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Stability of abstract dynamic equations on time scales by Lyapunov's second method

Alaa HAMZA[1,2], **Karima ORABY**[3,*]
[1]Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Cairo University, Cairo, Egypt
[3]Department of Mathematics, Faculty of Science, Suez University, Suez, Egypt

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Abstract: In this paper, we use the Lyapunov's second method to obtain new sufficient conditions for many types of stability like exponential stability, uniform exponential stability, h-stability, and uniform h-stability of the nonlinear dynamic equation

$$x^{\Delta}(t) = A(t)x(t) + f(t,x),\ t \in \mathbb{T}_{\tau}^{+} := [\tau, \infty)_{\mathbb{T}},$$

on a time scale $\mathbb{T}$, where $A \in C_{rd}(\mathbb{T}, L(X))$ and $f : \mathbb{T} \times X \to X$ is rd-continuous in the first argument with $f(t,0) = 0$. Here X is a Banach space. We also establish sufficient conditions for the nonhomogeneous particular dynamic equation

$$x^{\Delta}(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}_{\tau}^{+},$$

to be uniformly exponentially stable or uniformly h-stable, where $f \in C_{rd}(\mathbb{T}, X)$, the space of rd-continuous functions from $\mathbb{T}$ to X. We construct a Lyapunov function and we make use of this function to obtain our stability results. Finally, we give illustrative examples to show the applicability of the theoretical results.

Key words: Lyapunov stability theory, dynamic equations, time scales

1. Introduction and preliminaries

One of the most important and useful tools for investigating the behavior of solutions of dynamic equations on a general time scale is Lyapunov's second method (Lyapunov's direct method), which was introduced by Lyapunov in 1892. Many studies used the Lyapunov technique to investigate various types of stability for the systems of dynamic equations on time scales; for instance, see [5–9, 11–17, 20]. Mukdasai and Niamsup [18] constructed appropriate Lyapunov functions and derived sufficient conditions for uniform stability, uniform exponential stability, ψ-uniform stability, and h-stability for linear time-varying systems with nonlinear perturbation on time scales. Cui [7] gave generalizations for the boundedness theorems on $\mathbb{R}^n$. Nasser et al. [19] established some sufficient conditions for the existence of the quadratic Lyapunov function that ensure the desired asymptotic convergence of trajectories. The difficulty of the Lyapunov technique is to construct a Lyapunov function. For equations with solutions with values in the Euclidean space $\mathbb{R}^n$, the situation is simpler. The Lyapunov function is usually chosen to be

$$V(t,x) = x^T P(t)x,$$

*Correspondence: koraby83@yahoo.com

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where $P(t)$ is an $n \times n$ matrix and x^T is the transpose of $x \in \mathbb{R}^n$. See [5, 19]. In the Hilbert space setting, the Lyapunov function is chosen to be

$$V(t,x) =< P(t)x, x >,$$

where $P(t)$ is a bounded linear operator on a Hilbert space and $< .,. >$ is its inner product. See [9]. In this paper, we define an appropriate Lyapunov function for the Banach space situation. See Section 3.

Throughout this paper we denote by

$$C_{rd}(\mathbb{T}, X) = \{f : \mathbb{T} \to X | f \text{ is rd-continuous } \},$$

$$C^1_{rd}(\mathbb{T}, X) = \{f : \mathbb{T} \to X | \, f, f^{\Delta} \text{ are rd-continuous } \},$$

$$C^1_{rd}(\mathbb{T} \times X, X) = \{f : \mathbb{T} \times X \to X | \, f, f^{\Delta} \text{ are rd-continuous in the first variable } \},$$

$$C_{rd} := C_{rd}(\mathbb{T}, \mathbb{R}),$$

$$C^1_{rd} := C^1_{rd}(\mathbb{T}, \mathbb{R}),$$

$$\mathcal{R} = \{f : \mathbb{T} \to \mathbb{R} | \, f \text{ is regressive, i.e. } 1 + \mu(t)f(t) \neq 0, t \in \mathbb{T}\},$$

$$\mathcal{R}^+ = \{f : \mathbb{T} \to \mathbb{R} | \, f \text{ is positively regressive, i.e. } 1 + \mu(t)f(t) > 0, t \in \mathbb{T}\},$$

and

$$\mathcal{R}^+ C_{rd} = \{f : \mathbb{T} \to \mathbb{R} | \, f \text{ is positively regressive and rd-continuous } \}.$$

The paper starts with the investigation of sufficient conditions for the boundedness of solutions and the exponential stability, uniform exponential stability, h-stability, and uniform h-stability of the abstract dynamic equation

$$x^{\Delta}(t) = F(t,x), \; x(\tau) = x_\tau \in X, \; t \in \mathbb{T}^+_\tau := [\tau, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $F : \mathbb{T} \times X \to X$ is rd-continuous in the first argument with $F(t,0) = 0$. Here, $\mathbb{T}$ is a time scale and X is a Banach space. Thereafter, we construct a Lyapunov function and make use of this function to study the stability of the abstract homogeneous equation

$$x^{\Delta}(t) = A(t)x(t), \; t \in \mathbb{T}^+_\tau, \tag{1.2}$$

and its perturbed equation of the form

$$x^{\Delta}(t) = A(t)x(t) + f(t,x), \; t \in \mathbb{T}^+_\tau, \tag{1.3}$$

where $A(\cdot) \in C_{rd}(\mathbb{T}, L(X))$ and $f : \mathbb{T} \times X \to X$ is rd-continuous in the first argument with $f(t,0) = 0$. Also, we establish sufficient conditions for the nonhomogeneous particular dynamic equation

$$x^{\Delta}(t) = A(t)x(t) + f(t), \, t \in \mathbb{T}^+_\tau, \tag{1.4}$$

where $f \in C_{rd}(\mathbb{T}, X)$, to be uniformly exponentially stable or uniformly h-stable. For the theory of dynamic equations on time scales, we refer the reader to the very interesting monographs [2] and [3]. We organize this paper as follows. Section 2 is devoted to establishing characterizations for many types of stability like exponential stability and uniform exponential stability in the sense of Lyapunov's second method. These characterizations are inspired by those in [17] and [21], but for Eq. (1.1) in Banach spaces with small modifications. We end this section by obtaining new sufficient conditions for h-stability and uniform h-stability of the abstract Eq. (1.1). In Section 3, we obtain our main results. We construct a Lyapunov function to study the uniform exponential stability and uniform h-stability for Eqs. (1.2), (1.3), and (1.4). We supply this paper with illustrative examples to demonstrate the applicability of the theoretical results.

