. Lemma 2.4 proves a sufficient continuity. Similar Lemmas can be seen in a thesis [16], but since the nonlinear term is Burgers' nonlinearity they present different proofs.

Lemma 2.1 *Let* $u, v \in L^2(R)$*. Then* $\|\partial_x(1-\partial_x^2)^{-1}(uv)\|_{L^2} \leq \|u\|_{L^2}\|v\|_{L^2}$*.*

Proof See ([18]). □

Lemma 2.2 *Let* $u, v \in H^s(R)$*. Then* $\|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{H^s} \leq K\|u\|_{H^s}\|v\|_{H^s}$*.*

Proof See ([10]). Expansion in H^s norm and using Lemma 2.1 for each expanded terms yield Lemma 2.2. □

Lemma 2.3 *If* $\|u_0\|_{H^s} \leq M$*, then there exists* $\bar{t}(M) > 0$ *such that* $\|\Phi_B(u_0)\|_{H^s} \leq 2M$ *for* $0 \leq t \leq \bar{t}(M)$*.*

Proof A similar proof to [11, 12, 16] is followed. Assume that $w(t) = \Phi_B^t(u_0)$, which satisfies the equality

$$\begin{aligned}
\|w\|_{H^s}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^s} =& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_B^t(u_0)\|_{H^s}^2 = (w, w_t)_{H^s} \\
=& \sum_{j=0}^{s}\int \partial_x^j w \partial_x^j\left(\frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(w^2)\right)\mathrm{d}x \\
=& \frac{1}{2}\sum_{j=0}^{s}\int \partial_x^j w(1-\partial_x^2)^{-1}\partial_x\left(\sum_{k=0}^{j}\binom{s}{k}\partial_x^k w \partial_x^{j-k} w\right)\mathrm{d}x.
\end{aligned}$$

Each of the terms can be bounded using Lemma 2.2 for each $j \leq s$ and yields

$$\left|\int \partial_x^j w(1-\partial_x^2)^{-1}\partial_x\left(\sum_{k=0}^{j}\binom{j}{k}\partial_x^k w \partial_x^{j-k} w\right)\mathrm{d}x\right| \leq C\|w\|_{H^s}^3,$$

where C is any constant. Moreover, we obtain

$$\frac{d}{\mathrm{d}t}\|w\|_{H^s} \leq C\|w\|_{H^s}^2,$$

whose result follows by comparing with the differential equation $y' = cy^2$. □

Lemma 2.4 *If* $\|u_0\|_{H^s} \leq M$ *then there exists* $\bar{t}$ *depending on* M *such that the solution of BBM-type equations (1) with initial data* u_0*,* $w(t) = \Phi_B^t(u_0)$ *satisfies*

$$w \in C^2([0, \bar{t}], H^s). \tag{3}$$

Proof Recall Lemma 2.3; if $\|u_0\|_{H^s} \leq M$ then $\|w(t)\|_{H^s} = \|\Phi_B(u_0)\|_{H^s} \leq 2M$ for $t \in [0, \bar{t}]$ and we can define

$$\tilde{w}(t) = u_0 + tB(u_0) + \int_0^t (t-s)dB\big(w(s)\big)[B\big(w(s)\big)]\mathrm{d}s,$$

where $dB(w)[B(w)] = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x\big(w(1-\partial_x^2)^{-1}\partial_x(w^2)\big)$. Since $\tilde{w}_{tt} = dB(w)[B(w)] = B(w)_t$, $\tilde{w}_t(0) = B(u_0) = w_t(0)$ and $\tilde{w}(0) = u_0 = w(0)$, we have $\tilde{w} = w$. Now we have to show that $\tilde{w} \in C^2([0,\tilde{t}], H^s)$. Start with

$$\begin{aligned}
\|\tilde{w}_{tt}\|_{H^s} = \|dB(w)[B(w)]\|_{H^s} =& \frac{1}{2}\|(1-\partial_x^2)^{-1}\partial_x\big(w(1-\partial_x^2)^{-1}\partial_x(w^2)\big)\|_{H^s} \\
\leq& \frac{K_1}{2}\|w\|_{H^s}\|(1-\partial_x^2)^{-1}\partial_x(w^2)\|_{H^s} \\
\leq& \frac{K_2}{2}\|w\|_{H^s}\|w\|_{H^s}\|w\|_{H^s}.
\end{aligned}$$

Hence $\|\tilde{w}_{tt}\|_{H^s} \leq \frac{K_2}{2}\|w\|_{H^s}^3$ and Lemma 2.3 completes the proof. □

3. Stability analysis

In this section, we present the stability of Lie–Trotter splitting method when applied to the BBM-type equations (1).

