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Generalized geometry of Goncharov and configuration complexes

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Abstract: In this article, a generalized geometry of Goncharov's complex and the Grassmannian complex will be proposed. First, all new homomorphisms will be defined, and then they will be used extensively to connect the Bloch–Suslin and the Grassmannian complex for weight $n = 2$ and then Goncharov's complex with Grassmannian complex for weight $n = 3$, up to $n = 6$. Lastly, and most importantly, generalized morphisms will be presented to cover the geometry of the Goncharov and Grassmannian complex when weight $n = N$. Associated diagrams will be exhibited, proven to be commutative.

Key words: Classical polylogarithms, Grassmannian, chain complexes, generalization

1. Introduction

A free abelian group, generated by all possible projective configurations of m points in an n-dimensional vector space, is denoted by $G_m(n)$. Suslin [13] was the first mathematician to introduce a chain complex of these free abelian groups, which was named the Grassmannian complex. For this chain complex, Suslin used two types of differential morphisms, one called the differential d and the other called projection map p. The associated diagram is proven to be bicomplex and each square commutative.

Leibniz introduced classical polylogarithms function $Li_p(Z)$ by an infinite series, which was absolutely convergent in a unit disk. The dilogarithm series $Li_2(Z)$ was studied by Spence, Abel, Kummer, Lobachesky, Hill, Roger, and Ramanujan, but the most important of all was Abel's functional equation, known as Abel's five-term relation [6].

A chain complex of dilogarithm group $\mathcal{B}_2(F)$ generated by a five-term relation of the cross ratio of four points is called the Bloch–Suslin complex, which is also an exact sequence [4]. Later, Goncharov defined the trilogarithmic group $\mathcal{B}_3(F)$, generated by a seven-term relation of the triple cross ratio of six points. A momentous volume of his work addresses the generalization of the triple ratio. Goncharov also generalized the Bloch–Suslin complex for weight n [4], giving the resulting complex his own name. The author then connected the Bloch–Suslin complex with the Grassmannian complex for weight $n = 2$ and proved that the associated diagram is bicomplex and commutative. As a result, Goncharov found morphisms between the Grassmannian complex and Goncharov's complex for weight $n = 3$ [4, 6], proving the associated diagram to be bicomplex and each square to be commutative.

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Cathelineau [1, 2] introduced an infinitesimal analogy of Goncharov's complex. Naming this Cathelineau's complex, it introduced the F-vector space as $\beta_2(F)$, which was generated by a four-term relation, and $\beta_3(F)$, which was generated by a 22-term relation. Ultimately, Cathelineau generalized this group as $\beta_n(F)$. Siddiqui [11] used the Kähler differential to define a variant of the Cathelineau complex. For this new complex, the author introduced groups that had the same functional equations as Bloch groups, such as $\beta_2^D(F)$, which was generated by a five-term relation, and $\beta_3^D(F)$, which was generated by a seven-term relation. Siddiqui connected a variant of Cathelineau's complex with the Grassmannian complex for weight $n = 2$ and $n = 3$. The associated diagrams were bicomplex and commutative.

Khalid et al. [8, 9] defined new morphisms between the Grassmannian complex and a variant of the Cathelineau complex up to weight $n = N$. They furthermore defined second- and third-order homomorphisms in the Grassmannian complex [10] and generalized them as nth-order morphisms [7].

In this paper, Section two introduces the background of the Grassmannian complex, polylogarithmic groups, the Bloch–Suslin complex, and Goncharov's complex in detail. The third section presents the concept of the geometry of the Grassmannian chain complex in relation with both Bloch–Suslin and Goncharov complexes for weight $n = 2$ and 3. The last section defines new homomorphisms between the Grassmannian and the Bloch–Suslin complex for weight $n = 2$ and shows that the associated diagram is commutative as well as bicomplex. Moreover, the subcomplexes of Goncharov's complex and the Grassmannian complex are connected for weight $n = 3$ up to $n = 6$. Finally, the morphisms are generalized for the generalization of the geometry of Goncharov's complex and the Grassmannian complex, for weight $n = N$, alongside proving that the associated diagram is commutative.

2. Preliminary and background

This section discusses the groundwork that is relevant to the research being conducted. The details of the Grassmannian chain complex, cross ratio, Siegel cross ratio property, polylogarithmic functions, scissor congruence groups, Bloch–Suslin complex, and Goncharov's complex are presented here, all of which are crucial as preliminary background for this paper.

2.1. Grassmannian complex

Following is the Grassmannian chain complex:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_{n+4}(n+2) & \xrightarrow{d} & G_{n+3}(n+2) & \xrightarrow{d} & G_{n+2}(n+2) \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_{n+3}(n+1) & \xrightarrow{d} & G_{n+2}(n+1) & \xrightarrow{d} & G_{n+1}(n+1) \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_{n+2}(n) & \xrightarrow{d} & G_{n+1}(n) & \xrightarrow{d} & G_n(n)
\end{array}
\qquad \text{(A)}
$$

d is a differential morphism, defined as

$$d:(v_0,\dots,v_n)=\sum_{i=0}^{n}(-1)^i(v_0,\dots,\hat{v}_i,\dots,v_n), \tag{1}$$

and projection morphism p is defined as

$$p:(v_0,\dots,v_n)=\sum_{i=0}^{n}(-1)^i(v_i|v_0,\dots,\hat{v}_i,\dots,v_n). \tag{2}$$

Lemma 2.1 $d\circ d=p\circ p=0$ (see [13]).

Lemma 2.2 $d\circ p=p\circ d$ (see [13]).

2.2. Siegel cross ratio property

Let us define the cross ratio of 4 points as $r(v_0,v_1,v_2,v_3)=\frac{\triangle(v_0,v_3)\triangle(v_1,v_2)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}$. Siegel [12] defined the following most important property of this ratio:

$$1=\frac{\triangle(v_0,v_3)\triangle(v_1,v_2)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}+\frac{\triangle(v_0,v_1)\triangle(v_2,v_3)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}, \tag{3}$$

or

$$\frac{\triangle(v_0,v_2)\triangle(v_1,v_3)-\triangle(v_0,v_3)\triangle(v_1,v_2)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}=\frac{\triangle(v_0,v_1)\triangle(v_2,v_3)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}. \tag{4}$$

2.3. Classical polylogarithmic functions and groups

The classical p-logarithm is an absolutely convergent series defined as $Li_p(z)=\sum_{n=0}^{\infty}\frac{z^n}{n^p}$, in the unit disk $z\leq 1$, introduced a decade ago. For $p=1$, $Li_1(z)=-Li(1-z)$ with formula $\log x+\log y=\log xy$. In this article, field will be denoted by F and $F^{\bullet\bullet}=F-\{0,1\}$, and $Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ is a free abelian group generated by [x].

Definition 2.1 *$\mathcal{B}(F)$ is a scissor congruence group [3, 5, 11]. It is a quotient of $Z[\mathbf{P}^1_F]$ and its subgroup is generated by Abel's five-term relation $[x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-y^{-1}}{1-x^{-1}}\right]+\left[\frac{1-y}{1-x}\right]$ where $x\neq y$ and $x,y\neq 0,1$*

2.4. Bloch–Suslin and Goncharov complexes

2.4.1. Weight-1

$\mathsf{R}_1(F)\subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ is generated by $\{xy\}-\{x\}-\{y\}$ $(x,y\in F^{\times})$, $\mathcal{B}_1(F)=Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\mathsf{R}_1(F)$ and a map $\delta_1:\mathcal{B}_1(F)\to F^{\times}$ is defined as $\delta_1:[x]\to x$. This map is called isomorphism. Therefore, $\mathcal{B}_1(F)\cong F^{\times}$ [4].