2. Lyapunov stability theory

In this section, we introduce the concepts of exponential stability, uniform exponential stability, h-stability, and uniform h-stability. See [1, 2, 4, 5, 8, 10, 17]. These concepts involve the boundedness of solutions of the nonregressive dynamic equations. We develop the theory of stability of systems of dynamic equations on time scales to dynamic equations in Banach spaces. We obtain new sufficient conditions for the types of stability mentioned above.

Definition 2.1 *Consider the dynamic equation*

$$x^{\Delta}(t) = F(t,x),\ x(\tau) = x_\tau \in X,\ t \in \mathbb{T}^+_\tau, \tag{2.1}$$

where $F : \mathbb{T} \times X \to X$ *is rd-continuous in the first argument with* $F(t,0) = 0$.

(i) A solution $x(t)$ *of Eq. (2.1) is said to be bounded if there is a constant* $\vartheta(\tau, x_\tau)$ *that depends on* τ *and* x_τ *such that*

$$\| x(t) \| \leq \vartheta(\tau, x_\tau),\ t \in \mathbb{T}^+_\tau.$$

(ii) We say that the family of solutions of Eq. (2.1) is uniformly bounded if ϑ *is independent on* τ.

(iii) Eq. (2.1) is called exponentially stable if there exists $\alpha > 0$ *with* $-\alpha \in \mathcal{R}^+$ *and there is* $\beta : X \times \mathbb{T} \to \mathbb{R}^+$ *nonnegative function such that any solution* $x(t) = x(t, \tau, x_\tau)$ *of Eq. (2.1) satisfies*

$$\| x(t) \| \leq \beta(x_\tau, \tau) e_{-\alpha}(t, \tau),\ t \in \mathbb{T}^+_\tau.$$

(iv) Eq. (2.1) is called uniformly exponentially stable if β *is independent on* $\tau \in \mathbb{T}$.

(v) Let $h : \mathbb{T} \to \mathbb{R}$ *be a positive bounded function. We say that Eq. (2.1) is* h*-stable if there exists* $\theta : X \times \mathbb{T} \to \mathbb{R}^{\geq 1}$ *such that any solution* $x(t)$ *of Eq. (2.1) satisfies*

$$\| x(t) \| \leq \theta(x_\tau, \tau) h(t) h(\tau)^{-1},\ t \in \mathbb{T}^+_\tau$$

for any initial value x_τ.

(vi) Let $h : \mathbb{T} \to \mathbb{R}$ *be a positive bounded function. We say that Eq. (2.1) is uniformly* h*-stable if* θ *is independent on* $\tau \in \mathbb{T}$.

2.1. Boundedness of solutions

Liu in [17] showed that the results of [20] are true when $X = \mathbb{R}^n$. In this section our aim is to ensure that these results are true when X is a general Banach space. We generalize and improve the results of [9, 17, 20, 21].

Theorem 2.2 *Let* p *and* s *be positive constants. Assume there exists a positive definite function* $V \in C^1_{rd}(\mathbb{T} \times X, \mathbb{R}^+)$ *that satisfies the following conditions:*

(i) $\lambda(t)\|x\|^p \leq V(t,x)$, *for some positive nondecreasing function* λ*;*

(ii) $V^{\Delta}(t,x) \leq -b(t)V^s(t,x) + l(t)$, *for some positive function* b *with* $-b \in \mathcal{R}^+C_{rd}$ *and* $l \in C_{rd}$*;*

(iii) $V(t,x) - V^s(t,x) \leq \gamma$, *for some* $\gamma \geq 0$*;*

(iv) $\int_{\tau}^{t} l(u)e_{\ominus(-\omega)}(\sigma(u),t)\Delta u \leq L$, *for some nonnegative constant* L*, where* $\omega := \inf_{t\in\mathbb{T}} b(t) > 0$*.*

Then all solutions of Eq. (2.1) are bounded.

Proof Let $x(t)$ be a solution of Eq. (2.1). Conditions (ii) and (iii) imply

$$\begin{aligned}
[V(t,x)e_{\ominus(-\omega)}(t,\tau)]^{\Delta} &= V^{\Delta}(t,x)e_{\ominus(-\omega)}(\sigma(t),\tau) + \ominus(-\omega)V(t,x)e_{\ominus(-\omega)}(t,\tau)\\
&\leq [-b(t)V^s(t,x) + l(t) + \omega V(t,x)]\frac{e_{\ominus(-\omega)}(t,\tau)}{1-\omega\mu(t)} \qquad (2.2)\\
&\leq [-b(t)V^s(t,x) + l(t) + \omega(V^s(t,x)+\gamma)]\frac{e_{\ominus(-\omega)}(t,\tau)}{1-\omega\mu(t)}\\
&\leq \frac{l(t)+\gamma\omega}{1-\omega\mu(t)}e_{\ominus(-\omega)}(t,\tau),\ t\in\mathbb{T}_{\tau}^{+}.
\end{aligned}$$

Then

$$\begin{aligned}
V(t,x)e_{\ominus(-\omega)}(t,\tau) &\leq V(\tau,x_\tau)e_{\ominus(-\omega)}(\tau,\tau) + \gamma(e_{\ominus(-\omega)}(t,\tau)-1) + \int_{\tau}^{t} l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u\\
&\leq V(\tau,x_\tau) + \gamma e_{\ominus(-\omega)}(t,\tau) + \int_{\tau}^{t} l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u.
\end{aligned}$$