Lemma 3.1 *Let v, w be the Lie–Trotter splitting solutions of the BBM-type equations (1) with initial data $v_0, w_0 \in H^s$, respectively. Then*

$$\|v-w\|_{H^s} \leq \mathrm{e}^{L\Delta t}\|v_0-w_0\|_{H^s}, \tag{4}$$

where $L = \frac{K}{2}\max\{\|v\|_{H^s}, \|w\|_{H^s}\}$.

Proof Since the linear flow is preserved, we only concentrate on nonlinear flow. Let v, w be the nonlinear flows satisfying the initial value problems

$$\begin{aligned}
v_t =& \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(v^2), \quad v(0) = v_0 \\
w_t =& \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(w^2), \quad w(0) = w_0.
\end{aligned}$$

After subtraction and integrating from 0 to t, it yields

$$v - w = v_0 - w_0 + \frac{1}{2}\int_0^t (1-\partial_x^2)^{-1}\partial_x(v-w)(v+w)\mathrm{d}s, \quad (v-w)(0) = v_0 - w_0.$$

After taking H^s norm by using Lemma 2.2 and by applying Grönwall's lemma, it yields

$$\|v-w\|_{H^s} \leq \mathrm{e}^{L\Delta t}\|v_0-w_0\|_{H^s},$$

where $L = \max\{\|v\|_{H^s}, \|w\|_{H^s}\}$. □

4. Local error analysis

In this section, the local error bound for Lie–Trotter splitting for the BBM equation is constructed. Proof is similar to [10], but they use Strang splitting, which is a three-step method. It requires more calculation and more computational time. That is why in this paper we prefer Lie–Trotter splitting, which is a two-step method.

Theorem 4.1 *The local error of Lie–Trotter splitting applied to the BBM-type equations (1) is*

$$\|\Theta^{\Delta t}(u_0) - \Phi^{\Delta t}(u_0)\|_{H^s} \leq C\Delta t^2, \tag{5}$$

where C depends on α.

Proof The BBM-type equations (1) can be written as

$$u_t = Au + B(u),$$

where $Au = (1-\partial_x^2)^{-1}K(\partial_x)u$ and $B(u) = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(u^2)$.
The exact solution on $[0, \Delta t]$ is

$$u(\Delta t) = \mathrm{e}^{\Delta t A}u_0 + \int_0^{\Delta t} \mathrm{e}^{(\Delta t - s)A} B\big(u(s)\big)\mathrm{d}s. \tag{6}$$

This is similar to formula $\varphi(t) - \varphi(0) = \int_0^t \dot{\varphi}(s)\mathrm{d}s$ when $\varphi(s) = \mathrm{e}^{(t-s)A}u(s)$.
The second part of Eq. (6) can be written by taking $\varphi(\rho) = B(\mathrm{e}^{(s-\rho)A}u(\rho))$; then we get

$$B(\mathrm{e}^{(s-t)A}u(t)) - B(\mathrm{e}^{sA}u_0) = \int_0^s dB(\mathrm{e}^{(s-\rho)A}u(\rho))[\mathrm{e}^{(s-\rho)A}B(u(\rho))]\mathrm{d}\rho, \tag{7}$$

or

$$B(u(s)) = B(\mathrm{e}^{sA}u_0) + \int_0^s dB(\mathrm{e}^{(s-\rho)A}u(\rho))[\mathrm{e}^{(s-\rho)A}B(u(\rho))]\mathrm{d}\rho. \tag{8}$$

After inserting Eq. (8) into Eq. (6), we get

$$u(\Delta t) = \mathrm{e}^{\Delta t A}u_0 + \int_0^{\Delta t} \mathrm{e}^{(\Delta t - A)} B(\mathrm{e}^{sA}u_0)\mathrm{d}s + E_1,$$

where

$$E_1 = \int_0^{\Delta t}\int_0^s \mathrm{e}^{(\Delta t - s)A} dB(\mathrm{e}^{(s-\rho)A}u(\rho))[\mathrm{e}^{(s-\rho)A}B(u(\rho))]\mathrm{d}\rho\mathrm{d}s.$$

