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Existence of positive periodic solution of second-order neutral differential equations

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Abstract: In this work, we consider two types of second-order neutral differential equations and we obtain sufficient conditions for the existence of positive ω-periodic solutions for these equations. We employ Krasnoselskii's fixed point theorem for the sum of a completely continuous and a contraction mapping. An example is included to illustrate our results.

Key words: Neutral equations, fixed point, second-order, positive periodic solution

1. Introduction

In the present article, we investigate the existence of positive ω-periodic solutions of the following second-order neutral differential equations:

$$[x(t) - c(t)x(t-\tau)]'' = a(t)x(t) - f(t, x(t-\tau)) \tag{1}$$

and

$$[x(t) - c(t)x(t-\tau)]'' = -a(t)x(t) + f(t, x(t-\tau)), \tag{2}$$

where $c \in C(\mathbb{R}, \mathbb{R})$, $a \in C(\mathbb{R}, (0, \infty))$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\tau > 0$, and a, c are ω-periodic functions and f is ω-periodic in the first variable, $\omega > 0$. These equations or their variations appear in a number of fields such as biology, physics, and mechanics; see [7, 8, 11]. In order to show that we have positive ω-periodic solutions of (1) and (2), we transform (1) and (2) into equivalent integral equations and we apply Krasnoselskii's fixed point theorem.

Recently, the existence of positive periodic solutions of first- and second-order neutral differential equations have been investigated by many authors, see; [2–6, 9, 10, 12] and references therein. In [5], existence of positive periodic solutions of

$$[x(t) - cx(t-\tau(t))]'' = a(t)x(t) - f(t, x(t-\tau(t)))$$

and

$$[x(t) - cx(t-\tau(t))]'' = -a(t)x(t) + f(t, x(t-\tau(t))),$$

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where $\tau(t) \in C(\mathbb{R},\mathbb{R})$, $a \in C(\mathbb{R},(0,\infty))$, $f \in C(\mathbb{R}\times[0,\infty),[0,\infty))$, and a,τ are ω-periodic functions and f is ω-periodic in the first variable, were investigated with $0 \leqslant c < 1$ and $-1 < c < 0$. In the present paper, we consider $1 < c(t) < \infty$, $-\infty < c(t) < -1$, $0 \leqslant c(t) < 1$, and $-1 < c(t) \leqslant 0$ as four different ranges for variable coefficient $c(t)$, which makes the results in this current paper more general than that of [5].

The rest of this paper is organized as follows. In Section 2, we introduce some notations and we state and modify some lemmas from [5]. Section 3 contains our main results on existence of positive ω-periodic solutions of (1) and (2), respectively, and an example.

2. Preliminaries

Let $\Phi = \{x(t) : x(t) \in C(\mathbb{R},\mathbb{R}), \quad x(t) = x(t+\omega), \quad t \in \mathbb{R}\}$ with the sup norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. It is clear that Φ is a Banach space. Define

$$C_\omega^+ = \{x(t) : x(t) \in C(\mathbb{R},(0,\infty)), \quad x(t) = x(t+\omega)\},$$
$$C_\omega^- = \{x(t) : x(t) \in C(\mathbb{R},(-\infty,0)), \quad x(t) = x(t+\omega)\}.$$

Let

$$M = \max\{a(t) : t \in [0,\omega]\}, \quad m = \min\{a(t) : t \in [0,\omega]\} \quad \text{and} \quad \beta = \sqrt{M}.$$

Lemma 2.1 *([5]) The equation*

$$y''(t) - My(t) = h(t), \quad h \in C_\omega^-$$

has a unique ω-periodic solution

$$y(t) = \int_t^{t+\omega} G_1(t,s)(-h(s))ds,$$

where

$$G_1(t,s) = \frac{\exp(-\beta(s-t)) + \exp(\beta(s-t-\omega))}{2\beta(1-\exp(-\beta\omega))}, \quad s \in [t,t+\omega].$$

Lemma 2.2 *([5]) $G_1(t,s) > 0$ and $\int_t^{t+\omega} G_1(t,s)ds = \frac{1}{M}$ for all $t \in [0,\omega]$ and $s \in [t,t+\omega]$.*

