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Remarks on the zero Toeplitz product problem in the Bergman and Hardy spaces

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Abstract: In this article, we are interested in the zero Toeplitz product problem: for two symbols $f, g \in L^{\infty}(\mathbb{D}, dA)$, if the product $T_f T_g$ is identically zero on $L_a^2(\mathbb{D})$, then can we claim T_f or T_g is identically zero? We give a particular solution of this problem. A new proof of one particular case of the zero Toeplitz product problem in the Hardy space $H^2(\mathbb{D})$ is also given.

Key words: Toeplitz operator, Bergman space, Hardy space, zero Toeplitz product, Berezin symbol

1. Introduction

Let $dA(\lambda) = \frac{1}{\pi} dxdy$ denote the Lebesgue area measure on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space $L_a^2 := L_a^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. It is well known that $L_a^2(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$.

Let $P : L^2(\mathbb{D}, dA) \to L_a^2(\mathbb{D})$ be the Bergman orthogonal projector. P is an integral operator represented by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\lambda)}{\left(1 - \overline{\lambda} z\right)^2} dA(\lambda).$$

For $f \in L^{\infty}(\mathbb{D}, dA)$, the Toeplitz operator T_f with symbol f is the operator on $L_a^2(\mathbb{D})$ defined by $T_f h = P(fh)$ for $h \in L_a^2(\mathbb{D})$. It is not difficult to to see that T_f is the zero operator if and only if the symbol f is zero almost everywhere (see, for instance, [1, p. 203]).

The Hardy space $H^2 = H^2(\mathbb{D})$ is defined as the space of all analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which the norm

$$\|f\|_2 = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 \right)^{1/2}$$

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is finite. The reproducing kernel of H^2 is the function

$$k_{H^2,\lambda}(z) = \frac{1}{1-\overline{\lambda}z}.$$

For a function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in H^2$ we have $\|f\|_2 = \left(\sum\limits_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$.

For a function $\varphi \in L^\infty(\mathbb{T})$ the corresponding Toeplitz operator T_φ on H^2 is defined by

$$T_\varphi f = P_+ \varphi f,\ f \in H^2,$$

where $P_+ : L^2(\mathbb{T}) \to H^2$ is the Riesz projector.

In this article, we are interested in the zero Toeplitz product problem:
for two symbols $f, g \in L^\infty(\mathbb{D}, dA)$, if the product $T_f T_g$ is identically zero on $L_a^2(\mathbb{D})$, then can we claim T_f or T_g is identically zero?

This is a nontrivial problem and the answer is not known. Here we give a particular answer to this question. We also give a new proof for the zero Toeplitz product problem in the Hardy space H^2 in one particular case, which does not use a deep result of Brown and Halmos [7] for the product of two Toeplitz operators on H^2 (see also Stroethoff [16], Aleman and Vukotič [3], and Guediri [11]). In general, this article is motivated mainly by the following conjecture raised by Čučkovič in his paper [8]:

Conjecture 1 *Let $f, g \in L^\infty(\mathbb{D}, dA)$ with g harmonic. Then $T_f T_g = 0$ on $L_a^2(\mathbb{D})$ has only a trivial solution, i.e. $T_f = 0$.*

In [1], Ahern and Čučkovič solved the zero-product problem for two Toeplitz operators with harmonic symbols. Some particular results are also proved in [13] and [14].

Before giving our results, let us introduce some necessary definitions and notations.

For $\lambda \in \mathbb{D}$, the Bergman reproducing kernel is the function $k_{L_a^2,\lambda}(z) \in L_a^2$ such that

$$f(\lambda) = \left\langle f, k_{L_a^2,\lambda} \right\rangle$$

for every $f \in L_a^2$. The normalized Bergman reproducing kernel $\widehat{k}_{L_a^2,\lambda}$ is the function $\frac{k_{L_a^2,\lambda}}{\left\|k_{L_a^2,\lambda}\right\|_2}$. (It is well known that $\widehat{k}_{L_a^2,\lambda}(z) = \frac{1-|\lambda|^2}{(1-\overline{\lambda}z)^2}$.) Here, as elsewhere in this article, the norm $\|.\|_2$ and the inner product $\langle ., . \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. The set of bounded linear operators on L_a^2 is denoted by $\mathcal{B}(L_a^2)$.

For $T \in \mathcal{B}(L_a^2)$, the Berezin symbol (or the Berezin transform) of T is the complex-valued function $\widetilde{T}$ on $\mathbb{D}$ defined by (see [6])

$$\widetilde{T}(\lambda) := \left\langle T\widehat{k}_{L_a^2,\lambda}, \widehat{k}_{L_a^2,\lambda} \right\rangle,\quad \lambda \in \mathbb{D}.$$

Often the behavior of the Berezin transform of an operator provides important information about the operator.