The Lie–Trotter splitting solution for $[0, \Delta t]$ interval can be written as

$$u_1 = \Theta^{\Delta t}(u_0) = \Phi_B^{\Delta t}(\mathrm{e}^{\Delta t A}u_0). \tag{9}$$

The first-order Taylor expansion yields

$$\Phi_B^{\Delta t}(v) = v + \Delta t B(v) + \Delta t^2 \int_0^1 (1-\theta) dB(\Phi_B^{\theta\Delta t}(v))[B(\Phi_B^{\theta\Delta t}(v))]d\theta,$$

where $v = \mathrm{e}^{\Delta t A}u_0 \in H^s$. Hence, Eq. (9) becomes

$$u_1 = \mathrm{e}^{\Delta t A}u_0 + \Delta t B(\mathrm{e}^{\Delta t A}u_0) + E_2$$

with

$$E_2 = \Delta t^2 \int_0^1 (1-\theta) dB(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta t A}u_0))[B(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta t A}u_0))]d\theta.$$

The local error is

$$u_1 - u(\Delta t) = \Delta t B(\mathrm{e}^{\Delta t A} u_0) - \int_0^{\Delta t} \mathrm{e}^{(\Delta t - A)} B(\mathrm{e}^{sA} u_0) \mathrm{d}s + (E_2 - E_1). \tag{10}$$

In order to represent the error bounds in H^s, we rearrange the differences of terms of Eq. (10). The difference of the first two terms can be written as a quadrature error in first-order Peano form, i.e.

$$\Delta t h(\Delta t) - \int_0^{\Delta t} h(s) \mathrm{d}s = \Delta t^2 \int_0^1 \kappa(\theta) h'(\theta \Delta t) d\theta, \tag{11}$$

where κ is a bounded kernel, $h(s) = \mathrm{e}^{(\Delta t - A)} B(\mathrm{e}^{sA} u_0)$ and $h'(s) = -\mathrm{e}^{(\Delta t - s)A}[A, B](\mathrm{e}^{sA} u_0)$. Here, the Lie commutator $[A, B]$ is

$$[A, B](v) = dA(v)[B(v)] - dB(v)[Av],$$

where $v = \mathrm{e}^{sA} u_0$. Each of the terms is equal to

$$\begin{aligned} dA(v)[B(v)] =& \frac{1}{2}(1 - \partial_x^2)^{-1} K(\partial_x)(1 - \partial_x^2)^{-1} \partial_x (v^2), \\ dB(v)[Av] =& (1 - \partial_x^2)^{-1} \partial_x (v(1 - \partial_x^2)^{-1} K(\partial_x) v), \end{aligned}$$

respectively. After taking H^s norm of each of the terms with the help of Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned} \|dA(v)[B(v)]\|_{H^s} =& \|\frac{1}{2}(1 - \partial_x^2)^{-1} K(\partial_x)(1 - \partial_x^2)^{-1} \partial_x (v^2)\|_{H^s} \\ \leq& \frac{1}{2}\|(1 - \partial_x^2)^{-1} \partial_x (v^2)\|_{H^{s+d-2}} \\ \leq& C_1 \|v\|_{H^{s+d-2}}^2, \end{aligned}$$

$$\begin{aligned} \|dB(v)[Av]\|_{H^s} =& \|(1 - \partial_x^2)^{-1} \partial_x (v(1 - \partial_x^2)^{-1} K(\partial_x) v)\|_{H^s} \\ \leq& C_2 \|v\|_{H^s} \|(1 - \partial_x^2)^{-1} K(\partial_x) v\|_{H^s} \\ \leq& C_2 \|v\|_{H^{s+d-2}}^2, \end{aligned}$$

where C_1 and C_2 are any constants. Since e^{tA} does not increase the Sobolev norms, it follows that

$$\|h'(s)\| \leq C \|u_0\|_{H^{s+d-2}}^2$$

Thus, the integral (11) is bounded as

$$\Delta t^2 \int_0^1 \kappa(\theta) h'(\theta \Delta t) d\theta \leq C \|u_0\|_{H^{s+2}}^2 \Delta t^2.$$