Lemma 2.3 *The equation*

$$y''(t) - a(t+\tau)y(t) = h(t), \quad h \in C_\omega^- \tag{3}$$

has a unique positive ω-periodic solution

$$y(t) = (P_1h)(t) = (I - T_1B_1)^{-1}(T_1h)(t),$$

where $T_1, B_1 : \Phi \to \Phi$ defined such that

$$(T_1h)(t) = \int_t^{t+\omega} G_1(t,s)(-h(s))ds, \quad (B_1y)(t) = [-M + a(t+\tau)]y(t)$$

and P_1 is completely continuous and satisfies

$$0 < (T_1h)(t) \leqslant (P_1h)(t) \leqslant \frac{M}{m}\|T_1h\|, \quad h \in C_\omega^-.$$

Proof Rewrite (3) as

$$y''(t) - My = h(t) + [-M + a(t+\tau)]y(t).$$

Then, by Lemma 2.1,

$$y(t) = (T_1h)(t) + (T_1B_1y)(t) \tag{4}$$

is a solution of (3). It is obvious that (4) yields to

$$y(t) = (P_1h)(t) = (I - T_1B_1)^{-1}(T_1h)(t).$$

Moreover, it is clear that T_1, B_1 are completely continuous and $(T_1h)(t) > 0$. It can be shown that $\|T_1\| \leqslant \frac{1}{M}$ and $\|B_1\| \leqslant (M-m)$. Therefore,

$$\|T_1B_1\| \leqslant \|T_1\|\|B_1\| \leqslant \frac{(M-m)}{M} = 1 - \frac{m}{M} < 1. \tag{5}$$

By Neumann expansion of P_1, we have

$$\begin{aligned} P_1 &= (I - T_1B_1)^{-1}T_1 \\ &= (I + T_1B_1 + (T_1B_1)^2 + \ldots + (T_1B_1)^n + \ldots)T_1 \\ &= T_1 + T_1B_1T_1 + (T_1B_1)^2T_1 + \ldots + (T_1B_1)^nT_1 + \ldots. \end{aligned} \tag{6}$$

Since T_1 and B_1 are completely continuous, so is P_1. By using (5) and (6), we obtain

$$0 < (T_1h)(t) \leqslant (P_1h)(t) \leqslant \frac{M}{m}\|T_1h\|, \quad h \in C_\omega^-.$$

□

Lemma 2.4 *([5]) The equation*

$$y''(t) + My(t) = h(t), \quad h \in C_\omega^+$$

has a unique ω-periodic solution

$$y(t) = \int_t^{t+\omega} G_2(t,s)h(s)ds,$$

where

$$G_2(t,s) = \frac{\cos(\beta(\frac{\omega}{2} + t - s))}{2\beta\sin(\frac{\beta\omega}{2})}, \quad s \in [t, t+\omega].$$

Lemma 2.5 *([5])* $\int_t^{t+\omega} G_2(t,s)ds = \frac{1}{M}$. *Furthermore, if* $M < (\frac{\pi}{\omega})^2$, *then* $G_2(t,s) > 0$ *for all* $t \in [0,\omega]$ *and* $s \in [t, t+\omega]$.

Lemma 2.6 *Let* $M < (\frac{\pi}{\omega})^2$. *The equation*

$$y''(t) + a(t+\tau)y(t) = h(t), \quad h \in C_\omega^+$$

has a unique positive ω*-periodic solution*

$$y(t) = (P_2 h)(t) = (I - T_2 B_2)^{-1}(T_2 h)(t),$$

where $T_2, B_2 : \Phi \to \Phi$ *defined such that*

$$(T_2 h)(t) = \int_t^{t+\omega} G_2(t,s)h(s)ds, \quad (B_2 y)(t) = [M - a(t+\tau)]y(t)$$

and P_2 *is completely continuous and satisfies*

$$0 < (T_2 h)(t) \leqslant (P_2 h)(t) \leqslant \frac{M}{m}\|T_2 h\|, \quad h \in C_\omega^+.$$

Proof The proof is similar to that of Lemma 2.3, so it is omitted. □

Lemma 2.7 *(Krasnoselskii's fixed point theorem [1]). Let* X *be a Banach space, let* Ω *be a bounded closed and convex subset of* X, *and let* S_1, S_2 *be maps of* Ω *into* X *such that* $S_1 x + S_2 y \in \Omega$ *for every pair* $x, y \in \Omega$. *If* S_1 *is contractive and* S_2 *is completely continuous, then the equation*

$$S_1 x + S_2 x = x$$

has a solution in Ω.

