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On small covers over a product of simplices

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Abstract: In this paper, we give a formula for the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a product of simplices. We also give an upper bound for the number of small covers over a product of simplices up to homeomorphism.

Key words: Small cover, equivariant homeomorphism, polytope, acyclic digraph

1. Introduction

A small cover is a smooth closed manifold M^n that admits a locally standard $\mathbb{Z}_2^n$-action whose orbit space is a simple convex polytope. The notion of a small cover was introduced by Davis and Januszkiewicz [5] as a generalization of real toric manifolds. In [5], it was shown that every small cover over a simple convex polytope P^n can be obtained from a characteristic function on the set of facets of P^n. There is a free action of the general linear group $GL(n, \mathbb{Z}_2)$ on the set of characteristic functions and the orbit space of this action is in one-to-one correspondence with the Davis–Januszkiewicz equivalence classes of small covers. Recently, several studies have been done to calculate the number of Davis–Januszkiewicz equivalence classes of small covers over a specific polytope (see [1, 3, 6]). In [6], Garrison and Scott used a computer program to find the number of small covers over a dodecahedron up to Davis–Januszkiewicz equivalence. In [3], Choi constructed a bijection between the set of Davis–Januszkiewicz equivalence classes of small covers over an n-cube and the set of acyclic digraphs with n-labeled nodes. He also gave a formula for the number of small covers over a product of simplices up to Davis–Januszkiewicz equivalence in terms of acyclic digraphs with labeled nodes.

There is a standard action of the automorphism group of the face poset of P^n on the set of characteristic functions on P^n. Lü and Masuda [7] showed that there is a bijection between the set of orbits of this action and the set of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P^n. By Burnside's lemma, the number of orbits of an action is the average number of the points fixed by an element of the group. Therefore, one can find the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P^n by enumerating the number of fixed points of elements of the automorphism group. Using the Burnside lemma, Choi [3] gave a formula for the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a cube, which is the product of 1-simplices. When P^n is a product of simplices of dimension greater than 1, the action of the automorphism group of the face poset is free. Therefore, the number of equivariant small covers over a product of simplices of dimension greater than 1 is the quotient of the number of the small covers and the order of the automorphism

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group of the face poset. In [2], Chen and Wang directly counted the number of equivariant homeomorphism classes of small covers over $\Delta^1 \times \Delta^{n_1} \times \Delta^{n_2}$ and $\Delta_1 \times \Delta^{n_3}$, where Δ^{n_i} is an n_i-simplex with $n_i \geq 1$ for $1 \leq i \leq 3$. In this paper, we use Choi's argument to generalize these formulas to an arbitrary product of simplices.

The paper is organized as follows. In Section 2 we recall the basic theory about the small covers over a simple polytope and vector matrices. In Section 3 we obtain a formula for the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes over a product of simplices. In Section 4 we give an upper bound for the number of small covers over a product of equidimensional simplices up to homeomorphism.

2. Preliminaries

An n-dimensional convex polytope P is said to be simple if every vertex of P is the intersection of precisely n facets. A small cover over P is a smooth closed n-manifold M^n that admits a $\mathbb{Z}_2^n$-action that is locally isomorphic to a standard action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ and the orbit space of the action is P.

Given a simple convex polytope P of dimension n, let $\mathcal{F}(P) = \{F_1, \ldots, F_m\}$ be the set of facets of P. A function $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n$ is called a characteristic function if it satisfies the nonsingularity condition that whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_n}$ is nonempty, the set $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_n})\}$ forms a basis for $\mathbb{Z}_2^n$. For a given point $p \in P$, let $\mathbb{Z}_2^n(p)$ be the subgroup of $\mathbb{Z}_2^n$ generated by $\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})$ where the intersection $\bigcap_{j=1}^{k} F_{i_j}$ is the minimal face containing p in its relative interior. Then the manifold $M(\lambda) = (P \times \mathbb{Z}_2^n)/\sim$ where

$$(p, g) \sim (q, h) \text{ if } p = q \text{ and } g^{-1}h \in \mathbb{Z}_2^n(p)$$

is a small cover over P.

