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Connection between bi^{s}nomial coefficients and their analogs and symmetric functions

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Abstract: In this paper, on one hand, we propose a new type of symmetric function to interpret the bi^{s}nomial coefficients and their analogs. On other hand, according to this function, we give an interpretation of these coefficients by lattice paths and tiling. Some identities of these coefficients are also established. This work is an extension of the results of Belbachir and Benmezai's "*A q-analogue for bi^{s}nomial coefficients and generalized Fibonacci sequences*".

Key words: Bi^{s}nomial coefficients, symmetric functions, lattice paths, tiling

1. Introduction

There are many generalization possibilities of the Pascal triangle, such as:

- Pascal-pyramids. Let $r \geq 2$ denote an integer (the dimension) and consider the map $p : \mathbb{N}^r \to \mathbb{N}$,

$$(n_1, \ldots, n_r) \mapsto \binom{n_1 + \cdots + n_r}{n_1, \ldots, n_r}.$$

  The map p provides the number of ways of splitting a set of $n_1 + \cdots + n_r$ distinguishable objects into pairwise disjoint subsets S_i of cardinality $n_i, i = 1, \ldots, r$. When $r = 2$, the map returns the usual binomial coefficients in the Pascal triangle (for more details, see [6, 7]).

- Arithmetic triangles with Pascal's rule. It is the term that was used for the original triangle by Pascal himself. Now let the real sequences a_n and b_n be given with $a_0 = b_0$. Ensley [8] defined the object of the generalized arithmetic triangle for a_n and b_n as follows. Let $G(n, 0) = a_n, G(n, n) = b_n$, and

$$G(n, k) = G(n-1, k-1) + G(n-1, k) \text{ if } 1 \leq k \leq n-1.$$

  Belbachir and Szalay [5] established a more general concept of generalized arithmetic triangles.

- Hyperbolic Pascal triangles. Belbachir et al. [4] introduced a new generalization of the Pascal triangle. It is called the hyperbolic Pascal triangle since the mathematical background goes back to regular mosaics on

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the hyperbolic plane. They described precisely the procedure of how to obtain a given type of hyperbolic Pascal triangle from a mosaic. Németh [11] introduced this type in Pascal pyramids.

- s-Pascal triangles. The elements $\binom{n}{k}_s$ of this triangle have the following combinatorial interpretation. The term $\binom{n}{k}_s$ assigns the number of different ways of distributing k uniform objects among n boxes, where each box may contain at most s objects, $0 \leq k \leq sn$ (see, for instance, [3, 6, 7]).

A natural extension of the binomial coefficient is the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!}, \tag{1.1}$$

where $[m]_q! = [1]_q[2]_q \cdots [m]_q$ and $[i]_q = 1 + q + \cdots + q^{i-1}$.

A further generalization of the binomial coefficient is the p,q-binomial

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}, \tag{1.2}$$

where $[m]_{p,q}! = [1]_{p,q}[2]_{p,q} \cdots [m]_{p,q}$ and $[i]_{p,q} = p^{i-1} + p^{i-2}q + \cdots + q^{i-1}$.

Clearly, the p,q-binomial coefficient reduces to the q-binomial coefficient when $p = 1$.

The s-Pascal triangle is the triangle given by the ordinary multinomials or the bi^{s}nomial coefficients (see, for instance, [1, 6, 15]): let $s \geq 1$ and $n \geq 0$ be two integers. For $k = 0, 1, ..., sn$, the bi^{s}nomial coefficient $\binom{n}{k}_s$ is defined as the kth coefficient in the development

$$(1 + x + x^2 + \cdots + x^s)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k}_s x^k, \tag{1.3}$$

with $\binom{n}{k}_s = 0$ for $k > sn$ or $k < 0$.

Some readily well-known established properties are:

- the symmetry relation

$$\binom{n}{k}_s = \binom{n}{sn-k}_s, \tag{1.4}$$

- the longitudinal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^{s} \binom{n-1}{k-j}_s. \tag{1.5}$$

These coefficients, as for usual binomial coefficients, are built as for the Pascal triangle, known as an "s-Pascal triangle". One can find the first values of the s-Pascal triangle in the work of Sloane [14] as A027907 for $s = 2$.