The third and fourth terms can be bounded as follows:

$$\|E_1\|_{H^s} \leq \int_0^{\Delta t} \int_0^s \|\mathrm{e}^{(\Delta t - s)A} dB(\mathrm{e}^{(s-\rho)A} u(\rho))[\mathrm{e}^{(s-\rho)A} B(u(\rho))]\|_{H^s} \mathrm{d}\rho \mathrm{d}s,$$

where

$$dB(\mathrm{e}^{(s-\rho)A}u(\rho))[\mathrm{e}^{(s-\rho)A}B(u(\rho))] = (1-\partial_x^2)^{-1}\partial_x(\mathrm{e}^{2(s-\rho)A}u(\rho)B(u(\rho))).$$

After rearranging, we get

$$\begin{aligned}
\|E_1\|_{H^s} \leq &\int_0^{\Delta t}\int_0^s \|\mathrm{e}^{(\Delta t-s)A}(1-\partial_x^2)^{-1}\partial_x(\mathrm{e}^{2(s-\rho)A}u(\rho)B(u(\rho)))\|_{H^s}\mathrm{d}\rho\mathrm{d}s\\
\leq &\int_0^{\Delta t}\int_0^s \|\mathrm{e}^{(\Delta t-s)A}(1-\partial_x^2)^{-1}\partial_x(\mathrm{e}^{2(s-\rho)A}u(\rho)\frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(u(\rho)^2))\|_{H^s}\mathrm{d}\rho\mathrm{d}s\\
\leq &\int_0^{\Delta t}\int_0^s \|(1-\partial_x^2)^{-1}\partial_x(\mathrm{e}^{2(s-\rho)A}u(\rho)(1-\partial_x^2)^{-1}\partial_x(u(\rho)^2))\|_{H^s}\mathrm{d}\rho\mathrm{d}s\\
\leq &C\int_0^{\Delta t}\int_0^s \|u(\rho)\|_{H^s}\|(1-\partial_x^2)^{-1}\partial_x(u(\rho)^2))\|_{H^s}\mathrm{d}\rho\mathrm{d}s\\
\leq &C\int_0^{\Delta t}\int_0^s \|u(\rho)\|_{H^s}^3\mathrm{d}\rho\mathrm{d}s\\
\leq &C\Delta t^2\beta^3.
\end{aligned}$$

For the last term, taking H^s norm of E_2 yields

$$\|E_2\|_{H^s} \leq \Delta t^2\int_0^1(1-\theta)\|dB(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0))[B(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0))]\|_{H^s}d\theta,$$

where

$$dB(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0))[B(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0))] = (1-\partial_x^2)^{-1}\partial_x(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)B(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0))).$$

After rearranging, we get

$$\begin{aligned}
\|E_2\|_{H^s} \leq &\Delta t^2\int_0^1(1-\theta)\|(1-\partial_x^2)^{-1}\partial_x(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)B(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)))\|_{H^s}d\theta\\
\leq &\Delta t^2\int_0^1\|(1-\partial_x^2)^{-1}\partial_x(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)B(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)))\|_{H^s}d\theta\\
\leq &C\Delta t^2\int_0^1\|\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)\|_{H^s}\|B(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0))\|_{H^s}d\theta\\
\leq &C\Delta t^2\int_0^1\|\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)\|_{H^s}\|(1-\partial_x^2)^{-1}\partial_x(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0))^2\|_{H^s}d\theta\\
\leq &C\Delta t^2\|\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)\|_{H^s}^3,
\end{aligned}$$

where C is any constant and $\|(\Phi_B^{\theta\Delta t}(\mathrm{e}^{\Delta tA}u_0)\|_{H^s}$ is a bounded nonlinear flow.

Hence the quadrature error is $\mathcal{O}(\Delta t^2)$ in the H^s norm for $u_0 \in H^{s+d-2}$. □

5. Global error analysis

Theorem 5.1 *The global error of Lie–Trotter splitting applied to the BBM-type equations (1) is*

$$\|\Theta^{\Delta t}(u_{n-1}) - \Phi^{\Delta t}(u_{n-1})\|_{H^s} \leq G\Delta t, \tag{12}$$

where G depends on α, β and T.