Theorem 2.1 *([5]) For every small cover M over P, there is a characteristic function λ with $\mathbb{Z}_2^n$-homeomorphism $M(\lambda) \to M$ covering the identity on P.*

Two small covers M_1 and M_2 over P are said to be DJ-equivalent (Davis–Januszkiewicz equivalent) if there is a weakly $\mathbb{Z}_2^n$-homeomorphism $f : M_1 \to M_2$ covering the identity on P. Following [7], let $\Lambda(P)$ be the set of all characteristic functions on P. There is a free action of $GL(n, \mathbb{Z}_2)$ on $\Lambda(P)$ defined by $g \cdot \lambda = g \circ \lambda$. By the above theorem, DJ-equivalence classes of small covers over P bijectively correspond to the coset $GL(n, \mathbb{Z}_2)\backslash\Lambda(P)$. In particular, $|\Lambda(P)|$ is equal to the product of $|GL(n, \mathbb{Z}_2)\backslash\Lambda(P)|$ and $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^{n} (2^n - 2^{k-1})$.

On the other hand, the equivariant classes of small covers over P are characterized by the action of the automorphism group of the face poset of P. More precisely, let $\mathrm{Aut}(\mathcal{F}(P))$ be the group of bijections from the set of faces of P to itself, which preserves the poset structure. Then $\mathrm{Aut}(\mathcal{F}(P))$ acts on $\Lambda(P)$ on the right by $\lambda \cdot h = \lambda \circ h$. In [7], Lu and Masuda proved the following theorem.

Theorem 2.2 *The set of $\mathbb{Z}_2^n$-homeomorphism classes of small covers over P corresponds bijectively to the coset $\Lambda(P)/\mathrm{Aut}(\mathcal{F}(P))$.*

By the above theorem, to find the number of equivariant classes of small covers over P, we need to find the number of orbits of $\Lambda(P)$ under the action of $\mathrm{Aut}(\mathcal{F}(P))$. The Burnside lemma reduces this problem to the enumeration of fixed points

$$\Lambda(P)_h = \{\lambda \in \Lambda(P) \mid \lambda(h(F)) = \lambda(F) \text{ for all } F \in \mathcal{F}(P)\}$$

by elements $h \in \mathrm{Aut}(\mathcal{F}(P))$.

Lemma 2.3 (Burnside lemma) *Let G be a finite group acting on a set X. Then the number of G-orbits of X is equal to $\frac{1}{|G|}\sum_{g\in G}|X^g|$, where $X^g = \{x \in X \mid gx = x\}$.*

Therefore, one can find the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P^n by enumerating $\Lambda(P)_h$ for all $h \in \mathrm{Aut}(\mathcal{F}(P))$.

As a combination of the above theorems, we have the following result.

Theorem 2.4 *The number of weakly $\mathbb{Z}_2^n$-homeomorphism classes of small covers over P is the size of the double coset $GL(n,\mathbb{Z}_2)\backslash\Lambda(P)/\mathrm{Aut}(\mathcal{F}(P))$.*

3. The number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes

Let $P = \Delta^{n_1} \times \cdots \times \Delta^{n_m}$, where Δ^{n_i} is the standard n_i-simplex. Let $\mathcal{G}_m$ be the set of acyclic digraphs with m labeled nodes with labeled vertex set $V(G) = \{v_1,\ldots,v_m\}$. Here, a digraph is a graph with at most one edge directed from vertex v_i to v_j. A directed graph is said to be acyclic if there is no directed cycle. The outdegree $\mathrm{outdeg}(v)$ (the indegree $\mathrm{indeg}(v)$) of a vertex v is the number of edges directed from (to) v. In [3], Choi gave the following formula for the number of small covers over P.

Theorem 3.1 *(Theorem 2.8, [3]) The number of DJ-equivalence classes of small covers over $P = \Delta^{n_1} \times \cdots \times \Delta^{n_m}$ with $\sum_{i=1}^{m} n_i = n$ is*

$$|GL(n,\mathbb{Z}_2)\backslash\Lambda(P)| = \sum_{G\in\mathcal{G}_m}\prod_{v_i\in V(G)}(2^{n_i}-1)^{\mathrm{outdeg}(v_i)}.$$