Table 1. Triangle of trinomial coefficients: $s = 2$.

$n\backslash k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	9	*10*	*11*	*12*
0	1												
1	1	1	1										
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	90	126	141	126	90	50	21	6	1
7	1	7	28	77	161	266	357	393	357	266	161	77	$\cdots$

There are at least two different definitions of q-analogues for bi^{s}nomial coefficients. The first one was suggested by Andrews and Baxter [1], while the second one was given by Belbachir and Benmezai [2]. The latter authors proposed the q-bi^{s}nomial coefficient of the kth term of the product.

$$\prod_{j=0}^{n-1}\left(1+q^j z+\cdots+(q^j z)^s\right)=\sum_{k=0}^{sn}\begin{bmatrix} n \\ k \end{bmatrix}^{(s)} z^k. \tag{1.6}$$

The Young diagram (sometimes just called the diagram) of a partition λ is a left-justified array of squares, with λ_i squares in the ith row. For instance, the Young diagram of $(5,3,1)$ is as shown in Figure 1a.

(a) The Young diagram. (b) The Ferrer diagram.

Figure 1. The Young diagram and Ferrer diagram.

If dots are used instead of boxes, then the resulting diagram is called a Ferrer diagram. Thus, the Ferrer diagram of $(5,3,1)$ is as shown in Figure 1b. For more details, we refer the reader to [12, 16].

Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. The elementary homogeneous symmetric function of degree k in $x_1, x_2, \ldots, x_n$ is defined by

$$e_k(n) := e_k(x_1, x_2, \ldots, x_n) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k},$$

where $e_0(n) = 1$ and $e_k(n) = 0$ for $k > n$. Set $e_k(n) = 0$ unless $k, n \le 0$, and $e_k(0) = \delta_{0,k}$ where $\delta_{0,k}$ is the Kronecker delta. Then for $n \ge 1$ and $k \in \mathbb{Z}$,

$$e_k(n) = e_k(n-1) + x_n e_{k-1}(n-1). \tag{1.7}$$

For more details, see, for instance, [10, 13].

The q-binomial coefficients and p,q-binomial coefficients can be expressed respectively as specializations of symmetric function:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{-\binom{k}{2}} e_k(1, q, \ldots, q^{n-1}),$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{-\binom{k}{2}} q^{-\binom{k}{2}} e_k(p^{n-1}, p^{n-2}q, \ldots, q^{n-1}).$$

The object of this paper is to interpret the bi^{s}nomial coefficients and their analogues by symmetric functions, lattice paths, and tiling. In Section 2, we define a new symmetric function so that these coefficients can be expressed as specializations of this function. In Section 3, we interpret these coefficients by lattice path. In Section 4, we get some identities for q-bi^{s}nomial coefficients with simple proofs using lattices paths. In the last section, we give a tilling interpretation of q-bi^{s}nomial coefficients.

2. Connection between bi^{s}nomial coefficients and symmetric function

Several extensions and commentaries about these coefficients have been investigated in the literature; for example, Bondarenko [7] gave a combinatorial interpretation of the bi^{s}nomial coeffcients $\binom{n}{k}_s$ as the number of different ways of distributing "k" balls among "n" cells where each cell contains at most "s" balls.

If we denote by x_i the number of balls in a cell, the previous combinatorial interpretation given by Bondarenko is equivalent to evaluating the number of solutions of the system

$$\begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_n = k, \\ 0 \leq \alpha_1, \alpha_2, \ldots, \alpha_n \leq s; \end{cases} \tag{2.1}$$

see also [3].

Using (2.1) a new type of symmetric function is given to interpret the bi^{s}nomial coefficients and their analogues as follows.

Definition 2.1 *Let $s \geq 1$ be a positive integer. We define the generalized elementary symmetric function by*

$$E_k^{(s)}(n) := E_k^{(s)}(x_1, x_2, \ldots, x_n) = \sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_n=k \\ 0\leq\alpha_1,\alpha_2,\ldots,\alpha_n\leq s}} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \tag{2.2}$$

where $E_0^{(s)}(n) = 1$, $E_k^{(s)}(n) = 0$ unless $0 \leq k \leq sn$.