Proof The Lady Windermere's fan argument is used in the proof.
Here $u(t_n) = \Phi^{\Delta t}(u(t_{n-1}))$ is the exact solution at time t_n with initial data $u(t_{n-1})$, and $u_n = \Theta^{\Delta t}(u_{n-1})$ is the Lie–Trotter splitting solution, which can be written as

$$u_n = \Theta^{\Delta t}(u_{n-1}) = \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t}(u_{n-1}), \ \ n = 1, 2, 3, \cdots .$$

Subtraction of the exact solution from the Lie–Trotter solution yields

$$\Theta^{\Delta t}(u_{n-1}) - \Phi^{\Delta t}(u_{n-1}) = \sum_{k=0}^{n-1} \Phi^{(n-k-1)\Delta t}\Theta^{\Delta t}(u(t_k)) - \Phi^{(n-k-1)\Delta t}\Phi^{\Delta t}(u(t_k)). \tag{13}$$

After taking H^s norm of Eq. (13), using the local error bound given in Theorem 4.1, stability given in Lemma 3.1, and the boundedness assumptions given in Eq. (2) yields

$$\begin{aligned}
\|\Theta^{\Delta t}(u_{n-1}) - \Phi^{\Delta t}(u_{n-1})\|_{H^s} &\leq \sum_{k=0}^{n-1} \mathrm{e}^{L(n-k-1)\Delta t}\|\Theta^{\Delta t}(u(t_k)) - \Phi^{\Delta t}(u(t_k))\|_{H^s} \\
&\leq \sum_{k=0}^{n-1} \mathrm{e}^{LT} C\Delta t^2 \\
&\leq n\mathrm{e}^{LT} C\Delta t^2 \\
&\leq T\mathrm{e}^{LT} C\Delta t,
\end{aligned}$$

where $\mathrm{e}^{L(n-k-1)\Delta t} \leq \mathrm{e}^{LT}$ and $n\Delta t \leq T$. □

6. Numerical experiment

In this section, we focus on the numerical performance of Lie–Trotter splitting for BBM-type equations using MATLAB. Two examples are studied. We present error results in different norms and the convergence rates obtained by Lie–Trotter in tables and also CPU times are presented in seconds for various values of time step.

Example 6.1 *We consider a BBM–Burgers (BBMB) equation corresponding to the case $d = 2$ in Eq. (1), i.e.*

$$u_t - u_{xxt} - u_{xx} + u_x + uu_x = 0 \tag{14}$$

with the initial condition

$$u(x, 0) = \frac{1}{2} + \frac{1}{4}\sin(x). \tag{15}$$

and periodic boundary conditions in the space domain $[0, 2\pi]$. The Fourier transform is used in space discretization with $N = 256$ and the Lie–Trotter splitting method is used in time on $[0, T]$ interval. The exact solution is computed numerically by using classical explicit fourth-order Runge–Kutta methods relying on the method of integrating factors given in [20] for a sufficiently small time step.

In Table 1, we exhibit the L_1, L_2, and L_∞ errors of the Lie–Trotter splitting for the various time steps. Table 2 presents the convergence orders of Lie–Trotter splitting. It is confirmed that the expected convergence orders are obtained.

Figure 1(a) and Figure 1(b) present the reference and Lie–Trotter solutions of Eq. (14). Figure 2 presents the convergence orders taken with different time steps.

Table 1. Estimated errors using L_1, L_2, and L_∞ norms at $T = 1$.

Time steps	Lie–Trotter		
	L_1	L_2	L_∞
0.2000	0.0252	$1.7505e-003$	$1.5937e-004$
0.1250	0.0158	$1.0964e-003$	$9.992e-005$
0.0833	0.0105	$7.3188e-004$	$6.674e-005$
0.0556	$7.0324e-003$	$4.8835e-004$	$4.455e-005$
0.0370	$4.6911e-003$	$3.2576e-004$	$2.972e-005$
0.0244	$3.0905e-003$	$2.1461e-004$	$1.959e-005$
0.0161	$2.0443e-003$	$1.4196e-004$	$1.296e-005$
0.0108	$1.3631e-003$	$9.4660e-005$	$8.64e-006$

Table 2. Numerical convergence rates of Lie–Trotter splitting with different time steps at $T = 1$.