3. Main results

Theorem 3.1 *Suppose that* $1 < c_0 \leqslant c(t) \leqslant c_1 < \infty$ *and there exist positive constants* M_1 *and* M_2 *with* $0 < M_1 < M_2$ *such that*

$$(c_1 - 1)MM_1 \leqslant c(t)a(t)x - f(t,x) \leqslant (c_0 - 1)mM_2, \quad \forall (t,x) \in [0,\omega] \times [M_1, M_2]. \tag{7}$$

Then (1) has at least one positive ω*-periodic solution* $x(t)$ *such that* $M_1 \leqslant x(t) \leqslant M_2$.

Proof Let $\Omega = \{x \in \Phi : M_1 \leqslant x(t) \leqslant M_2,\ t \in [0,\omega]\}$. One can see that Ω is a bounded, closed, and convex subset of Φ. We show that

$$x(t) = \frac{1}{c(t+\tau)}\left[P_1\left[-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t))\right] + x(t+\tau)\right] \tag{8}$$

is a solution of (1). The equation

$$[c(t+\tau)x(t) - x(t+\tau)]'' = -a(t+\tau)x(t+\tau) + f(t+\tau, x(t))$$

or

$$\begin{aligned}[c(t+\tau)x(t)-x(t+\tau)]'' &- a(t+\tau)\left[c(t+\tau)x(t)-x(t+\tau)\right] \\ &= -c(t+\tau)a(t+\tau)x(t)+f(t+\tau,x(t)) \end{aligned} \tag{9}$$

is equivalent to (1). Let $y(t)=c(t+\tau)x(t)-x(t+\tau)$ in the the equation (9), and then we have

$$y''(t)-a(t+\tau)y(t)=-c(t+\tau)a(t+\tau)x(t)+f(t+\tau,x(t)).$$

Then, by Lemma 2.3, we have

$$y(t)=P_1(-c(t+\tau)a(t+\tau)x(t)+f(t+\tau,x(t))),$$

which is the same as (8).

We define two mappings S_1 and S_2 on Ω as follows:

$$\begin{aligned}(S_1x)(t) &= \frac{1}{c(t+\tau)}P_1\left[-c(t+\tau)a(t+\tau)x(t)+f(t+\tau,x(t))\right] \quad \text{and} \\ (S_2x)(t) &= \frac{x(t+\tau)}{c(t+\tau)}.\end{aligned}$$

It is obvious that S_1x and S_2x are continuous and ω-periodic, i.e we have $S_1(\Omega)\subset\Phi$ and $S_2(\Omega)\subset\Phi$. For all $x_1,x_2\in\Omega$ and $t\in\mathbb{R}$, from (7), Lemma 2.2, and Lemma 2.3, we get

$$\begin{aligned}(S_1x_1)(t)+(S_2x_2)(t) &= \frac{1}{c(t+\tau)}\Big[P_1\left[-c(t+\tau)a(t+\tau)x_1(t)+f(t+\tau,x_1(t))\right]+x_2(t+\tau)\Big] \\ &\leqslant \frac{M}{mc_0}\|T_1(-c(t+\tau)a(t+\tau)x_1(t)+f(t+\tau,x_1(t)))\|+\frac{M_2}{c_0} \\ &= \frac{M}{mc_0}\sup_{t\in[0,\omega]}\left|\int_t^{t+\omega}G_1(t,s)(c(s+\tau)a(s+\tau)x_1(s)-f(s+\tau,x_1(s)))ds\right|+\frac{M_2}{c_0} \\ &\leqslant \frac{M}{mc_0}\int_t^{t+\omega}G_1(t,s)(c_0-1)mM_2ds+\frac{M_2}{c_0}=M_2\end{aligned}$$

and

$$\begin{aligned}(S_1x_1)(t)+(S_2x_2)(t) &= \frac{1}{c(t+\tau)}\Big[P_1\left[-c(t+\tau)a(t+\tau)x_1(t)+f(t+\tau,x_1(t))\right]+x_2(t+\tau)\Big] \\ &\geqslant \frac{1}{c_1}\Big[T_1\left[-c(t+\tau)a(t+\tau)x_1(t)+f(t+\tau,x_1(t))\right]\Big]+\frac{x_2(t+\tau)}{c_1} \\ &\geqslant \frac{1}{c_1}\int_t^{t+\omega}G_1(t,s)(c(s+\tau)a(s+\tau)x_1(s)-f(s+\tau,x_1(s)))ds+\frac{M_1}{c_1} \\ &\geqslant \frac{1}{c_1}\int_t^{t+\omega}G_1(t,s)(c_1-1)MM_1ds+\frac{M_1}{c_1}=M_1,\end{aligned}$$