2.4.2. Weight-2

$\mathsf{R}_2(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ is generated by the five-term relation of cross ratio $\sum_{i=0}^{4}(-1)^i r(v_0,...,\hat{v_i},\ ...,v_4)$, and we define a map $\delta_2 : Z[\mathbf{P}^1_F/\{0,1,\infty\}] \to \wedge^2 F^\times$, where $\delta_2 : [x] \to (1-x)\wedge x$. Now define $\mathcal{B}_2(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\mathsf{R}_2(F)$ and connect $\mathcal{B}_2(F)$ with $\wedge^2 F^\times$ [4].

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times ,$$

where δ is an induced map defined as $\delta : [x]_2 \to (1-x)\wedge x$. The functional equations of the group $\mathcal{B}_2(F)$ are (i) the two-term relation $[x]_2 + [1-x]_2 = 0$, (ii) the inversion relation $[x]_2 + \left[\frac{1}{x}\right]_2 = 0$, and (iii) the five-term relation $[r(v_1,v_2,v_3,v_4)]_2 - [r(v_0,v_2,v_3,v_4)]_2 + [r(v_0,v_1,v_3,v_4)]_2 - [r(v_0,v_1,v_2,v_4)]_2 + [r(v_0,v_1,v_2,v_3)]_2 = 0$. The following chain is called the Bloch–Suslin complex, which is also an exact sequence [4]:

$$0 \xrightarrow{\delta} \mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times \xrightarrow{\delta} 0 .$$

2.4.3. Weight-3

Goncharov [4] introduced $\mathsf{R}_3(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ generated by the seven-term relation of the triple ratio, such that

$$\mathsf{R}_3(F) = \sum_{i=0}^{6}(-1)^i Alt_6\Big[\frac{(v_0,v_1,v_3)(v_1,v_2,v_4)(v_0,v_2,v_5)}{(v_0,v_1,v_4)(v_1,v_2,v_5)(v_0,v_2,v_3)}\Big], \tag{5}$$

and defined $\mathcal{B}_3(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\mathsf{R}_3(F)$ for weight 3. The chain complex of this group is given by

$$\mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F)\otimes F^\times \xrightarrow{\delta} \wedge^3 F^\times ,$$

where $\delta : [x]_3 \to [x]_2 \otimes x$ and the functional equation of group $\mathcal{B}_3(F)$ is the seven-term relation of the triple cross ratio of six points.

2.4.4. Weight-n

Goncharov [4] generalized the Bloch group, defined as $\mathcal{B}_n(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\mathsf{R}_n(F)$, where $\mathsf{R}_n(F) \subset Z[\mathbf{P}^1_F]$ is the kernel of the map $\delta_n : Z[\mathbf{P}^1_F/\{0,1,\infty\}] \to \mathcal{B}_{n-1}(F)\otimes F^\times$. Then Goncharov generalized the following chain complex, called Goncharov's complex:

$$\mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F)\otimes F^\times \xrightarrow{\delta} \mathcal{B}_{n-2}(F)\otimes \wedge^2(F) \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathcal{B}_2(F)\otimes \wedge^{n-2}(F) \xrightarrow{\delta} \frac{\wedge^n(F^\times)}{2-torsion}. \tag{6}$$

3. Geometry of Grassmannian and polylogarithmic complexes

For this geometry, Goncharov [4] used morphisms to connect the Grassmannian subcomplex with the Bloch–Suslin complex for weight $n=2$ and with his own complex for weight $n=3$.

3.1. Grassmannian and Bloch–Suslin complexes in weight-2

For weight $n=2$, Goncharov [4] defined the following geometry:

$$\begin{array}{ccccc} G_6(3) & \xrightarrow{d} & G_5(3) & \xrightarrow{d} & G_4(3) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_5(2) & \xrightarrow{d} & G_4(2) & \xrightarrow{d} & G_3(2) \\ & & \downarrow f_1^2 & & \downarrow f_0^2 \\ & & \mathcal{B}_2(F) & \xrightarrow{\delta} & \wedge^2 F^\times \end{array} \tag{B}$$

$$f_0^2 : (v_0, v_1, v_2) = -\triangle(v_1, v_2) \wedge \triangle(v_0, v_2) + \triangle(v_0, v_2) \wedge \triangle(v_1, v_2) - \triangle(v_0, v_1) \wedge \triangle(v_1, v_2)$$

and

$$f_1^2(v_0, v_1, v_2, v_3) = [r(v_0, ..., v_3)]_2. \tag{7}$$

Lemma 3.1 $f_0^2 \circ d = \delta \circ f_1^2$ (see [4]).

Lemma 3.2 $f_0^2 \circ p = f_1^2 \circ p = 0$ (see [4]).

3.2. Geometry of Grassmannian and Goncharov complexes in weight-3

As defined in [4, 6], Goncharov constructed the following diagram:

$$\begin{array}{ccccc} G_7(4) & \xrightarrow{d} & G_6(4) & \xrightarrow{d} & G_5(4) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_6(3) & \xrightarrow{d} & G_5(3) & \xrightarrow{d} & G_4(3) \\ \downarrow f_2^3 & & \downarrow f_1^3 & & \downarrow f_0^3 \\ \mathcal{B}_3(F) & \xrightarrow{\delta} & \mathcal{B}_2(F) \otimes F^\times & \xrightarrow{\delta} & \wedge^3 F^\times \end{array} \tag{C}$$

$$f_0^3(v_0, v_1, v_2, v_3) = \sum_{i=0}^{3} (-1)^i \bigwedge_{\substack{j \neq i \\ j=i+1}}^{3} \triangle(v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_3),$$

$$f_1^3(v_0, v_1, v_2, v_3, v_4) = -\frac{1}{3} \sum_{i=0}^{4} (-1)^{i+1} \Big[r(v_i | v_0, ..., \hat{v}_i, ..., v_4)\Big]_2 \otimes \prod_{\substack{j \neq i \\ j=i+1}}^{4} (v_i, v_j), \text{ and } f_2^3(v_0, ..., v_5) = \frac{2}{45} Alt_6 \Big[r(v_0, ..., v_5)\Big]_3.$$

Lemma 3.3 *(i)* $f_0^3 \circ d = \delta \circ f_1^3$ *(ii)* $f_1^3 \circ d = \delta \circ f_2^3$ *(see [4]).*

Lemma 3.4 $f_0^3 \circ p = f_1^3 \circ p = f_2^3 \circ p = 0$ (see [4]).

4. New morphisms and generalization

In this section, new homomorphisms will be introduced to connect the Grassmannian complex with the Bloch–Suslin complex for weight $n = 2$. These morphisms are further extended to connect the Grassmannian complex with Goncharov's complex for weight $n = 3$ up to weight $n = 6$. Lastly, a generalization of these morphisms will be used to generalize the geometry of the Grassmannian complex and Goncharov's complex for weight $n = N$.

4.1. Grassmannian and Bloch–Suslin complexes (weight-2)

First connect the subcomplexes of Grassmannian and Bloch–Suslin complexes.

$$\begin{array}{ccccc} G_6(3) & \xrightarrow{p} & G_5(2) & \xrightarrow{p} & G_4(1) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ G_5(3) & \xrightarrow{p} & G_4(2) & \xrightarrow{p} & G_3(1) \\ & & \downarrow{\scriptstyle g_1^2} & & \downarrow{\scriptstyle g_0^2} \\ & & \mathcal{B}_2(F) & \xrightarrow{\delta} & \wedge^2 F^\times \end{array} \tag{D}$$

Here,

$$g_0^2 : (v_0, v_1, v_2) \to -\triangle(v_1) \wedge \triangle(v_2) + \triangle(v_0) \wedge \triangle(v_2) - \triangle(v_0) \wedge \triangle(v_1) \pmod 3 \tag{8}$$

and

$$g_1^2 : (v_0, v_1, v_2, v_3) \to \sum_{i=0}^{3} (-1)^i [r(v_0, ..., v_3)]_2 \pmod 4. \tag{9}$$

g_1^2 is the same as f_1^2 defined in [4], so to show that g_0^2 is a well-defined morphism, see the proof of the lemma below.