The Berezin transform $\widetilde{f}$ of a function $f \in L^\infty(\mathbb{D}, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_f on L_a^2. In other words, $\widetilde{f} := \widetilde{T_f}$. Since $\widetilde{T_f}(\lambda) = \left\langle T_f \widehat{k}_{L_a^2,\lambda}, \widehat{k}_{L_a^2,\lambda} \right\rangle = \left\langle P\left(f\widehat{k}_{L_a^2,\lambda}\right), \widehat{k}_{L_a^2,\lambda} \right\rangle =$

$\left\langle f\widehat{k}_{L_a^2,\lambda},\widehat{k}_{L_a^2,\lambda}\right\rangle$, we obtain the formula

$$\widetilde{f}(\lambda)=\int_{\mathbb{D}}\left|\widehat{k}_{L_a^2,\lambda}(z)\right|^2 f(z)dA(z)=\int_{\mathbb{D}}\frac{\left(1-|\lambda|^2\right)^2}{\left|1-\overline{\lambda}z\right|^4}f(z)dA(z).$$

2. The results

Recall that every bounded harmonic function equals its Berezin transform. The converse also holds, so a function in $L^\infty(\mathbb{D},dA)$ equals its Berezin transform if and only if it is harmonic; for this deep result see Engliš [9] and Ahern et al. [2].

Now we can formulate and prove our results.

Theorem 1 *Let $f,g\in L^\infty(\mathbb{D},dA)$. The following are true:*

(a) If $\dim\overline{Range\,(T_g)}^{\perp}<+\infty$ and $T_fT_g=0$ on $L_a^2(\mathbb{D})$, then $T_f=0$.

(b) If g is harmonic with $g\left(e^{it}\right)\neq 0$ for almost all $t\in[0,2\pi)$, $Range\,(T_f)$ is closed, and $T_fT_g=0$, then $T_f=0$.

Proof (a) Since $T_fT_g=0$ on $L_a^2(\mathbb{D})$, we have that $\overline{Range\,(T_g)}\subset\ker(T_f)$. Hence, $\ker(T_f)^{\perp}\subset\overline{Range\,(T_g)}^{\perp}$, and thus $Range\left(T_f^*\right)\subset\overline{Range\,(T_g)}^{\perp}$; that is, $Range\left(T_{\overline{f}}\right)\subset\overline{Range\,(T_g)}^{\perp}$. By using the condition of the theorem, from this we assert that $T_{\overline{f}}$ is a finite rank Toeplitz operator on $L_a^2(\mathbb{D})$. However, by the well-known Luecking theorem [15], the only finite rank Toeplitz operator on the Bergman space $L_a^2(\mathbb{D})$ is the zero operator, which implies that $T_{\overline{f}}=0$, and hence $T_f=0$. This proves (a).

(b) Let $T_fT_g=0$. Then obviously $\widetilde{T_fT_g}=0$; that is, $\left\langle T_fT_g\widehat{k}_{L_a^2,\lambda},\widehat{k}_{L_a^2,\lambda}\right\rangle=0$ for all $\lambda\in\mathbb{D}$. Then, by considering that $\widetilde{g}=g$, we have

$$\begin{aligned}0&=\left\langle T_f\left(T_g\widehat{k}_{L_a^2,\lambda}-\widetilde{T_g}(\lambda)\widehat{k}_{L_a^2,\lambda}\right),\widehat{k}_{L_a^2,\lambda}\right\rangle+\left\langle T_f\left(\widetilde{T_g}(\lambda)\widehat{k}_{L_a^2,\lambda}\right),\widehat{k}_{L_a^2,\lambda}\right\rangle\\&=\left\langle T_g\widehat{k}_{L_a^2,\lambda}-g(\lambda)\widehat{k}_{L_a^2,\lambda},T_f^*\widehat{k}_{L_a^2,\lambda}\right\rangle+g(\lambda)\widetilde{T_f}(\lambda),\end{aligned}$$

and hence

$$\widetilde{f}(\lambda)\,g(\lambda)=\widetilde{T_f}(\lambda)g(\lambda)=-\left\langle T_{g-g(\lambda)}\widehat{k}_{L_a^2,\lambda},T_f^*\widehat{k}_{L_a^2,\lambda}\right\rangle$$

for all λ in $\mathbb{D}$. Since g is a bounded harmonic function, by using a result of Axler and Zheng [5, Corollary 3.7] and the Cauchy–Schwarz inequality, we obtain from the latter identity that

$$\left|\widetilde{f}(\lambda)\,g(\lambda)\right|\leq\|f\|_\infty\left\|T_{g-g(\lambda)}\widehat{k}_{L_a^2,\lambda}\right\|_2\to 0\ \text{ as }\ \lambda\to\partial\mathbb{D}$$