It is well known that the automorphism group of the face poset of Δ^n is the group of permutations on the set of facets, i.e. $\mathrm{Aut}(\mathcal{F}(\Delta^n)) \cong S_{n+1}$, where S_{n+1} is the symmetric group of degree $n+1$. To understand the automorphism group of $\mathcal{F}(P)$, we need to take the number of Δ^n occurring in P into account. For this reason, we write

$$P = \prod_{i=1}^{l} P_i, \text{ where } P_i = \Delta_1^{n_i} \times \cdots \times \Delta_{m_i}^{n_i},$$

with $1 \le n_1 < n_2 < \cdots < n_l$ and $\sum_{i=1}^{l} n_i m_i = n$. Then the set of facets of P^i is

$$\{f_{j,k}^i = \Delta_1^{n_i} \times \cdots \times \Delta_{j-1}^{n_i} \times \tilde{f}_{j,k}^i \times \Delta_{j+1}^{n_i} \times \cdots \times \Delta_{m_i}^{n_i} \mid 0 \le k \le n_i,\ 1 \le j \le m_i\}$$

where $\{\tilde{f}_{j,0}^i,\ldots,\tilde{f}_{j,n_i}^i\}$ is the set of facets of the simplex $\Delta_j^{n_i}$. Therefore, we have

$$\mathcal{F}(P) = \{F_{j,k}^i \mid 0 \le k \le n_i,\ 1 \le j \le m_i,\ 1 \le i \le l\}$$

where $F^i_{j,k} = P_1 \times \cdots \times P_{i-1} \times f^i_{j,k} \times P_{i+1} \times \cdots \times P_l$. Note that there are $(n+m)$-facets, where $m = \sum_{i=1}^{l} m_i$. Since $\mathrm{Aut}(\mathcal{F}(\Delta^n)) \cong S_{n+1}$, $\mathrm{Aut}(\mathcal{F}(P_i))$ is the wreath product of $S_{n_{i+1}}$ with S_{m_i}, where $\mu \in S_{m_i}$ sends $f^i_{j,k}$ to $f^i_{\mu(j),k}$. More precisely, $\mathrm{Aut}(\mathcal{F}(P_i)) = S_{n_i+1} \wr S_{m_i}$ is equal to $\underbrace{S_{n_i+1} \times \cdots \times S_{n_i+1}}_{m_i} \times S_{m_i}$ as a set where the group multiplication is defined by

$$(\sigma_1, \cdots, \sigma_{m_i}, \mu)(\sigma'_1, \cdots, \sigma'_{m_i}, \mu') = (\sigma_1 \sigma'_{\mu^{-1}(1)}, \cdots, \sigma_{m_i} \sigma'_{\mu^{-1}(m_i)}, \mu\mu')$$

for any $\sigma_i, \sigma'_i \in S_{n_i+1}$ and $\mu, \mu' \in S_{m_i}$. Since $n_1 < n_2 < \cdots < n_l$, we have the following.

Lemma 3.2 $\mathrm{Aut}(\mathcal{F}(P)) \cong \prod_{i=1}^{l} \Big(S_{n_i+1} \wr S_{m_i} \Big)$.

By the nonsingularity condition, a characteristic function must send any set obtained by taking n_i-many elements from $\{F^i_{j,k} | 0 \le k \le n_i\}$ for each $1 \le j \le m_i$ and $1 \le i \le l$ to a basis of $\mathbb{Z}_2^n$. When $1 < n_1$, more than one element is arbitrarily chosen from each set. However, for every nontrivial element g of $\mathrm{Aut}(\mathcal{F}(P))$, there exist $1 \le j \le m_i$ and $1 \le i \le l$ for which at least two elements from the set $\{F^i_{j,k} | 0 \le k \le n_i\}$ are not fixed by g. Therefore, g cannot fix any characteristic function. This means that the action of $\mathrm{Aut}(\mathcal{F}(P))$ on $\mathcal{F}(P)$ is free and hence the number of equivariant homeomorphism classes of small covers over P with $n_1 > 1$ is

$$\frac{|\Lambda(P)|}{|\mathrm{Aut}(\mathcal{F}(P))|} = \frac{|\Lambda(P)|}{\prod_{i=1}^{l} [(n_i+1)!]^{m_i} (m_i)!}.$$

Since $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^{n} (2^n - 2^{k-1})$, by the above theorem we have:

Corollary 3.3 *Let* $P = \prod_{i=1}^{l} \Delta_1^{n_i} \times \cdots \times \Delta_{m_i}^{n_i}$ *with* $\sum_{i=1}^{l} m_i = m$ *and* $\sum_{i=1}^{l} n_i m_i = n$. *Define a function* $n : \{1, \ldots, m\} \to \{n_1, \ldots, n_l\}$ *by* $n(s) = n_i$ *whenever* $k_1 + \cdots + k_{i-1} + 1 \le s \le k_1 + \cdots + m_i$. *Then the number of equivariant homeomorphism classes of small covers over* P *with* $n_1 > 1$ *is*

$$\frac{|\Lambda(P)|}{|\mathrm{Aut}(\mathcal{F}(P))|} = \frac{\Big(\prod_{k=1}^{n} (2^n - 2^{k-1})\Big) \Big(\sum_{G \in \mathcal{G}_m} \prod_{v_s \in V(G)} (2^{n(s)} - 1)^{\mathrm{outdeg}(v_s)}\Big)}{\prod_{i=1}^{l} [(n_i+1)!]^{m_i} (m_i)!}.$$

When $n_1 = 1$, the only elements of $\mathrm{Aut}(\mathcal{F}(P))$ that have a fixed point are the ones of the form

$$\chi_1^{\epsilon_1} \cdots \chi_{m_1}^{\epsilon_{m_1}}, \quad \epsilon_i \in \mathbb{Z}_2$$

where $\chi_1, \cdots, \chi_{m_1}$ are the reflections in $\mathrm{Aut}(\mathcal{F}(I^{m_1}))$. To count the number of elements in $\Lambda(P)_{\chi_1^{\epsilon_1} \cdots \chi_{m_1}^{\epsilon_{m_1}}}$, first note that it is a $GL(n, \mathbb{Z}_2)$-invariant subset of $\Lambda(P)$. Since the action of $GL(n, \mathbb{Z}_2)$ is free, we have

$$|\Lambda(P)_{\chi_1^{\epsilon_1} \cdots \chi_{m_1}^{\epsilon_{m_1}}}| = |GL(n, \mathbb{Z}_2)| \times |GL(n, \mathbb{Z}_2) \backslash \Lambda(P)_{\chi_1^{\epsilon_1} \cdots \chi_{m_1}^{\epsilon_{m_1}}}|.$$

To find $|GL(n,\mathbb{Z}_2)\backslash\Lambda(P)_{\chi_1^{\epsilon_1}\cdots\chi_{m_1}^{\epsilon_{m_1}}}|$ we use the correspondence given by Choi [3]. By the nonsingularity condition, for any $\lambda \in \Lambda(P)$, the vectors

$$\lambda(F_{1,1}^1),\ldots,\lambda(F_{m_1,1}^1),\lambda(F_{1,1}^2),\lambda(F_{1,2}^2),\ldots,\lambda(F_{1,n_2}^2),\lambda(F_{2,1}^2),\ldots,\lambda(F_{m_l,1}^l),\ldots,\lambda(F_{m_l,n_l}^l) \tag{1}$$

form a basis for $\mathbb{Z}_2^n$. For each coset in $GL(n,\mathbb{Z}_2)\backslash\Lambda(P)_{\chi_1^{e_1}\cdots\chi_{m_1}^{e_{m_1}}}$, choose a representative λ for which the vectors in (1) correspond to the standard basis elements

$$e_1 = (1,0,\ldots,0),\ \cdots,e_n = (0,\ldots,0,1),$$

respectively. More precisely, we have

$$\lambda(F_{j,k}^i) = e_{m_1n_1+\cdots+m_{i-1}n_{i-1}+(j-1)n_i+k}$$

for $1 \le i \le l$, $1 \le j \le m_i$ and $1 \le k \le n_i$. Let $A(\epsilon_1,\ldots,\epsilon_{m_1})$ be the set of such representatives. For the remaining facets, we write $F_{j,0}^i =: F_{m_1+\cdots+m_{i-1}+j}$ for $1 \le i \le l$ and $1 \le j \le m_i$. Then we have