Lemma 4.1 g_0^2 *does not depend on the volume formation of the vectors in* V_2.

Proof Let $g_0^2(v_0, v_1, v_2)$ be written as

$$g_0^2(v_0, v_1, v_2) = \frac{\triangle(v_0)}{\triangle(v_1)} \wedge \frac{\triangle(v_2)}{\triangle(v_1)}, \tag{10}$$

and so by changing the volume $V = \alpha V$ where $\alpha \in F$ and V is the volume element, the right side will remain unchanged. Hence, g_0^2 does not depend on the volume formation of the vectors in V_2. □

Lemma 4.2 $g_0^2 \circ p$ *does not depend on the volume formation of the vectors in* V_2.

Proof Let $g_0^2 \circ p(v_0, v_1, v_2, v_3)$ be written as

$$\begin{aligned} g_0^2 \circ p(v_0, ..., v_3) = & \frac{\triangle(v_0|v_1)}{\triangle(v_0|v_2)} \wedge \frac{\triangle(v_0|v_3)}{\triangle(v_0|v_2)} - \frac{\triangle(v_1|v_0)}{\triangle(v_1|v_2)} \wedge \frac{\triangle(v_1|v_3)}{\triangle(v_1|v_2)} \\ & + \frac{\triangle(v_2|v_0)}{\triangle(v_2|v_1)} \wedge \frac{\triangle(v_2|v_3)}{\triangle(v_2|v_1)} - \frac{\triangle(v_3|v_0)}{\triangle(v_3|v_1)} \wedge \frac{\triangle(v_3|v_2)}{\triangle(v_3|v_1)}, \end{aligned} \tag{11}$$

and so by changing the length of vectors $(v_0, v_1, v_2, v_3) = \alpha(v_0, v_1, v_2, v_3)$ $(\alpha \in F)$, due to fractions, the right side will remain unchanged. Therefore, $g_0^2 \circ p$ does not depend on the length of vectors. $\square$

Lemma 4.3 *The diagram D is bicomplex.*

$$G_4(1) \xrightarrow{d} G_3(1) \xrightarrow{g_0^2} \wedge^2 F^\times \quad i.e. \quad g_0^2 \circ d = 0$$

Proof If $(v_0, v_1, v_2, v_3) \in G_4(1)$, then $d(v_0, ..., v_3) = \sum_{i=0}^{3}(-1)^i(v_0, ...\hat{v}_i, ..., v_3)$. Applying morphism g_0^2,

$$\begin{aligned} g_0^2 \circ d(v_0, ..., v_3) = & - \triangle(v_2) \wedge \triangle(v_3) + \triangle(v_1) \wedge \triangle(v_3) - \triangle(v_1) \wedge \triangle(v_2) \\ & + \triangle(v_2) \wedge \triangle(v_3) - \triangle(v_0) \wedge \triangle(v_3) + \triangle(v_0) \wedge \triangle(v_2) \\ & - \triangle(v_1) \wedge \triangle(v_3) + \triangle(v_0) \wedge \triangle(v_3) - \triangle(v_0) \wedge \triangle(v_1) \\ & + \triangle(v_1) \wedge \triangle(v_2) - \triangle(v_0) \wedge \triangle(v_2) + \triangle(v_0) \wedge \triangle(v_1) = 0. \end{aligned} \tag{12}$$

$\square$

Lemma 4.4 *The lower square of the diagram D is commutative.*

Proof If $(v_0, v_1, v_2, v_3) \in G_4(2)$, and map p is applied,

$$p(v_0, ..., v_3) = \sum_{j=0}^{3}(-1)^j(v_j|v_0, ..., \hat{v}_j, ..., v_3). \tag{13}$$

Now apply morphism g_0^2:

$$g_0^2 \circ p(v_0, ..., v_3) = g_0^2\Big[\sum_{j=0}^{3}(-1)^j(v_j|v_0, ..., \hat{v}_j, ..., v_3)\Big]. \tag{14}$$

After simplification,

$$\begin{aligned} g_0^2 \circ p = & - \triangle(v_0, v_2) \wedge \triangle(v_0, v_3) + \triangle(v_0, v_1) \wedge \triangle(v_0, v_3) - \triangle(v_0, v_1) \wedge \triangle(v_0, v_2) \\ & + \triangle(v_1, v_2) \wedge \triangle(v_1, v_3) - \triangle(v_1, v_0) \wedge \triangle(v_1, v_3) + \triangle(v_1, v_0) \wedge \triangle(v_1, v_2) \\ & - \triangle(v_2, v_1) \wedge \triangle(v_2, v_3) + \triangle(v_2, v_0) \wedge \triangle(v_2, v_3) - \triangle(v_2, v_0) \wedge \triangle(v_2, v_1) \\ & + \triangle(v_3, v_1) \wedge \triangle(v_3, v_2) - \triangle(v_3, v_0) \wedge \triangle(v_3, v_2) + \triangle(v_3, v_0) \wedge \triangle(v_3, v_1). \end{aligned} \tag{15}$$

Take $(v_0, v_1, v_2, v_3) \in G_4(2)$ again and apply morphism g_1^2:

$$g_1^2(v_0, ..., v_3) = \Big[\frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}\Big]_2. \tag{16}$$

Now apply morphism δ:

$$\begin{aligned} \delta \circ g_1^2(v_0, ..., v_3) &= \Big(1 - \Big[\frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}\Big]\Big) \wedge \Big[\frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}\Big] \\ &= \Big(\frac{\triangle(v_0, v_2)\triangle(v_1, v_3) - \triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}\Big) \wedge \Big[\frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}\Big]. \end{aligned} \tag{17}$$

Applying the Siegel cross ratio property (see [12]) gives

$$\delta \circ g_1^2(v_0, ..., v_3) = \Big(\frac{\triangle(v_0,v_1)\triangle(v_2,v_3)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}\Big) \wedge \Big[\frac{\triangle(v_0,v_3)\triangle(v_1,v_2)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}\Big]. \tag{18}$$

Apply wedge properties:

$$\begin{aligned}\delta \circ g_1^2(v_0, ..., v_3) = & - \triangle(v_0,v_2)\wedge\triangle(v_0,v_3) + \triangle(v_0,v_1)\wedge\triangle(v_0,v_3) - \triangle(v_0,v_1)\wedge\triangle(v_0,v_2)\\ & + \triangle(v_1,v_2)\wedge\triangle(v_1,v_3) - \triangle(v_1,v_0)\wedge\triangle(v_1,v_3) + \triangle(v_1,v_0)\wedge\triangle(v_1,v_2)\\ & - \triangle(v_2,v_1)\wedge\triangle(v_2,v_3) + \triangle(v_2,v_0)\wedge\triangle(v_2,v_3) - \triangle(v_2,v_0)\wedge\triangle(v_2,v_1)\\ & + \triangle(v_3,v_1)\wedge\triangle(v_3,v_2) - \triangle(v_3,v_0)\wedge\triangle(v_3,v_2) + \triangle(v_3,v_0)\wedge\triangle(v_3,v_1). \end{aligned} \tag{19}$$

From Eq. (15) and Eq. (19), it is observed that $g_0^2 \circ p = \delta \circ g_1^2$. □

4.2. Grassmannian and Goncharov complexes in weight-3

For this weight, connect Grassmannian and Goncharov subcomplexes in weight-3.