Example 2.2 *We take $s = 2$.*

1. *For $n = 1$, we have:*

$$E_0^{(2)}(1) = 1, \quad E_1^{(2)}(1) = x_1, \quad E_2^{(2)}(1) = x_1^2.$$

2. *For $n = 2$, we obtain:*

$$E_0^{(2)}(2) = 1, \quad E_1^{(2)}(2) = x_1 + x_2, \quad E_2^{(2)}(2) = x_1^2 + x_2^2 + x_1x_2,$$

$$E_3^{(2)}(2) = x_1^2x_2 + x_1x_2^2, \qquad E_4^{(2)}(2) = x_1^2x_2^2.$$

It is easy to see from Definition 2.1 that the generalized elementary symmetric function satisfies the following recurrence relation and boundary condition:

Proposition 2.3

$$E_k^{(s)}(n) = \sum_{j=0}^{s} x_n^j E_{k-j}^{(s)}(n-1). \tag{2.3}$$

Proof By (2.2). □

Comparing this proposition with the definition of the bi^{s}nomial coefficients, q-bi^{s}nomial coefficients, and p,q-bi^{s}nomial coefficients as simple induction proves:

Corollary 2.4 *1.* $E_k^{(s)}(1,1,\dots,1) = \binom{n}{k}_s$,

2. $E_k^{(s)}(1,q,\dots,q^{n-1}) = \begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)}$,

3. $E_k^{(s)}(p^{n-1},p^{n-2}q,\dots,q^{n-1}). = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q}^{(s)}$.

Relation (2.3) is a restatement of the longitudinal relation

$$\binom{n}{k}_s = \sum_{j=0}^{s} \binom{n-1}{k-j}_s .$$

Taking $(x_1,x_2,\dots,x_n) = (1,q,\dots,q^{n-1})$ in Proposition 2.3, we obtain the result of Belbachir and Benmezai [2] (Theorem 2.5, Relation (13)).

Corollary 2.5 *The q-bi^{s}nomial coefficients satisfy the following recursion:*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} = \sum_{j=0}^{s} q^{(n-1)j} \begin{bmatrix} n-1 \\ k-j \end{bmatrix}_q^{(s)} . \tag{2.4}$$

These coefficients, as for usual bi^{s}nomial coefficients, are built through the s-Pascal triangle, known as the "q-analogue of the s-Pascal triangle".

As an illustration of recurrence relation (2.4), we establish the triangle of q-trinomial coefficients.

Table 2. Table values of q-trinomial coefficients.

n/k	0	1	2	3	4
0	1				
1	1	1	1		
2	1	$1+q$	$1+q+q^2$	$q+q^2$	q^2
3	1	$1+q+q^2$	$1+q+2q^2+q^3+q^4$	$\cdots$	$\cdots$
4	1	$1+q+q^2+q^3$	$1+q+2q^2+2q^3+2q^4+q^5+q^6$	$\cdots$	

By setting $(x_1,x_2,\dots,x_n) = (p^{n-1},p^{n-2}q,\dots,q^{n-1})$ in Proposition 2.3, we obtain the following result.

Corollary 2.6 *The p,q-bi^{s}nomial coefficients satisfy the following recursion*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}^{(s)} = \sum_{j=0}^{s} p^{k-j} q^{(n-1)j} \begin{bmatrix} n-1 \\ k-j \end{bmatrix}_{p,q}^{(s)}. \tag{2.5}$$

These coefficients, as for usual bi^{s}nomial coefficients, are built through the s-Pascal triangle, known as the "p,q-analogue of the s-Pascal triangle".

As an illustration of recurrence relation (2.5), we establish the triangle of p,q-trinomial coefficients.