Time steps	Lie–Trotter		
	L_1	L_2	L_∞
0.2000			
0.1250	0.9954	0.9954	0.9933
0.0833	0.9968	0.9969	0.9955
0.0556	0.9978	0.9978	0.9969
0.0370	0.9985	0.9985	0.9979
0.0244	0.9990	0.9990	0.9986
0.0161	0.9993	0.9993	0.9991
0.0108	0.9996	0.9996	0.9994

Example 6.2 *The next test problem is for the case $d = 3$ in Eq. (1), i.e.*

$$u_t - u_{xxt} + u_{xxx} + u_x + uu_x = 0 \tag{16}$$

with the initial condition

$$u(x, 0) = \mathrm{e}^{-10\sin^2(\frac{x}{2})} \tag{17}$$

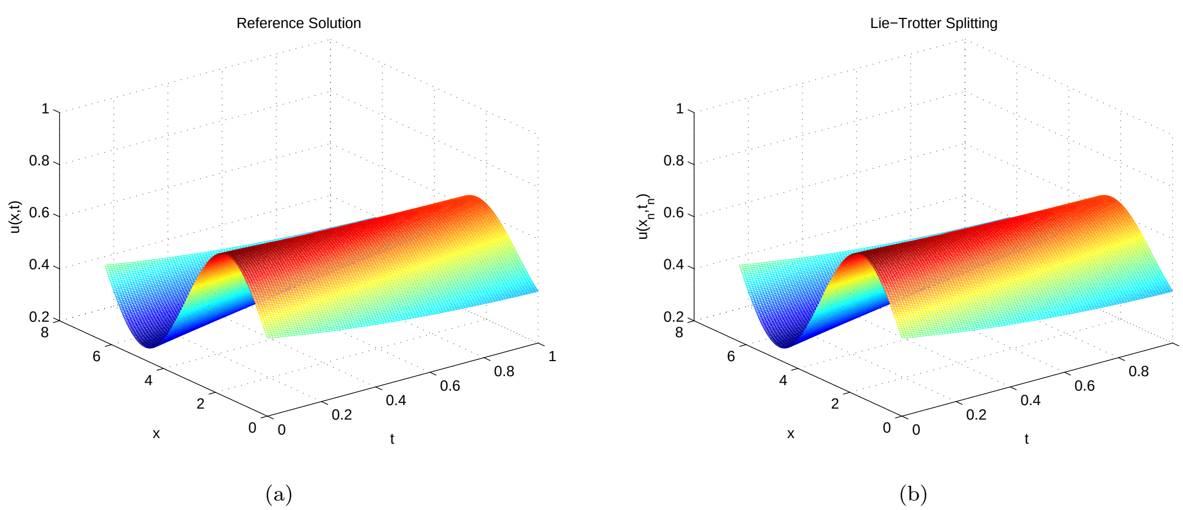


Figure 1. (a) Reference solutions generated by fourth-order Runge–Kutta method. (b) Lie–Trotter solutions.

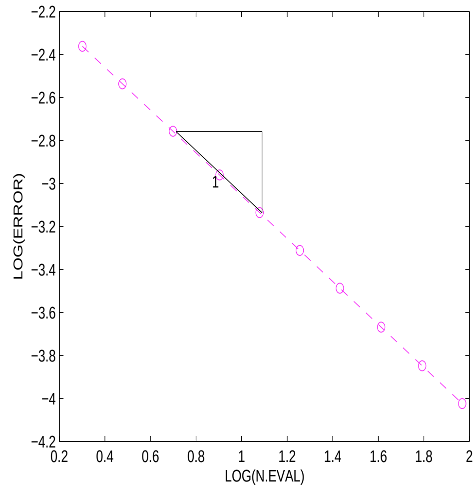


Figure 2. Convergence orders of the Lie–Trotter splitting solutions.

and periodic boundary conditions in the space domain $[0, 2\pi]$*. We use the same procedure as in the first example to obtain the reference and splitting solution.*

In Table 3, we show the L_1, L_2, and L_∞ errors of Lie–Trotter splitting for the various time steps. Table 4 presents the convergence orders of Lie–Trotter splitting. It is obtained that the numerical convergence rates for Δt followed the theoretical results.

Table 3. Estimated errors using L_1, L_2, and L_∞ norms at $T = 1$.

Time steps	Lie–Trotter		
	L_1	L_2	L_∞
0.2000	0.6252	0.0469	$6.9067e-003$
0.1250	0.3905	0.0292	$4.2693e-003$
0.0833	0.2603	0.0194	$2.8228e-003$
0.0556	0.1735	0.0129	$1.8782e-003$
0.0370	0.1157	$8.6115e-003$	$1.2487e-003$
0.0244	0.0752	$5.6679e-003$	$8.2078e-004$
0.0161	0.0504	$3.7468e-003$	$5.4212e-004$
0.0108	0.0149	$1.1057e-003$	$3.6112e-004$

Figure 3(a) and Figure 3(b) present the reference and Lie-Trotter solutions of Eq. (16). Figure 4 shows the convergence orders taken with different time steps.