$$\begin{array}{ccccc} G_7(3) & \xrightarrow{p} & G_6(2) & \xrightarrow{p} & G_5(1) \\ \downarrow d & & \downarrow d & & \downarrow d \\ G_6(3) & \xrightarrow{p} & G_5(2) & \xrightarrow{p} & G_4(1) \\ & & \downarrow g_1^3 & & \downarrow g_0^3 \\ & & \mathcal{B}_2(F)\otimes F^\times & \xrightarrow{\delta} & \wedge^3 F^\times \end{array} \tag{E}$$

Here,

$$g_0^3 : (v_0, v_1, v_2, v_3) \to \sum_{i=j+1}^{3} (-1)^{i+1} \bigwedge_{\substack{j\neq i\\ j=0}}^{3} \triangle(v_j) \quad (\mod 4) \tag{20}$$

and

$$g_1^3 : (v_0, v_1, v_2, v_3, v_4) \to -\frac{1}{3}\sum_{i=0}^{4}(-1)^i [r(v_0, ..., \hat{v}_i, ..., v_4)]_2 \otimes \prod_{\substack{m\neq i\\ m=i+1}}^{4} \triangle(v_i, v_m) \quad (\mod 5). \tag{21}$$

Lemma 4.5 *g_0^3 is not dependent on the volume formation of the vectors in V_2.*

Proof $g_0^3(v_0, v_1, v_2)$ can be written as

$$g_0^3(v_0, v_1, v_2, v_3) = -\frac{\triangle(v_0)}{\triangle(v_1)} \wedge \frac{\triangle(v_1)}{\triangle(v_2)} \wedge \frac{\triangle(v_2)}{\triangle(v_3)}, \tag{22}$$

so by changing volume $V = \alpha V$, where $\alpha \in F$, there will be no change on the right side. Hence, g_0^3 is independent of volume. □

Lemma 4.6 *$g_0^3 \circ p$ does not depend on the length of vectors in V_2.*

Proof The composition of morphisms can be written as

$$\begin{aligned}
g_0^3 \circ p(v_0, ..., v_4) = &- \frac{\triangle(v_0|v_1)}{\triangle(v_0|v_2)} \wedge \frac{\triangle(v_0|v_2)}{\triangle(v_0|v_3)} \wedge \frac{\triangle(v_0|v_3)}{\triangle(v_0|v_4)} \\
&+ \frac{\triangle(v_1|v_0)}{\triangle(v_1|v_2)} \wedge \frac{\triangle(v_1|v_2)}{\triangle(v_1|v_3)} \wedge \frac{\triangle(v_1|v_3)}{\triangle(v_1|v_4)} \\
&- \frac{\triangle(v_2|v_0)}{\triangle(v_2|v_1)} \wedge \frac{\triangle(v_2|v_1)}{\triangle(v_2|v_3)} \wedge \frac{\triangle(v_2|v_3)}{\triangle(v_2|v_4)} \\
&+ \frac{\triangle(v_3|v_0)}{\triangle(v_3|v_1)} \wedge \frac{\triangle(v_3|v_1)}{\triangle(v_3|v_2)} \wedge \frac{\triangle(v_3|v_2)}{\triangle(v_3|v_4)} \\
&- \frac{\triangle(v_4|v_0)}{\triangle(v_4|v_1)} \wedge \frac{\triangle(v_4|v_1)}{\triangle(v_4|v_2)} \wedge \frac{\triangle(v_4|v_2)}{\triangle(v_4|v_3)}.
\end{aligned} \tag{23}$$

Hence, by changing the length of vectors $(v_0, v_1, v_2, v_3) = \alpha(v_0, v_1, v_2, v_3)$ $(\alpha \in F)$, the difference becomes zero. □

Lemma 4.7 *g_1^3 does not depend on the volume formation of vectors in V_2.*

Proof The difference of volume V and αV $(\alpha \in F)$ is given by

$$g_1^3 = -\frac{1}{3} \sum_{i=0}^{4} (-1)^i [r(v_0, ..., \hat{v}_i, ..., v_4)]_2 \otimes \alpha^4. \tag{24}$$

Since $\sum_{i=0}^{4} (-1)^i [r(v_0, ..., \hat{v}_i, ..., v_4)]_2$ represents five-term relation $\in \mathcal{B}_2(F)$ and equals zero, therefore g_1^3 is independent of volume form. □

Lemma 4.8 *$g_1^3 \circ p$ does not depend on the length of vectors in V_2.*

Proof Letting $(v_0, v_1, v_2, v_3, v_4, v_5) = (\omega v_0, v_1, v_2, v_3, v_4, v_5)(\omega \in F)$, then $g_1^3 \circ p((v_0, v_1, v_2, v_3, \ v_4, v_5) - (\omega v_0, v_1, v_2, v_3, \ v_4, v_5)) = 0$. If v_0 is the projection element, the difference will be zero. Otherwise,

$$g_1^3 \circ p((v_0, ..., v_5) - (\omega v_0, ..., v_5)) = -\frac{1}{3} \sum_{\substack{i \neq j \\ i=j+1}}^{5} (-1)^i \sum_{\substack{j \neq i \\ j=0}}^{5} (-1)^j [r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)]_2 \otimes \omega^5, \tag{25}$$

where summation $\sum_{\substack{j \neq i \\ j=0}}^{5} (-1)^j [r(v_i|v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)]_2$ is the projected five-term relation $\in \mathcal{B}_2(F)$ and is equal to zero, and this is enough for the proof of Lemma (4.8). □

Lemma 4.9 *The diagram E is bicomplex.*

$$(i) \qquad G_5(1) \xrightarrow{d} G_4(1) \xrightarrow{g_0^3} \wedge^3 F^\times \ ; \ i.e. \ (g_0^3 \circ d = 0)$$

$$(ii) \qquad G_6(2) \xrightarrow{d} G_5(2) \xrightarrow{g_1^3} \mathcal{B}_2(F)\ ;\ i.e.\ (g_1^3 \circ d = 0)$$

Proof (i) Let us assume $(v_0, v_1, v_2, v_3, v_4)$ to be 5 points $\in G_5(1)$, and map d is applied, so then $d(v_0, ..., v_4) = \sum_{i=0}^{4}(-1)^i(v_0, ..., \hat{v}_i, ..., v_4)$. Now apply morphism g_0^3:

$$g_0^3 \circ d(v_0, ..., v_4) = g_0^3\Big[\sum_{i=0}^{4}(-1)^i(v_0, ..., \hat{v}_i ..., v_4)\Big], \tag{26}$$

$$\begin{aligned} g_0^3 \circ d(v_0, ..., v_4) = &- \triangle(v_2) \wedge \triangle(v_3) \wedge \triangle(v_4) + \triangle(v_1) \wedge \triangle(v_3) \wedge \triangle(v_4) \\ &- \triangle(v_1) \wedge \triangle(v_2) \wedge \triangle(v_4) + \triangle(v_1) \wedge \triangle(v_2) \wedge \triangle(v_3) \\ &+ \triangle(v_2) \wedge \triangle(v_3) \wedge \triangle(v_4) - \triangle(v_0) \wedge \triangle(v_3) \wedge \triangle(v_4) \\ &+ \triangle(v_0) \wedge \triangle(v_2) \wedge \triangle(v_4) - \triangle(v_0) \wedge \triangle(v_2) \wedge \triangle(v_3) \\ &- \triangle(v_1) \wedge \triangle(v_3) \wedge \triangle(v_4) + \triangle(v_0) \wedge \triangle(v_3) \wedge \triangle(v_4) \\ &- \triangle(v_0) \wedge \triangle(v_1) \wedge \triangle(v_4) + \triangle(v_0) \wedge \triangle(v_1) \wedge \triangle(v_3) \\ &+ \triangle(v_1) \wedge \triangle(v_2) \wedge \triangle(v_4) - \triangle(v_0) \wedge \triangle(v_2) \wedge \triangle(v_4) \\ &+ \triangle(v_0) \wedge \triangle(v_1) \wedge \triangle(v_4) - \triangle(v_0) \wedge \triangle(v_1) \wedge \triangle(v_2) \\ &- \triangle(v_1) \wedge \triangle(v_2) \wedge \triangle(v_3) + \triangle(v_0) \wedge \triangle(v_2) \wedge \triangle(v_3) \\ &- \triangle(v_0) \wedge \triangle(v_1) \wedge \triangle(v_3) + \triangle(v_0) \wedge \triangle(v_1) \wedge \triangle(v_2) \\ &= 0. \end{aligned} \tag{27}$$