from which we conclude that $M_1 \leqslant (S_1x_1)(t) + (S_2x_2)(t) \leqslant M_2$ for all $x_1, x_2 \in \Omega$ and $t \in \mathbb{R}$, i.e. we have $S_1x_1 + S_2x_2 \in \Omega$. For $x_1, x_2 \in \Omega$, we have

$$|(S_2x_1)(t) - (S_2x_2)(t))| = \left|\frac{x_1(t+\tau)}{c(t+\tau)} - \frac{x_2(t+\tau)}{c(t+\tau)}\right| \leqslant \frac{1}{c_0}\left|x_1(t+\tau) - x_2(t+\tau)\right| \leqslant \frac{1}{c_0}\|x_1 - x_2\|,$$

which implies that

$$\|S_2x_1 - S_2x_2\| \leqslant \frac{1}{c_0}\|x_1 - x_2\|.$$

Since $0 < \frac{1}{c_0} < 1$, S_2 is a contraction mapping on Ω.

From Lemma 2.3, we know that P_1 is completely continuous, and so is S_1. By Lemma 2.7, there is an $x \in \Omega$ such that $S_1x + S_2x = x$. It is easy to see that $x(t)$ is a positive ω-periodic solution of (1). This completes the proof. □

Theorem 3.2 *Suppose that* $-\infty < c_0 \leqslant c(t) \leqslant c_1 < -1$, $\frac{c_0}{c_1}M < -c_1m$, *and there exist positive constants* M_1 *and* M_2 *with* $0 < M_1 < M_2$ *such that*

$$(-c_0M_1 + \frac{c_0}{c_1}M_2)M \leqslant f(t,x) - c(t)a(t)x \leqslant -c_1mM_2, \quad \forall (t,x) \in [0,\omega] \times [M_1, M_2].$$

Then (1) has at least one positive ω*-periodic solution* $x(t)$ *such that* $M_1 \leqslant x(t) \leqslant M_2$.

Proof Let Ω be as in the proof of Theorem 3.1. We define $S_1, S_2 : \Omega \to \Phi$ as follows:

$$\begin{aligned}(S_1x)(t) &= \frac{-1}{c(t+\tau)}P_1\left[c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))\right] \quad \text{and} \\ (S_2x)(t) &= \frac{x(t+\tau)}{c(t+\tau)}.\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1, so it is omitted. □

Theorem 3.3 *Suppose that* $0 \leqslant c(t) \leqslant c_1 < 1$ *and there exist positive constants* M_1 *and* M_2 *with* $0 < M_1 < M_2$ *such that*

$$M_1M \leqslant f(t,x) - c(t)a(t)x \leqslant (1-c_1)M_2m, \quad \forall (t,x) \in [0,\omega] \times [M_1, M_2].$$

Then (1) has at least one positive ω*-periodic solution* $x(t)$ *such that* $M_1 \leqslant x(t) \leqslant M_2$.

Proof Let Ω be as in the proof of Theorem 3.1. We define the mappings $S_1, S_2 : \Omega \to \Phi$ as follows:

$$\begin{aligned}(S_1x)(t) &= P_1\left[c(t)a(t)x(t-\tau) - f(t, x(t-\tau))\right] \quad \text{and} \\ (S_2x)(t) &= c(t)x(t-\tau).\end{aligned}$$

The remaining part of the proof is similar to that of Theorem 3.1, so it is omitted. □

Theorem 3.4 *Suppose that* $-1 < c_0 \leqslant c(t) < 0$*,* $-c_0 M < m$ *and there exist positive constants* M_1 *and* M_2 *with* $0 < M_1 < M_2$ *such that*

$$(M_1 - c_0 M_2)M \leqslant f(t,x) - c(t)a(t)x \leqslant mM_2, \quad \forall (t,x) \in [0,\omega] \times [M_1, M_2].$$

Then (1) has at least one positive ω*-periodic solution* $x(t)$ *such that* $M_1 \leqslant x(t) \leqslant M_2$*.*