Table 3. Table values of p,q-trinomial coefficients.

n/k	0	1	2	3	4
0	1				
1	1	1	1		
2	1	$p+q$	p^2+pq+q^2	p^2q+pq^2	p^2q^2
3	1	p^2+pq+q^2	$p^4+p^3q+2p^2q^2+pq^3+q^4$	$\cdots$	$\cdots$
4	1	$p^3+p^2q+pq^2+q^3$	$\cdots$		

Remark 2.7 *By setting $s=1$ in Definition 2.1 we obtain immediately the elementary homogeneous symmetric function $e_k(n) = e_k(x_1, x_2, \dots, x_n)$, which gives the binomial coefficients and their analogues.*

3. Interpretation of bi^{s}nomial coefficients and their analogues by lattice paths

In this section, we first interpret the bi^{s}nomial coefficient $\binom{n}{k}_s$ using the number of lattice paths between two points as follows.

Theorem 3.1 *For $0 \le k \le sn$, let $u_1 = (0,0)$ and $v_1 = (k, n-1)$ be two points. The number of lattice paths from u_1 to v_1 taking at most s vertices in the eastern direction (east-north) is exactly the bi^{s}nomial coefficient $\binom{n}{k}_s$.*

Proof We know that the bi^{s}nomial coefficients are specializations of the generalized elementary symmetric function. Thus, it suffices to interpret this function by lattice paths.

It is easy to see that the generalized elementary symmetric function is a weight-generating function of lattice paths between two points. For each unit variable x_i in $E_k^{(s)}(n)$ we associate one unit horizontal (east) vertex, and if we suppose that each lattice path starting in $u_1 = (0,0)$ then it ends in $v_1 = (k, n-1)$ with at most s vertices in the eastern direction. Hence, the bi^{s}nomial coefficient is the number of lattice paths associated to $E_k^{(s)}(n)$. Figure 2 shows the lattice path interpretation for $n=3$, $s=2$, and $k=4$. □

By setting $s=1$ in Theorem 3.1, we will have the following result.

Corollary 3.2 *The number of paths from u_1 to v_1 taking at most $s=1$ vertices in the eastern direction is exactly the binomial coefficient $\binom{n}{k}$.*

Knuth [9] proved that the q-binomial coefficient is the polynomial in q obtained by q-counting partitions whose Ferrers diagram fits in a $k \times (n-k)$ box. That is,

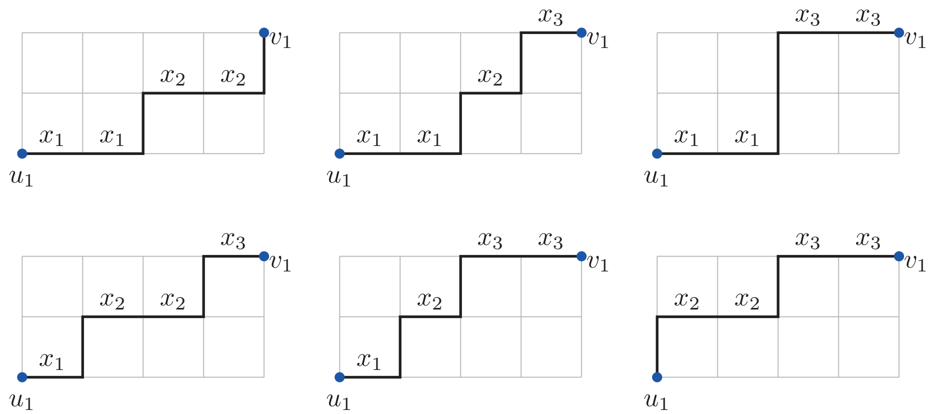

Figure 2. The six paths from $(0,0)$ to $(4,2)$. Note that these paths are associated to $E_4^{(2)}(3) = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2$.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subset (n-k)^k} q^{|\lambda|}, \tag{3.1}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, and $|\lambda| = \sum \lambda_i$.

Now suppose we have to place unit squares below (above, respectively) and to the right (left, respectively) in the lattice paths defined by Theorem 3.1. We establish two results as follows.