$$\lambda(F_p) = \sum_{q=1}^{n} a_{qp}e_q.$$

We can view the corresponding $(n\times m)$-matrix $\Lambda = [a_{pq}]$ as an $(m\times m)$-vector matrix $[\mathbf{v_{pq}}]$ whose entries in the pth row are vectors in $\mathbb{Z}_2^{n(p)}$ where $n(p)$ is defined as in Corollary 3.3. We refer reader to [4] for details. Let $\Lambda_{s_1\cdots s_m}$ be the $(m\times m)$-submatrix of Λ whose ith row is the s_ith row of $[\mathbf{v_{pq}}]$. Then λ satisfies the singularity condition if and only if every principal minor of $\Lambda_{s_1\cdots s_m}$ is 1 for any $1 \le s_1 \le n(1)$, $\cdots$, $1 \le s_m \le n(m)$.

Theorem 3.4 $|\Lambda(P)_{\chi_1^{\epsilon_1}\cdots\chi_{m_1}^{\epsilon_{m_1}}}| = \Big(\prod_{k=1}^{n}(2^n-2^{k-1})\Big)\Big(\sum_{G\in\mathcal{G}_m(\epsilon_1,\ldots,\epsilon_{m_1})}\prod_{v_i\in V(G)}(2^{n_i}-1)^{\mathrm{outdeg}(v_i)}\Big)$

where $\mathcal{G}_m(\epsilon_1,\ldots,\epsilon_{m_1})$ is the set of acyclic digraphs with m labeled nodes $\{v_1,\ldots,v_m\}$ such that $\mathrm{indeg}(v_i)=0$ whenever $\epsilon_i = 1$ for $1 \le i \le m_1$.

Proof Without loss of generality, we assume that $\epsilon_i = 1$ for $1 \le i \le t \le m_1$ and $\epsilon_i = 0$ for $t < i \le m_1$. Let $A = A(\underbrace{1,\ldots,1}_{t},0,\cdots,0)$. For $\lambda \in A$, let $\Lambda = [\mathbf{v_{ij}}]$ be the $(m\times m)$-vector matrix corresponding to λ. Let $B(\Lambda) =: [b_{ij}]$ be the $\mathbb{Z}_2$-matrix whose (i,j)th entry is 1 if $\mathbf{v_{ij}}$ is nonzero and 0 otherwise. By Lemma 5.1 in [4], Λ is conjugate to a unipotent upper triangular vector matrix. Therefore, $B(\Lambda) - I_m$, where I_m is the $(m\times m)$ identity matrix, is an adjacency matrix of an acyclic digraph. Define ϕ from A to $\mathcal{G}_m$ by $\phi(\lambda) = G$ where the adjacency matrix of G is $B(\Lambda) - I_m$.

Since $\lambda \in A$, $b_{ij} = 0$ for $i \ne j$ where $1 \le j \le t$ and $1 \le i \le n$. Therefore, the image of ϕ is indeed $\mathcal{G}_m(\epsilon_1,\ldots,\epsilon_{m_1})$. For $G \in \mathcal{G}_m(\epsilon_1,\ldots,\epsilon_{m_1})$, we have

$$|\phi^{-1}(G)| = \prod_{v_i\in V(G)}(2^{n_i}-1)^{\mathrm{outdeg}(v_i)},$$

as shown in the proof of Theorem 2.8 in [3]. □

Therefore, by the Burnside lemma, we have the following result.

Theorem 3.5 *The number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P with $n_1 = 1$ is*

$$\left(\frac{\sum\limits_{(\epsilon_1,\dots,\epsilon_{m_1})\in\{0,1\}^{m_1}}\sum\limits_{G\in\mathcal{G}_m(\epsilon_1,\dots,\epsilon_{m_1})}\prod\limits_{v_i\in V(G)}(2^{n_i}-1)^{\mathrm{outdeg}(v_i)}}{\prod\limits_{i=1}^{l}[(n_i+1)!]^{m_i}(m_i)!}\right)\cdot\prod_{k=1}^{n}(2^n-2^{k-1})$$

where $\mathcal{G}_m(\epsilon_1,\dots,\epsilon_{m_1})$ is the set of acyclic digraphs with m labeled nodes $\{v_1,\dots,v_m\}$ such that $\mathrm{indeg}(v_i)=0$ whenever $\epsilon_i = 1$ for $1 \le i \le m_1$.