Proof The proof is similar to that of Theorem 3.3, so it is omitted. □

Theorem 3.5 *Let* $M < (\frac{\pi}{\omega})^2$*. Suppose that* $1 < c_0 \leqslant c(t) \leqslant c_1 < \infty$ *and there exist positive constants* M_1 *and* M_2 *with* $0 < M_1 < M_2$ *such that*

$$(c_1 - 1)MM_1 \leqslant c(t)a(t)x - f(t,x) \leqslant (c_0 - 1)mM_2, \quad \forall (t,x) \in [0,\omega] \times [M_1, M_2]. \tag{10}$$

Then (2) has at least one positive ω*-periodic solution* $x(t)$ *such that* $M_1 \leqslant x(t) \leqslant M_2$*.*

Proof Let $\Omega = \{x \in \Phi : M_1 \leqslant x(t) \leqslant M_2,\ t \in [0,\omega]\}$. One can see that Ω is a bounded, closed, and convex subset of Φ. We show that

$$x(t) = \frac{1}{c(t+\tau)}\Bigg[P_2\left[c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))\right] + x(t+\tau)\Bigg] \tag{11}$$

is a solution of (2). The equation

$$[c(t+\tau)x(t) - x(t+\tau)]'' = a(t+\tau)x(t+\tau) - f(t+\tau, x(t))$$

or

$$\begin{aligned}[c(t+\tau)x(t) - x(t+\tau)]'' &+ a(t+\tau)\left[c(t+\tau)x(t) - x(t+\tau)\right] \\ &= c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))\end{aligned} \tag{12}$$

is equivalent to (2). Let $y(t) = c(t+\tau)x(t) - x(t+\tau)$ in equation (12), and then we have

$$y''(t) + a(t+\tau)y(t) = c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t)).$$

Then, by Lemma 2.6, we have

$$y(t) = P_2(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))),$$

which is the same as (11).

We define two mappings S_1 and S_2 on Ω as follows:

$$\begin{aligned}(S_1x)(t) &= \frac{1}{c(t+\tau)}P_2\left[c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))\right] \quad \text{and} \\ (S_2x)(t) &= \frac{x(t+\tau)}{c(t+\tau)}.\end{aligned}$$

It is clear that S_1x, S_2x are continuous and ω-periodic, i.e we have $S_1(\Omega) \subset \Phi$ and $S_2(\Omega) \subset \Phi$. For all $x_1, x_2 \in \Omega$ and $t \in \mathbb{R}$, from (10), Lemma 2.5, and Lemma 2.6, we get

$$\begin{aligned}
(S_1x_1)(t) + (S_2x_2)(t) &= \frac{1}{c(t+\tau)}\Bigg[P_2\left[c(t+\tau)a(t+\tau)x_1(t) - f(t+\tau, x_1(t))\right] + x_2(t+\tau)\Bigg] \\
&\leqslant \frac{M}{mc_0}\|T_2(c(t+\tau)a(t+\tau)x_1(t) - f(t+\tau, x_1(t)))\| + \frac{M_2}{c_0} \\
&= \frac{M}{mc_0}\sup_{t\in[0,\omega]}\left|\int_t^{t+\omega} G_2(t,s)(c(s+\tau)a(s+\tau)x_1(s) - f(s+\tau, x_1(s)))ds\right| + \frac{M_2}{c_0} \\
&\leqslant \frac{M}{mc_0}\int_t^{t+\omega} G_2(t,s)(c_0-1)mM_2 ds + \frac{M_2}{c_0} = M_2
\end{aligned}$$

and

$$\begin{aligned}
(S_1x_1)(t) + (S_2x_2)(t) &= \frac{1}{c(t+\tau)}\Bigg[P_2\left[c(t+\tau)a(t+\tau)x_1(t) - f(t+\tau, x_1(t))\right] + x_2(t+\tau)\Bigg] \\
&\geqslant \frac{1}{c_1}\Bigg[T_2\left[c(t+\tau)a(t+\tau)x_1(t) - f(t+\tau, x_1(t))\right]\Bigg] + \frac{x_2(t+\tau)}{c_1} \\
&\geqslant \frac{1}{c_1}\int_t^{t+\omega} G_2(t,s)(c(s+\tau)a(s+\tau)x_1(s) - f(s+\tau, x_1(s)))ds + \frac{M_1}{c_1} \\
&\geqslant \frac{1}{c_1}\int_t^{t+\omega} G_2(t,s)(c_1-1)MM_1 ds + \frac{M_1}{c_1} = M_1,
\end{aligned}$$