Proposition 3.3 *Let the nonnegative integers n, k, and $s \geq 1$. The q-bi^{s}nomial (resp. p,q-bi^{s}nomial) number is the generating function in the variable q (resp. in the two variables p and q) for the number of integer partitions with at most k parts of which there are at most s successive parts equal, the largest part at most $n-1$ and with into $k \times (n-1)$ boxes. That is,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} = \sum_{\lambda \subset (n-1)^k} q^{|\lambda|} \tag{3.2}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}^{(s)} = \sum_{\lambda \subset (n-1)^k} p^{|\lambda^c|} q^{|\lambda|}, \tag{3.3}$$

respectively, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq n-1$, $\lambda_i \neq \lambda_{i+s}$ for $i \geq 1$, $\lambda_i^c = (n-1) - \lambda_i$, and $|\lambda| = \sum \lambda_i$.

Proof We use an associated lattice path interpretation. Each partition that fits into a box of size $k \times (n-1)$ is uniquely determined by a lattice path from $(0,0)$ to $(k, n-1)$ defined in Theorem 3.1 and we simply take the weight of the lattice path to be the weight of its associated partition. Figure 3 shows the partition/lattice path interpretation for $n = 3$, $s = 2$, and $k = 4$. □

Hence, we obtain a new interpretation for the q-binomial (p,q-binomial respectively) coefficients as follows.

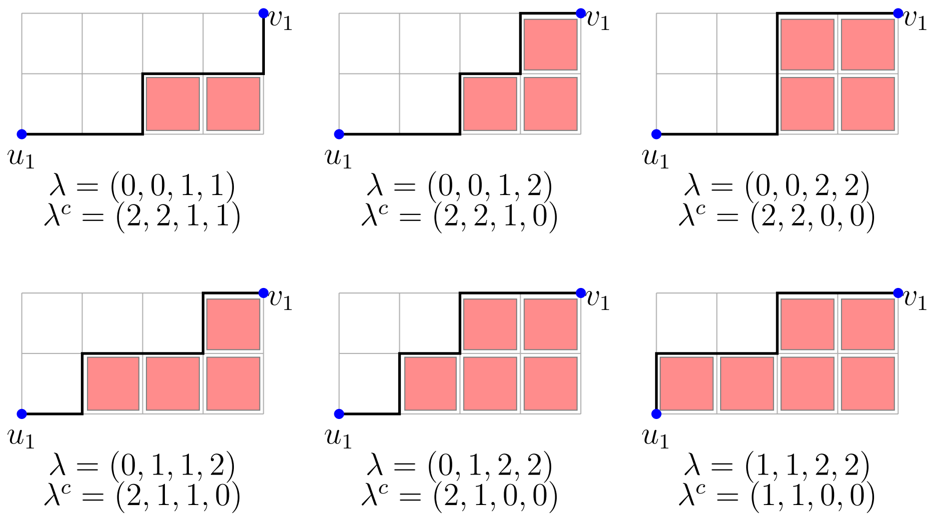

Figure 3. The six lattice paths from $(0,0)$ to $(4,2)$. The exponent on q (on p, respectively) in the weight of each path is given by counting the number of boxes that fit below (above, respectively) and to the right (left, respectively) of the lattice path.

Corollary 3.4 *The q-binomial (p,q-binomial respectively) number is the generating function for the number of integer partitions with at most k parts, the largest part at most $n-1$ and with into $k\times(n-1)$ boxes. That is,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{-\binom{k}{2}} \sum_{\lambda\subset(n-1)^k} q^{|\lambda|}, \tag{3.4}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{-\binom{k}{2}} q^{-\binom{k}{2}} \sum_{\lambda\subset(n-1)^k} p^{|\lambda^c|} q^{|\lambda|}, \tag{3.5}$$

where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ *with* $0\le\lambda_1<\lambda_2<\cdots<\lambda_k\le n-1$, $\lambda_i^c=(n-1)-\lambda_i$ *and* $\mid\lambda\mid=\sum\lambda_i$.

4. Some identities of q-bi^{s}nomial coefficients with simple proofs

In this section, we give two identities for the q-bi^{s}nomial coefficients whose proofs are straightforward using the lattice paths and Relation (1.6) for the first and the second identity, respectively.