Hence, by using (iv), we get

$$V(t,x) \leq V(\tau,x_\tau) + \gamma + L.$$

Consequently, (i) implies

$$\|x(t)\|^p \leq \frac{1}{\lambda(t)}(V(\tau,x_\tau)+L^*),$$

where $L^* = \gamma + L$. It follows that

$$\|x(t)\| \leq [\frac{1}{\lambda(\tau)}(V(\tau,x_\tau)+L^*)]^{\frac{1}{p}},\ t\in\mathbb{T}_{\tau}^{+}. \qquad (2.3)$$

Therefore, all solutions of Eq. (2.1) are bounded. □

Corollary 2.3 *Let* p, q, s, η_1*, and* η_2 *be positive constants. Assume there exists a positive definite function* $V \in C_{rd}^1(\mathbb{T}\times X,\mathbb{R}^+)$ *that satisfies the following conditions:*

(i) $\eta_1\|x\|^p \leq V(t,x) \leq \eta_2\|x\|^q$;

(ii) $V^{\Delta}(t,x) \leq -b(t)V^s(t,x) + l(t)$, *for some positive function* b *with* $-b \in \mathcal{R}^+C_{rd}$ *and* $l \in C_{rd}$*;*

(iii) $V(t,x) - V^s(t,x) \leq \gamma$, *for some* $\gamma \geq 0$;

(iv) $\displaystyle\int_\tau^t l(u)e_{\ominus(-\omega)}(\sigma(u),t)\Delta u \leq L$, *for some nonnegative constant* L, *where* $\omega := \inf_{t\in\mathbb{T}} b(t) > 0$.

Then the family of all solutions of Eq. (2.1) *is uniformly bounded with respect to the initial point* τ.

Proof From the inequality (2.3) and by using (i), we obtain

$$\|x(t)\| \leq [\frac{1}{\eta_1}(\eta_2\|x_\tau\|^q + L^*)]^{\frac{1}{p}},\, t \in \mathbb{T}_\tau^+,$$

where $L^* = \gamma + L$. Then the family of all solutions of Eq. (2.1) is uniformly bounded with respect to the initial point τ. □

Corollary 2.4 *Let* p, q, r, η_1, *and* η_2 *be positive constants. Assume there exists a positive definite function* $V \in C^1_{rd}(\mathbb{T}\times X, \mathbb{R}^+)$ *that satisfies the following conditions:*

(i) $\eta_1\|x\|^p \leq V(t,x) \leq \eta_2\|x\|^q$;

(ii) $V^\Delta(t,x) \leq -b(t)\|x\|^r + l(t)$, *for some positive function* b *with* $-\dfrac{b(\cdot)}{\eta_2^{\frac{r}{q}}} \in \mathcal{R}^+C_{rd}$ *and* $l \in C_{rd}$;

(iii) $V(t,x) - V^{\frac{r}{q}}(t,x) \leq \gamma$, *for some* $\gamma \geq 0$;

(iv) $\displaystyle\int_\tau^t l(u)e_{\ominus(-\omega)}(\sigma(u),t)\Delta u \leq L$, *for some nonnegative constant* L, *where* $\omega := \inf_{t\in\mathbb{T}} \dfrac{b(t)}{\eta_2^{\frac{r}{q}}} > 0$.

Then the family of all solutions of Eq. (2.1) *is uniformly bounded with respect to the initial point* τ.

Proof Let x be a solution of Eq. (2.1). By using (i) and (ii), we obtain

$$V^\Delta(t,x(t)) \leq -\frac{b(t)}{\eta_2^{\frac{r}{q}}}V^{\frac{r}{q}}(t,x(t)) + l(t).$$

By Corollary 2.3, the family of all solutions of Eq. (2.1) is uniformly bounded with respect to the initial point τ. □

Corollary 2.5 *Assume there exists a positive definite function* $V \in C^1_{rd}(\mathbb{T}\times X, \mathbb{R}^+)$ *that satisfies the following conditions:*

(i) $\eta_1\|x\|^2 \leq V(t,x) \leq \eta_2\|x\|^2$;

(ii) $V^\Delta(t,x) \leq -b(t)\|x\|^2$, *for some positive function* b *with* $-\frac{b(\cdot)}{\eta_2} \in \mathcal{R}^+C_{rd}$.

Then the family of all solutions of Eq. (2.1) *is uniformly bounded with respect to the initial point* τ.

2.2. Exponential stability

Now we develop and generalize the results of Liu [17] by establishing new sufficient conditions for the (uniform) exponential stability of Eq. (2.1) in a Banach space X in terms of $e_{-\delta}(t,\tau)$ (with constant $\delta>0$ and $-\delta\in\mathcal{R}^+$) instead of $e_{\ominus\delta}(t,\tau)$, by using Lyapunov's second method.

Theorem 2.6 *Assume that $\mathbb{T}$ is a time scale with bounded graininess. Let p and s be positive constants. Assume there exists a positive definite function $V\in C^1_{rd}(\mathbb{T}\times X,\mathbb{R}^+)$ that satisfies the following conditions:*

(i) $\lambda(t)\|x\|^p\le V(t,x)$, *for some positive nondecreasing function* λ*;*

(ii) $V^\Delta(t,x)\le -b(t)V^s(t,x)+l(t)$, *for some positive function b and some function $l\in C_{rd}$;*

(iii) $V(t,x)-V^s(t,x)\le\gamma e_{-\delta}(t,\tau)$, *for some $\gamma>0$ and $\delta>\omega:=\inf_{t\in\mathbb{T}}b(t)>0$, with $-\delta\in\mathcal{R}^+$;*

(iv) $\displaystyle\int_\tau^t l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u\le K$, *for some nonnegative constant K.*

Then Eq. (2.1) is exponentially stable and every solution x satisfies

$$\|x(t)\|\le\Big[\frac{1}{\lambda(\tau)}(V(\tau,x_\tau)+K^*)\Big]^{\frac{1}{p}}e_{-\alpha}(t,\tau),\ t\in\mathbb{T}^+_\tau.$$