Let A_{mr} be the number of acyclic digraphs with m labeled nodes and r edges where the labeled vertex set is $\{v_1,\dots,v_m\}$. For $\alpha \subseteq \{v_1,\cdots,v_m\}$, let A_m^α be the number of acyclic digraphs with m labeled nodes $\{v_1,\dots,v_m\}$ such that $\mathrm{indeg}(v)=0$ for all $v\in\alpha$ and A_{mr}^α be the number of such acyclic digraphs with r edges.

Corollary 3.6 *(Theorem 3.3, [3]) If $P = I^n$ then the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P is*

$$\left(\frac{\sum_{i=0}^n\binom{n}{i}2^{i(n-i)}A_i}{2^n n!}\right)\cdot\prod_{k=1}^{n}(2^n-2^{k-1}).$$

Proof Let $\alpha(\epsilon_1,\cdots,\epsilon_n)=\{v_i|\epsilon_i=1\}$. Then

$$|\Lambda(P)_{\chi_1^{\epsilon_1}\cdots\chi_n^{\epsilon_n}}| = \Big(\prod_{k=1}^{n}(2^n-2^{k-1})\Big)A_n^{\alpha(\epsilon_1,\cdots,\epsilon_n)}.$$

By (4) in [8], for any $\alpha\subseteq\{v_1,\cdots,v_n\}$,

$$A_n^\alpha=\sum_{r\ge 0}\sum_{k=0}^{r}\binom{|\alpha|(n-|\alpha|)}{r-k}A_{n-|\alpha|,k}.$$

Therefore, we have

$$\begin{aligned}
\sum_{(\epsilon_1,\cdots,\epsilon_n)\in\{0,1\}^n}A_n^{\alpha(\epsilon_1,\cdots,\epsilon_n)} &= \sum_{\alpha\subseteq\{v_1,\cdots,v_n\}}A_n^\alpha=\sum_{\alpha\subseteq\{v_1,\cdots,v_n\}}\sum_{r\ge0}\sum_{k=0}^{r}\binom{|\alpha|(n-|\alpha|)}{r-k}A_{n-|\alpha|,k}\\
&= \sum_{i=0}^{n}\binom{n}{i}\sum_{r\ge0}\sum_{k=0}^{r}\binom{i(n-i)}{r-k}A_{i,k}\\
&= \sum_{i=0}^{n}\binom{n}{i}\sum_{k\ge0}\Big(\sum_{r\ge k}\binom{i(n-i)}{r-k}\Big)A_{i,k}\\
&= \sum_{i=0}^{n}\binom{n}{i}\sum_{k\ge0}2^{i(n-i)}A_{i,k}=\sum_{i=0}^{n}\binom{n}{i}2^{i(n-i)}A_i
\end{aligned}$$

as desired. □

Let $P = I \times \Delta^n$ with $n \geq 2$. There are three acyclic digraphs with 2 labeled nodes $\{v_1, v_2\}$:

$$G_1 : \underset{v_1}{\bullet} \qquad \underset{v_2}{\bullet}, \qquad G_2 : \underset{v_1}{\bullet} \longrightarrow \underset{v_2}{\bullet}, \qquad \text{and} \qquad G_3 : \underset{v_1}{\bullet} \longleftarrow \underset{v_2}{\bullet}.$$

Since $m_1 = 1$ in the formula of Theorem 3.5, we have $\mathcal{G}(0) = \{G_1, G_2, G_3\}$ and $\mathcal{G}(1) = \{G_1, G_2\}$. Thus, we obtain:

Corollary 3.7 *(Theorem 4.2, [2]) If $P = I \times \Delta^n$ with $n \geq 2$, the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P is*

$$\Big(\frac{2^n + 3}{2(n+1)!}\Big) \prod_{t=1}^{n+1} (2^{n+1} - 2^{t-1}).$$

In a similar way, by listing the acyclic digraphs of 3 vertices, one can obtain the following result due to Chen and Wang.