Thus, $g_0^3 \circ d = 0$. □

Proof (ii) Let us suppose $(v_0, ..., v_5) \in G_6(2)$ and apply map d:

$$d(v_0, ..., v_5) = \sum_{i=0}^{5}(-1)^i(v_0, ..., \hat{v}_i, ..., v_5). \tag{28}$$

Now apply morphism g_1^3:

$$g_1^3 \circ d(v_0, ..., v_5) = -\frac{1}{3}\sum_{i=0}^{5}(-1)^i \sum_{\substack{j \neq i \\ j=i+1}}^{5} (-1)^j[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)]_2 \otimes \prod_{\substack{k \neq j \\ k=j+1}}^{5} \triangle(v_j, v_k). \tag{29}$$

The summation $\sum_{\substack{j \neq i \\ j=i+1}}^{5} (-1)^j[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)]_2$ is the functional equation of $\mathcal{B}_2(F)$ and is equal to zero, so $g_1^3 \circ d = 0$. □

Lemma 4.10 $g_0^3 \circ p = \delta \circ g_1^3$.

Proof Let us assume $(v_0, v_1, v_2, v_3, v_4) \in G_5(2)$ and apply map p:

$$p(v_0, ..., v_4) = \sum_{j=0}^{4} (v_j | v_0, ..., \hat{v}_j, ..., v_4). \tag{30}$$

Applying morphism g_0^3,

$$g_0^3 \circ p(v_0, ..., v_4) = g_0^3 \Big[\sum_{j=0}^{4} (v_j | v_0, ..., \hat{v}_j, ..., v_4) \Big]. \tag{31}$$

After simplifying the composition

$$\begin{aligned} g_0^3 \circ p(v_0, ..., v_4) = & - \triangle(v_0, v_2) \wedge \triangle(v_0, v_3) \wedge \triangle(v_0, v_4) + \triangle(v_0, v_1) \wedge \triangle(v_0, v_3) \wedge \triangle(v_0, v_4) \\ & - \triangle(v_0, v_1) \wedge \triangle(v_0, v_2) \wedge \triangle(v_0, v_4) + \triangle(v_0, v_1) \wedge \triangle(v_0, v_2) \wedge \triangle(v_0, v_3) \\ & + \triangle(v_1, v_2) \wedge \triangle(v_1, v_3) \wedge \triangle(v_1, v_4) - \triangle(v_1, v_0) \wedge \triangle(v_1, v_3) \wedge \triangle(v_1, v_4) \\ & + \triangle(v_1, v_0) \wedge \triangle(v_1, v_2) \wedge \triangle(v_1, v_4) - \triangle(v_1, v_0) \wedge \triangle(v_1, v_2) \wedge \triangle(v_1, v_3) \\ & - \triangle(v_2, v_1) \wedge \triangle(v_2, v_3) \wedge \triangle(v_2, v_4) + \triangle(v_2, v_0) \wedge \triangle(v_2, v_3) \wedge \triangle(v_2, v_4) \\ & - \triangle(v_2, v_0) \wedge \triangle(v_2, v_1) \wedge \triangle(v_2, v_4) + \triangle(v_2, v_0) \wedge \triangle(v_2, v_1) \wedge \triangle(v_2, v_3) \\ & + \triangle(v_3, v_1) \wedge \triangle(v_3, v_2) \wedge \triangle(v_3, v_4) - \triangle(v_3, v_0) \wedge \triangle(v_3, v_2) \wedge \triangle(v_3, v_4) \\ & + \triangle(v_3, v_0) \wedge \triangle(v_3, v_1) \wedge \triangle(v_3, v_4) - \triangle(v_3, v_0) \wedge \triangle(v_3, v_1) \wedge \triangle(v_4, v_2) \\ & - \triangle(v_4, v_1) \wedge \triangle(v_4, v_2) \wedge \triangle(v_4, v_3) + \triangle(v_4, v_0) \wedge \triangle(v_4, v_2) \wedge \triangle(v_4, v_3) \\ & - \triangle(v_4, v_0) \wedge \triangle(v_4, v_1) \wedge \triangle(v_4, v_3) + \triangle(v_4, v_0) \wedge \triangle(v_4, v_1) \wedge \triangle(v_4, v_2), \end{aligned} \tag{32}$$

take $(v_0, v_1, v_2, v_3, v_4) \in G_5(2)$,

$$\begin{aligned} g_1^3(v_0, ..., v_4) = & -\frac{1}{3} \Big(\Big[r(v_1, v_2, v_3, v_4) \Big]_2 \otimes \triangle(v_0, v_1) \triangle(v_0, v_2) \triangle(v_0, v_3) \triangle(v_0.v_4) \\ & - \Big[r(v_0, v_2, v_3, v_4) \Big]_2 \otimes \triangle(v_1, v_0) \triangle(v_1, v_2) \triangle(v_1, v_3) \triangle(v_1, v_4) \\ & + \Big[r(v_0, v_1, v_3, v_4) \Big]_2 \otimes \triangle(v_2, v_0) \triangle(v_1, v_2) \triangle(v_2, v_3) \triangle(v_2, v_4) \\ & - \Big[r(v_0, v_1, v_2, v_4) \Big]_2 \otimes \triangle(v_3, v_0) \triangle(v_1, v_3) \triangle(v_3, v_2) \triangle(v_3, v_4) \\ & + \Big[r(v_0, v_1, v_2, v_3) \Big]_2 \otimes \triangle(v_4, v_0) \triangle(v_1, v_4) \triangle(v_4, v_2) \triangle(v_3, v_4) \Big), \end{aligned} \tag{33}$$

and then

$$\begin{aligned} \delta \circ g_1^3 = & -\frac{1}{3} \Big(1 - r(v_1, v_2, v_3, v_4) \otimes r(v_1, v_2, v_3, v_4) \wedge \triangle(v_0, v_1) \triangle(v_0, v_2) \triangle(v_0, v_3) \triangle(v_0.v_4) \\ & - (1 - r(v_0, v_2, v_3, v_4)) \otimes r(v_0, v_2, v_3, v_4) \wedge \triangle(v_1, v_0) \triangle(v_1, v_2) \triangle(v_1, v_3) \triangle(v_1, v_4) \\ & + (1 - r(v_0, v_1, v_3, v_4)) \otimes r(v_0, v_1, v_3, v_4) \wedge \triangle(v_2, v_0) \triangle(v_1, v_2) \triangle(v_2, v_3) \triangle(v_2, v_4) \\ & - (1 - r(v_0, v_1, v_2, v_4)) \otimes r(v_0, v_1, v_2, v_4) \wedge \triangle(v_3, v_0) \triangle(v_1, v_3) \triangle(v_3, v_2) \triangle(v_3, v_4) \\ & + (1 - r(v_0, v_1, v_2, v_3)) \otimes r(v_0, v_1, v_2, v_3) \wedge \triangle(v_4, v_0) \triangle(v_1, v_4) \triangle(v_4, v_2) \triangle(v_3, v_4) \Big). \end{aligned} \tag{34}$$