Proof Let x be a solution to Eq. (2.1). By using (ii)–(iii) and inequality (2.2), we have

$$\begin{aligned}[V(t,x(t))e_{\ominus(-\omega)}(t,\tau)]^\Delta&\le[-b(t)V^s(t,x(t))+l(t)+\omega(V^s(t,x(t))+\gamma e_{-\delta}(t,\tau))]\frac{e_{\ominus(-\omega)}(t,\tau)}{1-\omega\mu(t)}\\&\le l(t)\frac{e_{\ominus(-\omega)}(t,\tau)}{1-\omega\mu(t)}+\frac{\gamma\omega}{1-\omega\mu(t)}e_{-\delta\ominus(-\omega)}(t,\tau).\end{aligned}$$

This implies that

$$V(t,x(t))e_{\ominus(-\omega)}(t,\tau)\le V(\tau,x_\tau)+\frac{\gamma\omega}{\delta-\omega}+K.$$

Using (i), we obtain

$$\begin{aligned}\lambda(t)\|x(t)\|^p&\le V(t,x(t))\\&\le(V(\tau,x_\tau)+K^*)e_{-\omega}(t,\tau),\end{aligned}$$

where $K^*=\frac{\gamma\omega}{\delta-\omega}+K$. Hence,

$$\|x(t)\|^p\le\frac{1}{\lambda(t)}(V(\tau,x_\tau)+K^*)e_{-\omega}(t,\tau),$$

and

$$\|x(t)\|\le\Big[\frac{1}{\lambda(\tau)}(V(\tau,x_\tau)+K^*)\Big]^{\frac{1}{p}}e_{\ominus(\frac{\omega}{p})}(t,\tau).$$

In view of

$$\ominus\frac{\omega}{p}\le-\frac{\frac{\omega}{p}}{1+\frac{\omega}{p}\|\mu\|_\infty},$$

we obtain $e_{\ominus\frac{\omega}{p}}(t,\tau) \leq e_{-\alpha}(t,\tau)$, where $\alpha = \frac{\omega}{p}/(1+\frac{\omega}{p}\|\mu\|_\infty)$. It follows that

$$\|x(t)\| \leq [\frac{1}{\lambda(\tau)}(V(\tau,x_\tau)+K^*)]^{\frac{1}{p}} e_{-\alpha}(t,\tau),\ t \in \mathbb{T}_\tau^+.$$

Therefore, Eq. (2.1) is exponentially stable. □

Corollary 2.7 *Assume that $\mathbb{T}$ is a time scale with bounded graininess. Let p, q, and r be positive constants. Assume there exists a positive definite function $V \in C^1_{rd}(\mathbb{T}\times X, \mathbb{R}^+)$ that satisfies the following conditions:*

(i) $\lambda_1(t)\|x\|^p \leq V(t,x) \leq \lambda_2(t)\|x\|^q$, *for some positive functions λ_1, λ_2 with nondecreasing λ_1;*

(ii) $V^\Delta(t,x) \leq -b(t)\|x\|^r + l(t)$, *for some positive function b and some function $l \in C_{rd}$;*

(iii) $V(t,x) - V^{\frac{r}{q}}(t,x) \leq \gamma e_{-\delta}(t,\tau)$, *for some $\gamma > 0$ and $\delta > \omega := \inf_{t\in\mathbb{T}} \frac{b(t)}{\lambda_2^{\frac{r}{q}}(t)} > 0$, with $-\delta \in \mathcal{R}^+$;*

(iv) $\int_\tau^t l(u) e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u \leq K$, *for some nonnegative constant K.*

Then Eq. (2.1) is exponentially stable.

Proof Let x be a solution of Eq. (2.1). By using (i) and (ii), we have

$$V^\Delta(t,x(t)) \leq -\frac{b(t)}{\lambda_2^{\frac{r}{q}}(t)} V^{\frac{r}{q}}(t,x(t)) + l(t).$$

By Theorem 2.6, Eq. (2.1) is exponentially stable. □

Corollary 2.8 *Assume that $\mathbb{T}$ is a time scale with bounded graininess. Let p, q, s, η_1, and η_2 be positive constants. Assume there exists a positive definite function $V \in C^1_{rd}(\mathbb{T}\times X, \mathbb{R}^+)$ that satisfies the following conditions:*

(i) $\eta_1\|x\|^p \leq V(t,x) \leq \eta_2\|x\|^q$;

(ii) $V^\Delta(t,x) \leq -b(t)V^s(t,x) + l(t)$, *for some positive function b and some function $l \in C_{rd}$;*

(iii) $V(t,x) - V^s(t,x) \leq \gamma e_{-\delta}(t,\tau)$, *for some $\gamma > 0$ and $\delta > \omega := \inf_{t\in\mathbb{T}} b(t) > 0$, with $-\delta \in \mathcal{R}^+$;*

(iv) $\int_\tau^t l(u) e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u \leq K$, *for some nonnegative constant K.*

Then Eq. (2.1) is uniformly exponentially stable.

Proof Let x be a solution of Eq. (2.1). By using (i) and Theorem 2.6, we obtain

$$\|x(t)\| \leq [\frac{1}{\eta_1}(\eta_2\|x_\tau\|^q + K^*)]^{\frac{1}{p}} e_{-\alpha}(t,\tau),\ t \in \mathbb{T}_\tau^+,$$

where $K^* = \frac{\gamma\omega}{\delta-\omega} + K$. Then Eq. (2.1) is uniformly exponentially stable. □

Corollary 2.9 *Assume that* $\mathbb{T}$ *is a time scale with bounded graininess. Let* p, q, r, η_1*, and* η_2 *be positive constants. Assume there exists a positive definite function* $V \in C^1_{rd}(\mathbb{T}\times X, \mathbb{R}^+)$ *that satisfies the following conditions:*

(i) $\eta_1\|x\|^p \le V(t,x) \le \eta_2\|x\|^q$;

(ii) $V^\Delta(t,x) \le -b(t)\|x\|^r + l(t)$, *for some positive function* b *and some* $l \in C_{rd}$;

(iii) $V(t,x) - V^{\frac{r}{q}}(t,x) \le \gamma e_{-\delta}(t,\tau)$, *for some* $\gamma > 0$ *and* $\delta > \omega := \inf_{t\in\mathbb{T}} \frac{b(t)}{\eta_2^{\frac{r}{q}}} > 0$, *with* $-\delta \in \mathcal{R}^+$;

(iv) $\displaystyle\int_\tau^t l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u \le K$, *for some nonnegative constant* K.