Corollary 3.8 *(Theorem 4.1, [2]) If $P = I \times \Delta^n \times \Delta^m$ then the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P is*

1. $\dfrac{\prod_{t=1}^{n+m+1}(2^{n+m+1} - 2^{t-1})}{2(n+1)!(m+1)!} \Big(2^{2n+m} + 2^{n+2m} + 2^{2n} + 2^{2m} + 3 \cdot 2^{n+1} + 3 \cdot 2^{m+1} - 2^{n+m} - 7\Big)$ *if* $1 < n < m$,

2. $\dfrac{\prod_{t=1}^{n+m+1}(2^{n+m+1} - 2^{t-1})}{4(n+1)!(m+1)!} \Big(2^{3n+1} + 2^{2n} + 3 \cdot 2^{n+2} - 7\Big)$ *if* $1 < n = m$,

3. $\dfrac{\prod_{t=1}^{n+2}(2^{n+2} - 2^{t-1})}{8(m+1)!} \Big(3 \cdot 2^{2m} + 3 \cdot 2^{m+2} + 8\Big)$ *if* $1 = n < m$.

4. The number of weakly equivariant homeomorphism classes

By Theorem 2.4 the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a simple polytope P is equal to the size of the double coset on $\Lambda(P)$ by $GL(n, \mathbb{Z}_2)$ and $\mathrm{Aut}(\mathcal{F}(P))$. Therefore, the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P = \prod_{i=1}^{l} P_i$, where $P_i = \Delta_1^{n_i} \times \cdots \times \Delta_{m_i}^{n_i}$ with $1 \leq n_1 < n_2 < \cdots < n_l$, $\sum_{i=1}^{l} m_i = m$ and $\sum_{i=1}^{l} n_i m_i = n$, is

$$|A(P) / \prod_{i=1}^{l} (S_{n_i+1} \wr S_{m_i})|$$

where $A(P) = A(0, \cdots, 0)$.

Consider the subgroup $H = \prod_{i=1}^{l} S_{m_i} \leq \mathrm{Aut}(\mathcal{F}(P))$. Note that an element $\mu \in S_{m_i}$ acts on $A(P)$ by $\lambda \cdot \mu = \lambda_\mu$ where $\lambda_\mu \in A(P)$ corresponds to a class represented by the characteristic function that sends $F_{q,r}^p$ to $\lambda(F_{\mu(q),r}^i)$ if $p = i$ and to $\lambda(F_{q,r}^p)$ otherwise. Let $\overline{\mu} \in S_n$ be the permutation that sends

$m_1n_1+\cdots+m_{i-1}n_{i-1}+(j-1)n_i+k$ to $m_1n_1+\cdots+m_{i-1}n_{i-1}+(\mu(j)-1)n_i+k$ for $1\le j\le m_i$, $1\le k\le n_i$ and fixes other elements. Then the matrix Λ_μ corresponding to λ_μ is

$$P(\overline{\mu})^{-1}\Lambda P(\mu)$$

where $P(\sigma)$ denotes the permutation matrix corresponding to a permutation σ. This is the conjugation action of $H=\prod_{i=1}^{l} S_{m_i}\le S_m$ on the set of $(m\times m)$-vector matrices. It corresponds to an action of $H\le S_m$ on the acyclic digraph with m-labeled nodes $\{v_1,\cdots,v_m\}$ given by

$$\mu\cdot v_j=\begin{cases} v_{\mu(j)}, & \text{if } m_1+\cdots+m_{i-1}+1\le j\le m_1+\cdots+m_i \\ v_j, & \text{otherwise}\end{cases}$$

for any $\mu\in S_{m_i}$. Therefore, when $l=1$, we have the following generalization of Theorem 4.1 in [3].

Theorem 4.1 *The number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P=\Delta_1^k\times\cdots\times\Delta_m^k$ with $mk=n$ is less than or equal to*

$$\sum_{G\in\overline{\mathcal{G}}_m}(2^k-1)^{|E(G)|}$$

where $\overline{\mathcal{G}}_m$ is the set of acyclic digraphs with m unlabeled nodes and $E(G)$ is the set of edges of the graph G.

Corollary 4.2 *The number of homeomorphism classes of small covers over $P=\Delta_1^k\times\cdots\times\Delta_m^k$ with $mk=n$ is less than or equal to*

$$\sum_{G\in\overline{\mathcal{G}}_m}(2^k-1)^{|E(G)|}$$

where $\overline{\mathcal{G}}_m$ is the set of acyclic digraphs with m unlabeled nodes.

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