Proposition 4.1 *For $n\ge 0$ and $s\ge 1$ we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} = q^{(n-1)k-\frac{sn(n-1)}{2}} \begin{bmatrix} n \\ sn-k \end{bmatrix}_q^{(s)}. \tag{4.1}$$

Proof Let

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} = \sum_P q^{w(P)}, \tag{4.2}$$

$$\begin{bmatrix} n \\ sn-k \end{bmatrix}_q^{(s)} = \sum_{P'} q^{w(P')}, \tag{4.3}$$

where:

- $w(P)$ is the weight of each path P from point $(0,0)$ to point $(k,n-1)$ with at most s steps in the eastern direction,
- $w(P')$ is the weight of each path P' from point $(0,0)$ to point $(sn-k,n-1)$ with at most s steps in the eastern direction.

Assume that:

- the highest weight is associated to path starting in $(0,0)$ and ending in $(sn,n-1)$, i.e. $q^{\frac{sn(n-1)}{2}}$;
- the total weight of the grid between point $(0,0)$ and point $(k,n-1)$ is $q^{(n-1)k}$.

Hence, for each path P there exists only path P' where

$$q^{w(P')} = q^{\frac{sn(n-1)}{2}-((n-1)k-w(P))}.$$

Since $\sum_P = \sum_{P'}$, by Theorem 3.1 and relation (1.4), one gets

$$\begin{aligned}
\begin{bmatrix} n \\ sn-k \end{bmatrix}_q^{(s)} &= \sum_{P'} q^{w(P')} = \sum_P q^{\frac{sn(n-1)}{2}-((n-1)k-w(P))} \\
&= q^{\frac{sn(n-1)}{2}-(n-1)k} \sum_P q^{w(P)} \\
&= q^{\frac{sn(n-1)}{2}-(n-1)k} \begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)},
\end{aligned}$$

and thus

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} = q^{(n-1)k-\frac{sn(n-1)}{2}} \begin{bmatrix} n \\ sn-k \end{bmatrix}_q^{(s)}.$$

□

The q-Chu–Vandermonde identity can be extended to the q-analogue of the bi^{s}nomial coefficients as follows.

Proposition 4.2 *We have*

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q^{(s)} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_q^{(s)} \begin{bmatrix} m \\ k-j \end{bmatrix}_q^{(s)} q^{n(k-j)}, \tag{4.4}$$

or

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q^{(s)} = \sum_{j=0}^{k} \begin{bmatrix} n \\ k-j \end{bmatrix}_q^{(s)} \begin{bmatrix} m \\ j \end{bmatrix}_q^{(s)} q^{nj}. \tag{4.5}$$

Proof Consider the coefficient polynomial of x^k in the product
$\prod_{j=0}^{n+m-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right)$.

By relation (2.4) this coefficient polynomial is $\begin{bmatrix} n+m \\ k \end{bmatrix}_q^{(s)}$. We also have

$$\prod_{j=0}^{n+m-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right) = \prod_{j=0}^{n-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right) \prod_{j=0}^{(m-1)}\left(1+q^{n+j}z+\cdots+\left(q^{n+j}z\right)^s\right),$$

and by applying relation (2.4) twice, on the right-hand side, we find

$$\begin{aligned}\prod_{j=0}^{n+m-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right) &= \prod_{j=0}^{n-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right) \\ &\quad \prod_{j=0}^{m-1}\left(1+q^{n+j}z+\cdots+\left(q^{n+j}z\right)^s\right) \\ &= \sum_{k=0}^{ns}\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} z^k \sum_{j=0}^{ms}\begin{bmatrix} m \\ j \end{bmatrix}_q^{(s)} (zq^n)^j \\ &= \sum_{k=0}^{2(n+m)s} z^k \sum_{j=0}^{k}\begin{bmatrix} n \\ j \end{bmatrix}_q^{(s)}\begin{bmatrix} m \\ k-j \end{bmatrix}_q^{(s)} q^{n(k-j)},\end{aligned}$$

or

$$\prod_{j=0}^{n+m-1}\left(1+q^jz+\cdots+\left(q^jz\right)^s\right) = \sum_{k=0}^{2(n+m)s} z^k \sum_{j=0}^{k}\begin{bmatrix} m \\ j \end{bmatrix}_q^{(s)}\begin{bmatrix} n \\ k-j \end{bmatrix}_q^{(s)} q^{nj}.$$