Apply the Siegel cross ratio property:

$$
\begin{aligned}
=&-\frac{1}{3}\Big(\frac{\triangle(v_1,v_2)\triangle(v_3,v_4)}{\triangle(v_1,v_3)\triangle(v_2,v_4)}\otimes\frac{\triangle(v_1,v_4)\triangle(v_2,v_3)}{\triangle(v_1,v_3)\triangle(v_2,v_4)}\wedge\triangle(v_0,v_1)\triangle(v_0,v_2)\triangle(v_0,v_3)\triangle(v_0,v_4)\\
&-\frac{\triangle(v_0,v_2)\triangle(v_3,v_4)}{\triangle(v_0,v_3)\triangle(v_2,v_4)}\otimes\frac{\triangle(v_0,v_4)\triangle(v_2,v_3)}{\triangle(v_0,v_3)\triangle(v_2,v_4)}\wedge\triangle(v_1,v_0)\triangle(v_1,v_2)\triangle(v_1,v_3)\triangle(v_1,v_4)\\
&+\frac{\triangle(v_0,v_1)\triangle(v_3,v_4)}{\triangle(v_0,v_3)\triangle(v_1,v_4)}\otimes\frac{\triangle(v_0,v_4)\triangle(v_1,v_3)}{\triangle(v_0,v_3)\triangle(v_1,v_4)}\wedge\triangle(v_2,v_0)\triangle(v_1,v_2)\triangle(v_2,v_3)\triangle(v_2,v_4)\\
&-\frac{\triangle(v_0,v_1)\triangle(v_2,v_4)}{\triangle(v_0,v_2)\triangle(v_1,v_4)}\otimes\frac{\triangle(v_0,v_4)\triangle(v_1,v_2)}{\triangle(v_0,v_2)\triangle(v_1,v_4)}\wedge\triangle(v_3,v_0)\triangle(v_1,v_3)\triangle(v_2,v_3)\triangle(v_3,v_4)\\
&+\frac{\triangle(v_0,v_1)\triangle(v_2,v_3)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}\otimes\frac{\triangle(v_0,v_3)\triangle(v_1,v_2)}{\triangle(v_0,v_2)\triangle(v_1,v_3)}\wedge\triangle(v_4,v_0)\triangle(v_1,v_4)\triangle(v_2,v_4)\triangle(v_3,v_4)\Big).
\end{aligned}
\tag{35}
$$

Apply wedge and tensor properties:

$$
\begin{aligned}
\delta\circ g_1^3(v_0,...,v_4)=&-\triangle(v_0,v_2)\wedge\triangle(v_0,v_3)\wedge\triangle(v_0,v_4)+\triangle(v_0,v_1)\wedge\triangle(v_0,v_3)\wedge\triangle(v_0,v_4)\\
&-\triangle(v_0,v_1)\wedge\triangle(v_0,v_2)\wedge\triangle(v_0,v_4)+\triangle(v_0,v_1)\wedge\triangle(v_0,v_2)\wedge\triangle(v_0,v_3)\\
&+\triangle(v_1,v_2)\wedge\triangle(v_1,v_3)\wedge\triangle(v_1,v_4)-\triangle(v_1,v_0)\wedge\triangle(v_1,v_3)\wedge\triangle(v_1,v_4)\\
&+\triangle(v_1,v_0)\wedge\triangle(v_1,v_2)\wedge\triangle(v_1,v_4)-\triangle(v_1,v_0)\wedge\triangle(v_1,v_2)\wedge\triangle(v_1,v_3)\\
&-\triangle(v_2,v_1)\wedge\triangle(v_2,v_3)\wedge\triangle(v_2,v_4)+\triangle(v_2,v_0)\wedge\triangle(v_2,v_3)\wedge\triangle(v_2,v_4)\\
&-\triangle(v_2,v_0)\wedge\triangle(v_2,v_1)\wedge\triangle(v_2,v_4)+\triangle(v_2,v_0)\wedge\triangle(v_2,v_1)\wedge\triangle(v_2,v_3)\\
&+\triangle(v_3,v_1)\wedge\triangle(v_3,v_2)\wedge\triangle(v_3,v_4)-\triangle(v_3,v_0)\wedge\triangle(v_3,v_2)\wedge\triangle(v_3,v_4)\\
&+\triangle(v_3,v_0)\wedge\triangle(v_3,v_1)\wedge\triangle(v_3,v_4)-\triangle(v_3,v_0)\wedge\triangle(v_3,v_1)\wedge\triangle(v_4,v_2)\\
&-\triangle(v_4,v_1)\wedge\triangle(v_4,v_2)\wedge\triangle(v_4,v_3)+\triangle(v_4,v_0)\wedge\triangle(v_4,v_2)\wedge\triangle(v_4,v_3)\\
&-\triangle(v_4,v_0)\wedge\triangle(v_4,v_1)\wedge\triangle(v_4,v_3)+\triangle(v_4,v_0)\wedge\triangle(v_4,v_1)\wedge\triangle(v_4,v_2).
\end{aligned}
\tag{36}
$$

From Eq. (32) and Eq. (36), it is observed that $g_0^3\circ p=\delta\circ g_1^3$. □

4.3. Weight n = 4

The geometry between the subcomplexes of Goncharov in weight-4 and the Grassmannian is as constructed below:

$$
\begin{array}{ccccc}
G_8(3) & \xrightarrow{p} & G_7(2) & \xrightarrow{p} & G_6(1)\\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d}\\
G_7(3) & \xrightarrow{p} & G_6(2) & \xrightarrow{p} & G_5(1)\\
 & & \downarrow{\scriptstyle g_1^4} & & \downarrow{\scriptstyle g_0^4}\\
 & & \mathcal{B}_2(F)\otimes\wedge^2F^\times & \xrightarrow{\delta} & \wedge^4F^\times
\end{array}
\tag{F}
$$

Here,

$$g_0^4 : (v_0, ..., v_4) \to \sum_{i=j+1}^{4} (-1)^i \bigwedge_{\substack{j\neq i\\ j=0}}^{4} \triangle(v_j) \quad (\mod 5) \tag{37}$$

and

$$g_1^4 : (v_0, ..., v_5) \to \frac{1}{6} \sum_{\substack{i\neq j\\ i=0\\ j=i+1}}^{5} (-1)^{i+1}[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)]_2 \otimes \prod_{\substack{m\neq i\\ m=i+1}}^{5} \triangle(v_i, v_m)\wedge$$

$$\prod_{\substack{m\neq j\\ m=j+1}}^{5} \triangle(v_j, v_m) \quad (\mod 6). \tag{38}$$

Lemma 4.11 *The diagram F is commutative.*

Proof Let us assume $(v_0, ..., v_5) \in G_6(2)$ and apply homomorphism p:

$$p(v_0, ..., v_5) = \sum_{j=0}^{5} (-1)^j (v_j|v_0, ..., \hat{v}_j, ..., v_5). \tag{39}$$

Now apply map g_0^4:

$$g_0^4 \circ p(v_0, ..., v_5) = \sum_{i=j+1}^{5} (-1)^i \sum_{j=0}^{5} (-1)^j \bigwedge_{\substack{m\neq j\\ m=j+1}}^{5} \triangle(v_j, v_m). \tag{40}$$

Take $(v_0, ..., v_5) \in G_6(2)$ again and apply morphism g_1^4:

$$g_1^4(v_0, ..., v_5) = \frac{1}{6} \sum_{\substack{i\neq j\\ i=0\\ j=i+1}}^{5} (-1)^{i+1}[r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)]_2 \otimes \prod_{\substack{m\neq i\\ m=i+1}}^{5} \triangle(v_i, v_m) \wedge \prod_{\substack{m\neq j\\ m=j+1}}^{5} \triangle(v_j, v_m). \tag{41}$$