Then Eq. (2.1) *is uniformly exponentially stable.*

Proof Let x be a solution of Eq. (2.1). By using (i) and (ii), we have

$$V^\Delta(t,x(t)) \le -\frac{b(t)}{\eta_2^{\frac{r}{q}}}V^{\frac{r}{q}}(t,x(t)) + l(t).$$

Then, by Corollary 2.8, Eq. (2.1) is uniformly exponentially stable. □

Theorem 2.10 *Assume that* $\mathbb{T}$ *is a time scale with bounded graininess. Let* p *and* s *be positive constants. Assume there exists a positive definite function* $V \in C^1_{rd}(\mathbb{T}\times X, \mathbb{R}^+)$ *that satisfies the following conditions:*

(i) $\lambda(t)\|x\|^p \le V(t,x)$, *for some positive nondecreasing function* λ;

(ii) $V^\Delta(t,x) \le -b(t)V^s(t,x) + l(t)$, *for some positive function* b, *that satisfies* $\omega := \inf_{t\in\mathbb{T}} b(t) > 0$ *and* $-\omega \in \mathcal{R}^+$, *and some function* $l \in C_{rd}$;

(iii) $V(t,x) - V^s(t,x) \le 0$;

(iv) $\displaystyle\int_\tau^t l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u \le K$, *for some nonnegative constant* K.

Then Eq. (2.1) is exponentially stable.

Proof Let x be a solution of Eq. (2.1). By using (ii)–(iii), and inequality (2.2), we have

$$\begin{aligned}[V(t,x(t))e_{\ominus(-\omega)}(t,\tau)]^\Delta &\le [-b(t)V^s(t,x(t)) + l(t) + \omega V^s(t,x(t))]\frac{e_{\ominus(-\omega)}(t,\tau)}{1-\omega\mu(t)}\\ &\le l(t)\frac{e_{\ominus(-\omega)}(t,\tau)}{1-\omega\mu(t)}.\end{aligned}$$

This implies that

$$V(t,x(t))e_{\ominus(-\omega)}(t,\tau) \le V(\tau,x_\tau) + K.$$

Using (i), we obtain

$$\begin{aligned}\lambda(t)\|x(t)\|^p &\le V(t,x(t))\\ &\le (V(\tau,x_\tau)+K)e_{-\omega}(t,\tau).\end{aligned}$$

Hence,

$$\|x(t)\| \leq [\frac{1}{\lambda(\tau)}(V(\tau, x_\tau) + K)]^{\frac{1}{p}} e_{\ominus(\frac{\omega}{p})}(t, \tau).$$

Again we can see $e_{\ominus\frac{\omega}{p}}(t,\tau) \leq e_{-\alpha}(t,\tau)$, where $\alpha = \frac{\omega}{p}/(1+\frac{\omega}{p}\|\mu\|_\infty)$. It follows that

$$\|x(t)\| \leq [\frac{1}{\lambda(\tau)}(V(\tau, x_\tau) + K)]^{\frac{1}{p}} e_{-\alpha}(t, \tau),\ t \in \mathbb{T}_\tau^+.$$

Therefore, Eq. (2.1) is exponentially stable. □

Corollary 2.11 *Assume that $\mathbb{T}$ is a time scale with bounded graininess. Let p, q, r, η_1, and η_2 be positive constants. Assume there exists a positive definite function $V \in C_{rd}^1(\mathbb{T}\times X, \mathbb{R}^+)$ that satisfies the following conditions:*

(i) $\eta_1\|x\|^p \leq V(t,x) \leq \eta_2\|x\|^q$;

(ii) $V^\Delta(t,x) \leq -b(t)\|x\|^r + l(t)$, *for some positive function* b, *that satisfies* $\omega := \inf_{t\in\mathbb{T}} \frac{b(t)}{\eta_2^{\frac{r}{q}}} > 0$ *and* $-\omega \in \mathcal{R}^+$, *and some* $l \in C_{rd}$*;*

(iii) $V(t,x) - V^{\frac{r}{q}}(t,x) \leq 0$;

(iv) $\int_\tau^t l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u \leq K$, *for some nonnegative constant* K*.*

Then Eq. (2.1) *is uniformly exponentially stable.*

Proof Let x be a solution of Eq. (2.1). By using (i) and (ii), we have

$$V^\Delta(t,x(t)) \leq -\frac{b(t)}{\eta_2^{\frac{r}{q}}}V^{\frac{r}{q}}(t,x(t)) + l(t).$$

By using (i) and Theorem 2.10, we obtain

$$\|x(t)\| \leq [\frac{1}{\eta_1}(\eta_2\|x_\tau\|^q + K)]^{\frac{1}{p}} e_{-\alpha}(t,\tau),\ t \in \mathbb{T}_\tau^+.$$

Then Eq. (2.1) is uniformly exponentially stable. □

Corollary 2.12 *Assume that $\mathbb{T}$ is a time scale with bounded graininess. Let η_1 and η_2 be positive constants. Assume there exists a positive definite function $V \in C_{rd}^1(\mathbb{T}\times X, \mathbb{R}^+)$ that satisfies the following conditions:*

(i) $\eta_1\|x\|^2 \leq V(t,x) \leq \eta_2\|x\|^2$;

(ii) $V^\Delta(t,x) \leq -b(t)\|x\|^2$, *for some positive function* b *with* $-\frac{b(\cdot)}{\eta_2} \in \mathcal{R}^+C_{rd}$.