Hence, we find

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q^{(s)} = \sum_{j=0}^{k}\begin{bmatrix} n \\ j \end{bmatrix}_q^{(s)}\begin{bmatrix} m \\ k-j \end{bmatrix}_q^{(s)} q^{n(k-j)},$$

or

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q^{(s)} = \sum_{j=0}^{k}\begin{bmatrix} n \\ k-j \end{bmatrix}_q^{(s)}\begin{bmatrix} m \\ j \end{bmatrix}^{(s)} q^{nj}.$$

□

Corollary 4.3 *For $n \geq 0$ and $s \geq 1$, we have*

$$\begin{bmatrix} 2n \\ sn \end{bmatrix}_q^{(s)} = q^{\frac{sn(n-1)}{2}} \sum_{k=0}^{sn}\left(\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)}\right)^2 q^k. \tag{4.6}$$

Proof By Proposition 4.2 and Proposition 4.1, we have

$$\begin{bmatrix} 2n \\ sn \end{bmatrix}_q^{(s)} = \sum_{k=0}^{sn} \begin{bmatrix} n \\ sn-k \end{bmatrix}_q^{(s)} \begin{bmatrix} n \\ k \end{bmatrix}^{(s)} q^{nk} = q^{\frac{sn(n-1)}{2}} \sum_{k=0}^{sn} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} \right)^2 q^k.$$

□

5. A tiling interpretation of q-bi^{s}nomial coefficients

Let $T^s_{n,k}$ be the set of all tilings of an $(n+k-1)$-board using exactly k red squares and $n-1$ green squares with at most s red squares successively. Also let q^{w_T} be the weight of tiling T. For each $T \in T^s_{n,k}$, we calculate w_T as follows:

1. Assign a weight to each individual square in the tiling. A green square always receives a weight of 1. A red square has weight q^m where m is equal to the number of green squares to the left of that red square in the tiling.
2. Calculate w_T by multiplying the weight q^m of all the red squares.

Theorem 5.1 *The q-bi^{s}nomial coefficient is created by summing the weights of all tilings of $T^s_{n,k}$. That is,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} = \sum_{T \in T^s_{n,k}} q^{w_T}. \tag{5.1}$$

Proof Note that there is an obvious bijection between this tiling interpretation and the boxed partition interpretation of the q-bi^{s}nomial coefficients. To each $(n+k-1)$-tiling using k red squares and $n-1$ green squares with at most s red squares successively, create an associated lattice path from $(0,0)$ to $(k, n-1)$. Each green tile represents a move of one unit up and each red square represents a move of one unit right (see Figure 4).

Figure 4. Here we see a sample tiling and its associated lattice path. Note that the tiling receives weight q^2 and the lattice path corresponds to a partition of the number 2.

This bijection clearly gives the same number of tilings and boxed partitions. It just remains to show that the weight of the tiling and its associated lattice path are the same. To see this, note that we can calculate the weight of the lattice path by summing one column at a time. That is, since each column corresponds to a right move, for each right move, the weight of that column is given by the number of preceding up moves in the path. This is precisely how we calculate the weight of our tilings, since the weight of each red tile is determined by the number of green tiles before it. Therefore, the bijection between partitions/lattice paths and tilings is weight-preserving.

Hence, since $\left[{n \atop k}\right]_q^{(s)}$ counts the number of partitions that will fit into a box of size $k \times (n-1)$ weighted by the size of the partition, it also counts the number of $(n+k-1)$-tilings with k red squares and $n-1$ green squares weighted as described above. □

For example, the weight of the tiling rrgrrg is $q^{0+0+1+1} = q^2$, as shown in Figure 4.

Corollary 5.2 *The bi^{s}nomial coefficient $\binom{n}{k}_s$ counts the number of ways to tile a board of length $n+k-1$ using k red squares and $n-1$ green squares with at most s red squares successively.*

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