Now apply morphism δ, and then

$$\delta \circ g_1^4(v_0, ..., v_5) = \frac{1}{6} \sum_{\substack{i\neq j\\ i=0\\ j=i+1}}^{5} (-1)^{i+1}(1 - r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)) \wedge r(v_0, ..., \hat{v}_i, \hat{v}_j, ..., v_5)\wedge$$

$$\prod_{\substack{m\neq i\\ m=i+1}}^{5} \triangle(v_i, v_m) \wedge \prod_{\substack{m\neq j\\ m=j+1}}^{5} \triangle(v_j, v_m). \tag{42}$$

Using wedge and Siegel cross ratio properties [12],

$$\delta \circ g_1^4(v_0, ..., v_5) = \sum_{i=j+1}^{5} (-1)^i \sum_{j=0}^{5} (-1)^j \bigwedge_{\substack{m\neq j\\ m=j+1}}^{5} \triangle(v_j, v_m). \tag{43}$$

Thus, from Eq. (40) and Eq. (43), it is proved that the above diagram is commutative. □

4.4. Weight n = 5

By connecting the Grassmannian complex with the subcomplex of Goncharov's complex, we have:

$$\begin{array}{ccccc}
G_9(3) & \xrightarrow{p} & G_8(2) & \xrightarrow{p} & G_7(1) \\
\downarrow d & & \downarrow d & & \downarrow d \\
G_8(3) & \xrightarrow{p} & G_7(2) & \xrightarrow{p} & G_6(1) \\
 & & \downarrow g_1^5 & & \downarrow g_0^5 \\
 & & \mathcal{B}_2(F)\otimes \wedge^3 F^{\times} & \xrightarrow{\delta} & \wedge^5 F^{\times}
\end{array} \tag{G}$$

Here,

$$g_0^5 : (v_0, ..., v_5) \to \sum_{i=j+1}^{5} (-1)^i \bigwedge_{\substack{j\neq i\\ j=0}}^{5} \triangle(v_j) \quad (\mod 6) \tag{44}$$

and

$$g_1^5 : (v_0, ..., v_6) \to -\frac{1}{10} \sum_{\substack{i\neq j\neq k\\ i=0\\ j=i+1\\ k=i+2}}^{6} (-1)^{i+1} [r(v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_6)]_2 \otimes \prod_{\substack{m\neq i\\ m=i+1}}^{6} \triangle(v_i, v_m)$$

$$\wedge \prod_{\substack{m\neq j\\ m=j+1}}^{6} \triangle(v_j, v_m) \wedge \prod_{\substack{m\neq k\\ m=k+1}}^{6} \triangle(v_k, v_m) \quad (\mod 7). \tag{45}$$

Lemma 4.12 *The above diagram G is commutative.*

Proof Let us suppose $(v_0, ..., v_6)$ to be 7 points $\in G_7(2)$ and apply homomorphism p:

$$p(v_0, ..., v_6) = \sum_{j=0}^{6} (-1)^j (v_j | v_0, ..., \hat{v}_j, ..., v_6). \tag{46}$$

Now apply function g_0^5:

$$g_0^5 \circ p(v_0, ..., v_6) = \sum_{i=j+1}^{5} (-1)^i \sum_{j=0}^{6} (-1)^j \bigwedge_{\substack{m\neq j\\ m=j+1}}^{6} \triangle(v_j, v_m). \tag{47}$$

Take $(v_0, ..., v_6) \in G_7(2)$ again and apply morphism g_1^5:

$$g_1^5(v_0, ..., v_6) = -\frac{1}{10} \sum_{\substack{i\neq j\neq k\\ i=0\\ j=i+1\\ k=i+2}}^{6} (-1)^{i+1} [r(v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_6)]_2 \otimes \prod_{\substack{m\neq i\\ m=i+1}}^{6} \triangle(v_i, v_m)$$

$$\wedge \prod_{\substack{m\neq j\\ m=j+1}}^{6} \triangle(v_j, v_m) \wedge \prod_{\substack{m\neq k\\ m=k+1}}^{6} \triangle(v_k, v_m). \tag{48}$$

Now, by applying morphism δ,

$$\delta \circ g_1^5(v_0,...,v_6) = -\frac{1}{10}\sum_{\substack{i\neq j\neq k\\ i=0\\ j=i+1\\ k=i+2}}^{6}(-1)^{i+1}(1-r(v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)) \wedge r(v_0,...,\hat{v}_i,\hat{v}_j,$$

$$\hat{v}_k,...,v_6) \wedge \prod_{\substack{m\neq i\\ m=i+1}}^{6}\triangle(v_i,v_m) \wedge \prod_{\substack{m\neq j\\ m=j+1}}^{6}\triangle(v_j,v_m) \wedge \prod_{\substack{m\neq k\\ m=k+1}}^{6}\triangle(v_k,v_m). \tag{49}$$

Using wedge and Siegel cross ratio properties,

$$\delta \circ g_1^5(v_0,...,v_6) = \sum_{i=j+1}^{6}(-1)^i\sum_{j=0}^{6}(-1)^j\bigwedge_{\substack{m\neq j\\ m=j+1}}^{6}\triangle(v_j,v_m). \tag{50}$$

Thus, from Eq. (47) and Eq. (50), the diagram G is proved to be commutative. □

4.5. Weight n = 6

The geometry for this weight is given as follows:

$$\begin{array}{ccccc}
G_{10}(3) & \xrightarrow{p} & G_9(2) & \xrightarrow{p} & G_8(1) \\
\downarrow{d} & & \downarrow{d} & & \downarrow{d} \\
G_9(3) & \xrightarrow{p} & G_8(2) & \xrightarrow{p} & G_7(1) \\
 & & \downarrow{g_1^6} & & \downarrow{g_0^6} \\
 & & \mathcal{B}_2(F)\otimes\wedge^4F^\times & \xrightarrow{\delta} & \wedge^6F^\times
\end{array} \tag{H}$$

Here,

$$g_0^6:(v_0,...,v_6)\to\sum_{i=j+1}^{6}(-1)^i\bigwedge_{\substack{j\neq i\\ j=0}}^{6}\triangle(v_j) \quad (\mod 7) \tag{51}$$

and

$$g_1^6:(v_0,...,v_7)\to\frac{1}{15}\sum_{\substack{i\neq j\neq k\neq l\\ i=0\\ j=i+1\\ k=i+2\\ l=i+3}}^{7}(-1)^{i+1}[r(v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,\hat{v}_l,...,v_7)]_2\otimes\prod_{\substack{m\neq i\\ m=i+1}}^{7}\triangle(v_i,v_m)$$

$$\wedge\prod_{\substack{m\neq j\\ m=j+1}}^{7}\triangle(v_j,v_m)\wedge\prod_{\substack{m\neq k\\ m=k+1}}^{7}(v_k,v_m)\wedge\prod_{\substack{m\neq l\\ m=l+1}}^{7}\triangle(v_l,v_m) \quad (\mod 8). \tag{52}$$

Lemma 4.13 *The diagram H is commutative.*

Proof Let us assume $(v_0, ..., v_7)$ are 8 points of dimension two $\in G_8(2)$ and apply homomorphism p:

$$p(v_0, ..., v_7) = \sum_{j=0}^{7} (-1)^j (v_j | v_0, ...\hat{v}_j, ..., v_7). \tag{53}$$

Now apply function g_0^6:

$$g_0^6 \circ p(v_0, ..., v_7) = \sum_{i=j+1}^{7} (-1)^i \sum_{j=0}^{7} (-1)^j \bigwedge_{\substack{m \neq j \\ m=j+1}}^{7} \triangle(v_j, v_m). \tag{54}$$