Then Eq. (2.1) *is uniformly exponentially stable.*

Example 2.13 *Consider the dynamic equation*

$$x^{\Delta}(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}_{\tau}^{+},\ x(\tau) = x_{\tau} \in H, \tag{2.4}$$

where $f \in C_{rd}(\mathbb{T}, H)$, H *is a Hilbert space, and* $A : \mathbb{T} \to L(H)$, *which satisfies* $\eta_1\|x\|^2 \le \langle A(t)x, x\rangle \le \eta_2\|x\|^2$, *for some* $\eta_1, \eta_2 \in \mathbb{R}$. *If the following conditions hold:*

(1) $-\frac{\delta}{2} - (\frac{1}{4} + \frac{1}{2}\|\mu\|_{\infty}N^2) < \eta_2 < -(\frac{1}{4} + \frac{1}{2}\|\mu\|_{\infty}N^2), t \in \mathbb{T}$ *for some* $\delta \in (0, \frac{1}{\|\mu\|_{\infty}})$ ($\delta \in (0,\infty)$ *when* $\|\mu\|_{\infty} = 0$);

(2) $[\|\mu\|_{\infty} + 2(1 + \|\mu\|_{\infty}N)^2]\|f(t)\|^2 \le l(t)$, *for some* $l \in C_{rd}$ *and* N *is any bound of* $\{\|A(t)\| : t \in \mathbb{T}\}$;

(3) $\int_{\tau}^{t} l(u)e_{\ominus(-c)}(\sigma(u), \tau)\Delta u \le K$, *for some nonnegative* K, *where* $c = -(2\eta_2 + \frac{1}{2} + \|\mu\|_{\infty}N^2)$,

then Eq. (2.4) is uniformly exponentially stable.

Proof We show that, under the assumptions, the conditions of Corollary 2.8 are satisfied. Let $V(t,x) = \langle x, x\rangle$. Then

$$\begin{aligned} V^{\Delta}(t, x(t)) &= \langle x(t), x(t)\rangle^{\Delta} \\ &= \langle x^{\Delta}(t), x(t)\rangle + \mu(t)\langle x^{\Delta}(t), x^{\Delta}(t)\rangle + \langle x(t), x^{\Delta}(t)\rangle \qquad (2.5) \\ &= \langle A(t)x(t), x(t)\rangle + \langle f(t), x(t)\rangle + \mu(t)\|A(t)x(t) + f(t)\|^2 + \langle x(t), A(t)x(t)\rangle + \langle x(t), f(t)\rangle \\ &\le (2\eta_2 + \|\mu\|_{\infty}N^2)\|x(t)\|^2 + (2\|f(t)\| + 2\|f(t)\|\|\mu\|_{\infty}N)\|x(t)\| + \|f(t)\|^2\|\mu\|_{\infty}. \end{aligned}$$

By using Young's inequality ($wz \le \frac{w^p}{p} + \frac{z^q}{q}$ if $\frac{1}{p} + \frac{1}{q} = 1$), we get

$$\|x(t)\|(2\|f(t)\| + 2\|f(t)\|\|\mu\|_{\infty}N) \le \frac{\|x(t)\|^2}{2} + \frac{(2\|f(t)\| + 2\|f(t)\|\|\mu\|_{\infty}N)^2}{2}.$$

This implies that

$$\begin{aligned} V^{\Delta}(t, x(t)) &\le (2\eta_2 + \|\mu\|_{\infty}N^2 + \frac{1}{2})\|x(t)\|^2 + [\frac{(2\|f(t)\| + 2\|f(t)\|\|\mu\|_{\infty}N)^2}{2} + \|f(t)\|^2\|\mu\|_{\infty}] \\ &\le -c\|x(t)\|^2 + l(t) = -cV(t, x(t)) + l(t). \end{aligned}$$

Condition (1) implies that $\delta > c > 0$. Therefore, by Corollary 2.8, Eq. (2.4) is uniformly exponentially stable.

We consider the following concrete cases:

Case 1: If $\mathbb{T} = \mathbb{R}^{\ge 0}$, then $\mu(t) = 0$ and $\|\mu\|_{\infty} = 0$. Therefore, Eq. (2.4) is uniformly exponentially stable if

$$-\frac{\delta}{2} - \frac{1}{4} \le \eta_2 < -\frac{1}{4},$$

$$2\|f(t)\|^2 \le l(t),$$

and

$$\int_{\tau}^{t} l(u)e^{c(u-\tau)}du \leq K.$$

Case 2: If $\mathbb{T} = h\mathbb{Z}^{\geq 0}$, $h > 0$, then $\mu(t) = h$ and $\|\mu\|_\infty = h$. Therefore, Eq. (2.4) is uniformly exponentially stable if

$$-\frac{\delta}{2} - (\frac{1}{4} + \frac{h}{2}N^2) \leq \eta_2 < -(\frac{1}{4} + \frac{h}{2}N^2),$$

and conditions (2) and (3) hold.

Case 3: $\mathbb{T} = \bigcup_{k=0}^{\infty}[k(l+h), k(l+h)+l]$, l, h are positive constants. Then $\inf_{t\in\mathbb{T}} \mu(t) = 0$ and $\|\mu\|_\infty = \sup_{t\in\mathbb{T}} \mu(t) = h$. Therefore, Eq. (2.4) is uniformly exponentially stable if

$$-\frac{\delta}{2} - (\frac{1}{4} + \frac{h}{2}N^2) \leq \eta_2 < -(\frac{1}{4} + \frac{h}{2}N^2),$$

and conditions (2) and (3) hold.

For example, assuming $H = \mathbb{R}^2$ and

$$A(t) = \begin{pmatrix} e_{\ominus 8}(t,0) - 4 & 0 \\ 0 & e_{\ominus 8}(t,0) - 4 \end{pmatrix}, t \in h\mathbb{Z}^{\geq 0},$$

one can see that $-4\|x\|^2 \leq \langle A(t)x, x\rangle \leq -3\|x\|^2$, and $\|A(t)\| \leq 4$. Then Eq. (2.4) is uniformly exponentially stable when $h < 11/32$. □

2.3. h-Stability

In this part, under appropriate conditions, we establish certain estimate of solutions of Eq. (2.1). The following results are more general than the boundedness theorems. We extend and generalize the results of [7].