nontangentially at almost every point of $\partial\mathbb{D}$. This implies that $\overline{\lim}_{\substack{\lambda\to\partial\mathbb{D}\\ \text{nontangentially}}}\left|\widetilde{f}(\lambda)\,g(\lambda)\right|=0$, and hence $\lim_{\substack{\lambda\to\partial\mathbb{D}\\ \text{nontangentially}}}\left(\widetilde{f}(\lambda)\,g(\lambda)\right)=0$ for a.e. $t\in[0,2\pi)$. However, since by hypothesis g is a nonzero harmonic bounded function such that $g\left(e^{it}\right) \neq 0$ for a.e. $t \in [0, 2\pi)$, we deduce that the nontangential boundary value $\widetilde{f}(e^{it})$ of the function $\widetilde{f}$ exists for a.e. $t \in [0, 2\pi)$ and $\widetilde{f}(e^{it}) = 0$ for a.e. $t \in [0, 2\pi)$. From this, by applying the Axler–Zheng theorem (see [4, Theorem 2.2]), we assert that T_f is a compact operator on L_a^2. Therefore, since by hypothesis the range of T_f is closed, we have that T_f is a finite rank operator. Hence, by Luecking's theorem [15] we conclude that T_f must be zero, which proves (b). The theorem is proven. □

The arguments used in the proof of (b) allow us also to prove the following well-known theorem [7] by using a different method.

Theorem 2 *Let $f, g \in L^\infty(\mathbb{T})$ be two functions such that $f\left(e^{it}\right) \neq 0$ and $g\left(e^{it}\right) \neq 0$ almost everywhere on $\mathbb{T}$, and T_f, T_g be two associated Toeplitz operators on the Hardy space H^2. Then $T_f T_g \neq 0$.*

Proof Suppose that $T_f T_g = 0$. It is well known that (see Engliš [10]) $\widetilde{T_f} = \widetilde{f}$, $\widetilde{T_g} = \widetilde{g}$. By hypothesis,

$$f(\xi)\, g(\xi) \neq 0 \tag{1}$$

for almost all $\xi \in \mathbb{T}$. On the other hand, we have from $T_f T_g = 0$ that $\widetilde{T_f T_g} = 0$; that is,

$$\left\langle T_f T_g \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle = 0$$

for all $\lambda \in \mathbb{D}$. Then, as in the proof of (b) in Theorem 1, we have that

$$\widetilde{T}_f(\lambda)\widetilde{T}_g(\lambda) = -\left\langle T_g \widehat{k}_{H^2,\lambda} - \widetilde{g}(\lambda)\widehat{k}_{H^2,\lambda}, T_{\overline{f}} \widehat{k}_{H^2,\lambda} \right\rangle$$

and hence

$$\widetilde{f}(\lambda)\widetilde{g}(\lambda) = -\left\langle T_g \widehat{k}_{H^2,\lambda} - \widetilde{g}(\lambda)\widehat{k}_{H^2,\lambda}, T_{\overline{f}} \widehat{k}_{H^2,\lambda} \right\rangle$$

for all $\lambda \in \mathbb{D}$. From this, by using the known fact that (see Engliš [10, Theorem 6] and Karaev [12, Lemma 1.1]) $\left\| T_g \widehat{k}_{H^2,\lambda} - \widetilde{g}(\lambda)\widehat{k}_{H^2,\lambda} \right\| \to 0$ as $\lambda \to \mathbb{T}$ radially at almost every point of $\mathbb{T}$, we obtain that

$$\left| \widetilde{f}(\lambda)\widetilde{g}(\lambda) \right| \leq \|f\|_\infty \left\| T_g \widehat{k}_{H^2,\lambda} - \widetilde{g}(\lambda)\widehat{k}_{H^2,\lambda} \right\| \to 0 \tag{2}$$

as $\lambda \to \mathbb{T}$ radially at almost every point of $\mathbb{T}$. Since $\widetilde{f}$ and $\widetilde{g}$ are harmonic, by Fatou's theorem they have boundary values at almost every point of $\mathbb{T}$. It follows from the relation (2) by passing to the upper limit, as in the proof of Theorem 1, (b), that $\widetilde{f}(\lambda)\widetilde{g}(\lambda) \to 0$ as $\lambda \to \mathbb{T}$ radially, and hence $f(\xi)\, g(\xi) = 0$ for almost every $\xi \in \mathbb{T}$. This contradicts (1) and proves the theorem. □

We remark that, of course, Theorem 2 can also be stated under the condition that $f(\xi)\, g(\xi) \neq 0$ almost everywhere on $\mathbb{T}$.

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