from which we conclude that $M_1 \leqslant (S_1x_1)(t) + (S_2x_2)(t) \leqslant M_2$ for all $x_1, x_2 \in \Omega$ and $t \in \mathbb{R}$, i.e. we have $S_1x_1 + S_2x_2 \in \Omega$. For $x_1, x_2 \in \Omega$, we obtain

$$|(S_2x_1)(t) - (S_2x_2)(t))| = \left|\frac{x_1(t+\tau)}{c(t+\tau)} - \frac{x_2(t+\tau)}{c(t+\tau)}\right| \leqslant \frac{1}{c_0}\Big|x_1(t+\tau) - x_2(t+\tau)\Big| \leqslant \frac{1}{c_0}\|x_1 - x_2\|,$$

which implies that

$$\|S_2x_1 - S_2x_2\| \leqslant \frac{1}{c_0}\|x_1 - x_2\|.$$

Since $0 < \frac{1}{c_0} < 1$, S_2 is a contraction mapping on Ω.

From Lemma 2.6, we know that P_2 is completely continuous, and so is S_1. By Lemma 2.7, there is an $x \in \Omega$ such that $S_1x + S_2x = x$. It is easy to see that $x(t)$ is a positive ω-periodic solution of (2). This completes the proof. □

The proofs of the next three theorems are similar to that of Theorem 3.2–Theorem 3.4, respectively, so they are omitted.

Theorem 3.6 *Let $M < (\frac{\pi}{\omega})^2$. Suppose that $-\infty < c_0 \leqslant c(t) \leqslant c_1 < -1$, $\frac{c_0}{c_1}M < -c_1 m$, and there exist positive constants M_1 and M_2 with $0 < M_1 < M_2$ such that*

$$(-c_0M_1 + \frac{c_0}{c_1}M_2)M \leqslant f(t,x) - c(t)a(t)x \leqslant -c_1 mM_2, \quad \forall (t,x) \in [0,\omega] \times [M_1, M_2].$$

Then (2) has at least one positive ω-periodic solution $x(t)$ such that $M_1 \leqslant x(t) \leqslant M_2$.

Theorem 3.7 *Let $M < (\frac{\pi}{\omega})^2$. Suppose that $0 \leqslant c(t) \leqslant c_1 < 1$ and there exist positive constants M_1 and M_2 with $0 < M_1 < M_2$ such that*

$$M_1 M \leqslant f(t,x) - c(t)a(t)x \leqslant (1-c_1)M_2 m, \quad \forall (t,x) \in [0,\omega] \times [M_1, M_2].$$

Then (2) has at least one positive ω-periodic solution $x(t)$ such that $M_1 \leqslant x(t) \leqslant M_2$.

Theorem 3.8 *Let $M < (\frac{\pi}{\omega})^2$. Suppose that $-1 < c_0 \leqslant c(t) < 0$, $-c_0 M < m$ and there exist positive constants M_1 and M_2 with $0 < M_1 < M_2$ such that*

$$(M_1 - c_0 M_2)M \leqslant f(t,x) - c(t)a(t)x \leqslant m M_2, \quad \forall (t,x) \in [0,\omega] \times [M_1, M_2].$$

Then (2) has at least one positive ω-periodic solution $x(t)$ such that $M_1 \leqslant x(t) \leqslant M_2$.

Example 3.1 *Consider the first-order neutral differential equation*

$$\begin{aligned} [x(t) - 3\exp(\cos t/10)x(t-6\pi)]'' &= (3+\sin t)x(t) - 3\exp(\cos t/10)(3+\sin t)x(t-6\pi) \\ &+ 5 - \exp(\sin t) - \sin(x^3(t-6\pi)). \end{aligned} \tag{13}$$

Note that (13) of the form (1) with $\omega = 2\pi$, $c(t) = 3\exp(\cos t/10)$, $a(t) = 3+\sin t$, $f(t,x) = 3\exp(\cos t/10)(3+\sin t)x - 5 + \exp(\sin t) + \sin x^3$, and $\tau = 6\pi$. It is easy to verify that the conditions of Theorem 3.1 are satisfied with $M_1 = 0.1$, $M_2 = 2$. Thus, (13) has at least one positive ω-periodic solution.

Remark 3.1 *Since $c(t) = 3\exp(\cos t/10)$ in Example 3.1 is not constant and $c(t) > 1$, we can not apply the results in [5].*

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