Theorem 2.14 *Let p, s be positive constants. Assume there exist a positive definite function $V \in C^1_{rd}(\mathbb{T} \times X, \mathbb{R}^+)$ and a bounded positive differentiable function $h : \mathbb{T} \to \mathbb{R}^+$ with nonnegative (nonpositive) derivative h^Δ that satisfy the following conditions:*

(i) $\lambda(t)\|x(t)\|^p \leq V(t, x(t))$, *for some positive nondecreasing function* λ;

(ii) $V^\Delta(t, x(t)) \leq \dfrac{h^\Delta(t)V^s(t,x)}{h(t)} + l(t)$, *for some* $l \in C_{rd}$;

(iii) $V^s(t, x(t)) - V(t, x(t)) \leq \gamma \quad (V^s(t, x(t)) - V(t, x(t)) \geq \gamma)$, *for some* $\gamma \geq 0$;

(iv) $\displaystyle\int_{\tau}^{t} \frac{l(u)h(\tau)}{h(\sigma(u))}\Delta u \leq L$, *for some nonnegative constant* L.

Then all solutions of Eq. (2.1) satisfy the estimate

$$\|x(t)\| \leq (\frac{\frac{h(t)}{h(\tau)}[V(\tau, x_\tau) + L + \gamma]}{\lambda(\tau)})^{\frac{1}{p}}, \; t \in \mathbb{T}^+_\tau. \tag{2.6}$$

Proof Let x be a solution to Eq. (2.1). Then we have

$$[\frac{V(t,x)}{h(t)}]^{\Delta} = \frac{V^{\Delta}(t,x)h(t) - V(t,x)h^{\Delta}}{h(t)h(\sigma(t))}$$

$$\leq \frac{[\frac{h^{\Delta}(t)V^{s}(t,x)}{h(t)} + l(t)]h(t) - V(t,x)h^{\Delta}(t)}{h(t)h(\sigma(t))}$$

$$= \frac{h^{\Delta}(t)[V^{s}(t,x) - V(t,x)] + l(t)h(t)}{h(t)h(\sigma(t))} \tag{2.7}$$

$$\leq \gamma\frac{h^{\Delta}(t)}{h(t)h(\sigma(t))} + \frac{l(t)}{h(\sigma(t))}$$

$$= -\gamma[\frac{1}{h(t)}]^{\Delta} + \frac{l(t)}{h(\sigma(t))}.$$

It follows that

$$\frac{V(t,x)}{h(t)} \leq \frac{V(\tau,x_{\tau})}{h(\tau)} + \gamma\frac{1}{h(\tau)} + \int_{\tau}^{t}\frac{l(u)}{h(\sigma(u))}\Delta u.$$

This implies that

$$V(t,x) \leq \frac{h(t)}{h(\tau)}[V(\tau,x_{\tau}) + \gamma + L].$$

Consequently,

$$\lambda(t)\|x(t)\|^{p} \leq \frac{h(t)}{h(\tau)}[V(\tau,x_{\tau}) + \gamma + L].$$

Since $\lambda(t) \geq \lambda(\tau)$, $t \in \mathbb{T}_{\tau}^{+}$, then

$$\|x(t)\| \leq (\frac{\frac{h(t)}{h(\tau)}[V(\tau,x_{\tau}) + \gamma + L]}{\lambda(\tau)})^{\frac{1}{p}}, \; t \in \mathbb{T}_{\tau}^{+}.$$

□

Corollary 2.15 *Let p, q, η_1, η_2, s be positive constants be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume there exist a positive definite function $V \in C_{rd}^{1}(\mathbb{T} \times X, \mathbb{R}^{+})$ and a bounded positive differentiable function $h : \mathbb{T} \to \mathbb{R}^{+}$ with nonnegative (nonpositive) derivative that satisfy the following conditions:*

(i) $\eta_1\|x(t)\|^{p} \leq V(t,x(t)) \leq \eta_2\|x(t)\|^{q}$;

(ii) $V^{\Delta}(t,x(t)) \leq \dfrac{h^{\Delta}(t)V^{s}(t,x)}{h(t)} + l(t)$, *for some* $l \in C_{rd}$;

(iii) $V^s(t,x(t)) - V(t,x(t)) \leq \gamma \quad (V^s(t,x(t)) - V(t,x(t)) \geq \gamma)$, *for some* $\gamma \geq 0$;

(iv) $\displaystyle\int_\tau^t \frac{l(u)h(\tau)}{h(\sigma(u))}\Delta u \leq L$, *for some nonnegative constant* L.

Then Eq. (2.1) is uniformly h*-stable.*

Proof By using conditions (i)–(iv) and Theorem 2.14, we obtain

$$\|x(t)\| \leq \left(\frac{\frac{h(t)}{h(\tau)}[\eta_2\|x_\tau\|^q + L + \gamma]}{\eta_1}\right)^{\frac{1}{p}},\ t \in \mathbb{T}_\tau^+.$$

Then Eq. (2.1) is uniformly h-stable. □

Corollary 2.16 *Assume there exist a positive definite function* $V \in C_{rd}^1(\mathbb{T} \times X, \mathbb{R}^+)$ *and a bounded positive differentiable function* $h : \mathbb{T} \to \mathbb{R}^+$ *that satisfy the following conditions*

(i) $\eta_1\|x(t)\|^2 \leq V(t,x(t)) \leq \eta_2\|x(t)\|^2$;

(ii) $V^\Delta(t,x(t)) \leq \dfrac{h^\Delta(t)V(t,x)}{h(t)}$.

Then Eq. (2.1) is uniformly h*-stable.*

Example 2.17 *Consider the dynamic equation*

$$x^\Delta(t) = A(t)x(t),\ t \in \mathbb{T}_\tau^+,\ x(\tau) = x_\tau \in H, \tag{2.8}$$

where $A : \mathbb{T} \to L(H)$, *which satisfies* $\rho_1\|x\|^2 \leq \langle A(t)x, x\rangle \leq \rho_2\|x\|^2$, *for some* $\rho_1, \rho_2 \in \mathbb{R}$. *Assuming there is a positive bounded differentiable function* $h : \mathbb{T} \to \mathbb{R}^+$ *such that*

$$(2\rho_2 + \|\mu\|_\infty N^2)h(t) \leq h^\Delta(t), \tag{2.9}$$

holds, then Eq. (2.8) is uniformly h*-stable. Here,* $N > 0$ *is any bound of* $\{\|A(t)\| : t \in \mathbb{T}\}$.