Take $(v_0, ..., v_7) \in G_8(2)$ again and apply map g_1^6:

$$g_1^6(v_0, ..., v_7) = \frac{1}{15} \sum_{\substack{i \neq j \neq k \neq l \\ i=0 \\ j=i+1 \\ k=i+2 \\ l=i+3}}^{7} (-1)^{i+1} [r(v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, \hat{v}_l, ..., v_7)]_2 \otimes \prod_{\substack{m \neq i \\ m=i+1}}^{7} \triangle(v_i, v_m)$$
$$\wedge \prod_{\substack{m \neq j \\ m=j+1}}^{7} \triangle(v_j, v_m) \wedge \prod_{\substack{m \neq k \\ m=k+1}}^{7} \triangle(v_k, v_m) \wedge \prod_{\substack{m \neq l \\ m=l+1}}^{7} \triangle(v_l, v_m). \tag{55}$$

By applying morphism δ,

$$\delta \circ g_1^6(v_0, ..., v_7) = \frac{1}{15} \sum_{\substack{i \neq j \neq k \neq l \\ i=0 \\ j=i+1 \\ k=i+2 \\ l=i+3}}^{7} (-1)^{i+1} (1 - r(v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, \hat{v}_l, ..., v_7)) \wedge r(v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k,$$
$$\hat{v}_l, ..., v_7) \wedge \prod_{\substack{m \neq i \\ m=i+1}}^{7} \triangle(v_i, v_m) \wedge \prod_{\substack{m \neq j \\ m=j+1}}^{7} \triangle(v_j, v_m) \wedge \prod_{\substack{m \neq k \\ m=k+1}}^{7} \triangle(v_k, v_m) \wedge \prod_{\substack{m \neq l \\ m=l+1}}^{7} \triangle(v_l, v_m). \tag{56}$$

Using wedge and Siegel cross ratio properties, then

$$\delta \circ g_1^6(v_0, ..., v_7) = \sum_{i=j+1}^{7} (-1)^i \sum_{j=0}^{7} (-1)^j \bigwedge_{\substack{m \neq j \\ m=j+1}}^{7} \triangle(v_j, v_m). \tag{57}$$

Eq. (54) and Eq. (57) prove that diagram H is commutative. □

4.6. Weight n = N (generalization)

Through the prior calculated work, the following generalized diagram of Grassmannian and Goncharov complexes is constructed:

$$
\begin{array}{ccccc}
G_{N+4}(3) & \xrightarrow{p} & G_{N+3}(2) & \xrightarrow{p} & G_{N+2}(1) \\
\downarrow d & & \downarrow d & & \downarrow d \\
G_{N+3}(3) & \xrightarrow{p} & G_{N+2}(2) & \xrightarrow{p} & G_{N+1}(1) \\
 & & \downarrow g_1^N & & \downarrow g_0^N \\
 & & \mathcal{B}_2(F)\otimes\wedge^{N-2}F^\times & \xrightarrow{\delta} & \wedge^N F^\times
\end{array} \qquad (N \geq 2) \tag{I}
$$

Here,

$$g_0^N : (v_0, ..., v_N) \to \sum_{i=j+1}^{N} (-1)^i \bigwedge_{\substack{j\neq i \\ j=0}}^{N} \triangle(v_j) \quad (\mod N+1) \tag{58}$$

and

$$g_1^N : (v_0, ..., v_{N+1}) \to \frac{1}{{}^N C_2} (-1)^N \sum_{\substack{i_0=0 \\ i_1=i_0+1 \\ i_2=i_0+2 \\ \vdots \\ i_{N-3}=i_0+N-3}}^{N+1} (-1)^{i+1} [r(v_0, ..., \hat{v}_{i_0}, \hat{v}_{i_1}, ..., \hat{v}_{i_{N-3}}, ..., v_{N+1})]_2$$

$$\otimes \prod_{\substack{m\neq i_0 \\ m=i_0+1}}^{N+1} \triangle(v_{i_0}, v_m) \wedge \prod_{\substack{m\neq i_1 \\ m=i_1+1}}^{N+1} \triangle(v_{i_1}, v_m) \wedge ... \wedge \prod_{\substack{m\neq i_{N-3} \\ m=1+i_{N-3}}}^{N+1} \triangle(v_{i_{N-3}}, v_m) \quad (\mod N+2). \tag{59}$$

Theorem 4.1 $\delta \circ g_1^N = g_0^N \circ p$.

Proof Let us assume $(v_0, ..., v_{N+1})$ are $(n+2)$ points $\in G_{N+2}(2)$ and apply homomorphism p:

$$p(v_0, ..., v_{N+1}) = \sum_{j=0}^{N+1} (-1)^j (v_j | v_0, ..., \hat{v}_j, ..., v_{N+1}). \tag{60}$$

Now apply morphism g_0^N:

$$g_0^N \circ p(v_0, ..., v_{N+1}) = \sum_{i=j+1}^{N+1} (-1)^i \sum_{j=0}^{N+1} (-1)^j \bigwedge_{\substack{m\neq j \\ m=j+1}}^{N+1} \triangle(v_j, v_m). \tag{61}$$

Take $(v_0, ..., v_{N+1}) \in G_{N+2}(2)$ again and apply morphism g_1^N:

$$g_1^N(v_0, ..., v_{N+1}) = \frac{1}{{}^N C_2}(-1)^N \sum_{\substack{i_0=0 \\ i_1=i_0+1 \\ i_2=i_0+2 \\ \vdots \\ i_{N-3}=i_0+N-3}}^{N+1} (-1)^{i+1}[r(v_0, ..., \hat{v}_{i_0}, \hat{v}_{i_1}, ..., \hat{v}_{i_{N-3}}, ..., v_{N+1})]_2$$

$$\otimes \prod_{\substack{m \neq i_0 \\ m=i_0+1}}^{N+1} \triangle(v_{i_0}, v_m) \wedge \prod_{\substack{m \neq i_1 \\ m=i_1+1}}^{N+1} \triangle(v_{i_1}, v_m) \wedge ... \wedge \prod_{\substack{m \neq i_{N-3} \\ m=1+i_{N-3}}}^{N+1} \triangle(v_{i_{N-3}}, v_m). \tag{62}$$

Now apply map δ:

$$\delta \circ g_1^N(v_0, ..., v_{N+1}) = \frac{1}{{}^N C_2}(-1)^N \sum_{\substack{i_0=0 \\ i_1=i_0+1 \\ i_2=i_0+2 \\ \vdots \\ i_{N-3}=i_0+N-3}}^{N+1} (-1)^{i+1}(1 - r(v_0, ..., \hat{v}_{i_0}, \hat{v}_{i_1}, ..., \hat{v}_{i_{N-3}}, ...,$$

$$v_{N+1})) \wedge r(v_0, ..., \hat{v}_{i_0}, \hat{v}_{i_1}, ..., \hat{v}_{i_{N-3}}, ..., v_{N+1}) \wedge \prod_{\substack{m \neq i_0 \\ m=i_0+1}}^{N+1} \triangle(v_{i_0}, v_m)$$

$$\wedge \prod_{\substack{m \neq i_1 \\ m=i_1+1}}^{N+1} \triangle(v_{i_1}, v_m) \wedge ... \wedge \prod_{\substack{m \neq i_{N-3} \\ m=1+i_{N-3}}}^{N+1} \triangle(v_{i_{N-3}}, v_m). \tag{63}$$

Using Siegel and wedge properties,

$$\delta \circ g_1^N(v_0, ..., v_{N+1}) = \sum_{i=j+1}^{N+1} (-1)^i \sum_{j=0}^{N+1} (-1)^j \bigwedge_{\substack{m \neq j \\ m=j+1}}^{N+1} \triangle(v_j, v_m), \tag{64}$$

from Eq. (61) and Eq. (64), to get $\delta \circ g_1^N = g_0^N \circ p$. □

5. Conclusion

In this research article, the generalization of the new morphisms is defined to connect Goncharov's complex with the Grassmannian complex up to weight $n = N$. This generalization is devoted to the study of the combinatorial aspect of simplicial complexes, or more exactly to some chain complexes, as they appear in algebraic number theory, algebraic Topology, topological K-theory, and classical polylogarithmic Group theory.

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