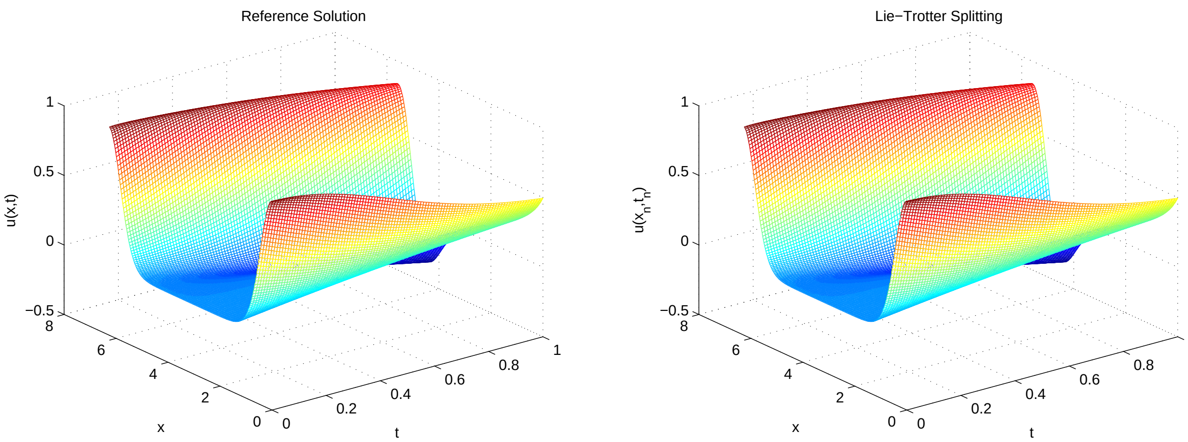
For the given numerical examples, CPU times of the method are illustrated for various values of time step in seconds in Table 5.

7. Conclusion

In this paper, the BBM equation was studied by using Lie–Trotter splitting. Theoretical results reveal that the method is stable and has a first-order convergence rate as expected. We confirm these theoretical results by considering two numerical test problems. In addition, Lie–Trotter splitting needs a shorter time of computation than Strang splitting does. This is because it has two subequations that need to be solved in each time step.

Table 4. Numerical convergence rates of Lie–Trotter splitting with different time steps at $T = 1$.

Time steps	Lie–Trotter		
	L_1	L_2	L_∞
0.2000			
0.1250	1.0010	1.0079	1.0235
0.0833	1.0002	1.0048	1.0151
0.0556	1.0001	1.0031	1.0101
0.0370	1.0000	1.0020	1.0067
0.0244	1.0000	1.0013	1.0045
0.0161	1.0000	1.0009	1.0029
0.0108	1.0000	1.0006	1.0020



(a) (b)

Figure 3. (a) Reference solutions generated by fourth-order Runge–Kutta method. (b) Lie–Trotter solutions.

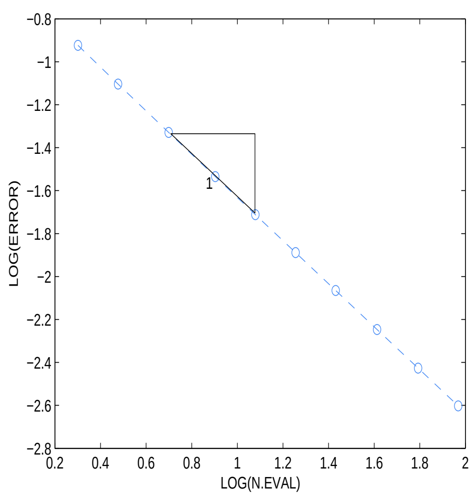


Figure 4. Convergence orders of the Lie–Trotter splitting solutions.

Table 5. CPU times in seconds of the Lie–Trotter splitting for various values of time step.

Time steps	CPU(s)	
	Example 6.1	Example 6.2
0.1	0.0156	0.0312
0.01	0.0312	0.0624
0.001	0.2028	0.2652
0.0001	2.0124	2.0592
0.00001	19.5001	19.7185
0.000001	196.7797	195.9060

This gives us a motivation to solve the BBM equation with Lie–Trotter splitting rather than Strang splitting. As a result, the Lie–Trotter method is an easier and more robust method to apply to variable nonlinear partial differential equations.

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