Proof We show that, under the assumptions, the conditions of Corollary 2.16 are satisfied. Let $V(t,x) = \langle x, x\rangle$. Conditions (i)–(ii) are satisfied when $\eta_1 = 1$, $\eta_2 = 1$. In view of equality (2.5) and relation (2.9), we have

$$\begin{aligned} V^\Delta(t,x) &= \langle Ax, x\rangle + \mu(t)\langle Ax, Ax\rangle + \langle x, Ax\rangle \\ &\leq (2\rho_2 + \|\mu\|_\infty N^2)\|x\|^2 \\ &\leq \frac{h^\Delta(t)}{h(t)}V(t,x). \end{aligned}$$

Therefore, by Corollary 2.16, Eq. (2.8) is uniformly h-stable. □

We consider the following concrete cases:

Case 1: *If* $h(t) = 5$, *then Eq. (2.8) is uniformly* 5*-stable if*

$$0 \leq \|\mu\|_\infty \leq \frac{-2\rho_2}{N^2}.$$

Case 2: *If* $h(t) = e_{-\lambda}(t,0)$, $\lambda > 0$, *then Eq. (2.8) is uniformly* $e_{-\lambda}$*-stable if the following condition holds:*

$$2\rho_2 + \|\mu\|_\infty N^2 \leq -\lambda.$$

For instance, let $\mathbb{T} = \mathbb{P}_{0.6,0.4} = \bigcup_{k=0}^{\infty}[k, k+0.6]$. *In this time scale*

$$\mu(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{k=0}^{\infty}[k, k+0.6), \\ 0.4, & \text{if } t \in \bigcup_{k=0}^{\infty}\{k+0.6\}. \end{cases}$$

Assume $A(t) = -a(t)I$, *where* $-\rho_2 \leq a(t) \leq -\rho_1$, $t \in \mathbb{T}$, $a(\cdot) \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, *and* ρ_1, $\rho_2 < 0$. *One can see that* $\rho_1\|x\|^2 \leq \langle Ax, x\rangle \leq \rho_2\|x\|^2$. *Therefore, Eq. (2.8) is uniformly* $e_{-\lambda}$*-stable when*

$$2(\rho_2 + \frac{1}{5}\rho_1^2) \leq -\lambda.$$

3. Main results

Our aim in this section is to establish the boundedness, the exponential stability, and the h-stability of the abstract homogeneous equation

$$x^\Delta(t) = A(t)x(t),\ t \in \mathbb{T}_\tau^+, \tag{3.1}$$

and its perturbed equations of the form

$$x^\Delta(t) = A(t)x(t) + f(t,x),\ t \in \mathbb{T}_\tau^+ \tag{3.2}$$

and

$$x^\Delta(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}_\tau^+, \tag{3.3}$$

under the initial condition $x(\tau) = x_\tau \in X$, where $A \in C_{rd}(\mathbb{T}, L(X))$ and $f : \mathbb{T} \times X \to X$ is rd-continuous in the first argument with $f(t,0) = 0$.

In the rest of the paper, we consider the operators $S(t)$, $W(t)$, $S_1(t) \in L(X, X^*)$ are defined as follows:

$$(S(t)x)y = P^\Delta(t)x(y) + \mu(t)P^\Delta(t)(A(t)x)(y) + \mu(t)P^\Delta(t)x(A(t)y) + \mu^2(t)P^\Delta(t)(A(t)x)(A(t)y),$$

$$(W(t)x)(y) = P(t)(A(t)x)(y) + \mu(t)P(t)(A(t)x)(A(t)y) + (P(t)x)(A(t)y),$$

and

$$(S_1(t)x)y = P^\Delta(t)x(y) + P^\Delta(t)y(x) + \mu(t)P^\Delta(t)(A(t)x)(y) + \mu(t)P^\Delta(t)y(A(t)x) + \mu(t)P^\Delta(t)(y)(y),$$

for $x, y \in X$, where $P(t) : X \to X^*$ is a linear mapping and P is differentiable with respect to $t \in \mathbb{T}$. Here X^* is the dual space of X. As usual, we say that an operator $P(t) : X \to X^*$ is greater than or equal to (resp. less than or equal to) a real number η if $P(t)x(x) \geq \eta\|x\|^2$ (resp. $P(t)x(x) \leq \eta\|x\|^2$). In this case we write $P(t) \geq \eta$ (resp. $P(t) \leq \eta$). In this section we denote the Lyapunov function by

$$V(t,x) = (P(t)x)x.$$

Theorem 3.1 *Assume that $\mathbb{T}$ is a time scale with bounded graininess. Let η_1, η_2 be positive constants and β be a positive function with $-\frac{\beta}{\eta_2} \in \mathcal{R}^+ C_{rd}$. Assume there exist $Q \in C_{rd}(\mathbb{T}, L(X, X^*))$ and $P \in C^1_{rd}(\mathbb{T}, L(X, X^*))$ such that the following conditions hold:*

(i) $\eta_1 \leq P(t) \leq \eta_2$,

(ii) $W(t) \leq -Q(t) \leq -\beta(t)$,

(iii) $S(t) \leq 0$,

Then Eq. (3.1) is uniformly exponentially stable.

Proof Let x be a solution of Eq. (3.1) and $g(t) = V(t, x(t))$. Condition (i) implies that

$$\eta_1 \|x\|^2 \leq V(t, x(t)) \leq \eta_2 \|x\|^2.$$

The delta derivative of $g(t)$ is given by

$$\begin{aligned} g^{\Delta}(t) &= (P(t)x(t))^{\Delta} x(\sigma(t)) + (P(t)x(t))(x^{\Delta}(t)) \\ &= P^{\Delta} x(t)(x(t)) + \mu(t) P^{\Delta}(t) x^{\Delta}(t) x(t) + \mu(t) P^{\Delta}(t) x(t) x^{\Delta}(t) + \mu^2(t) P^{\Delta}(t) x^{\Delta}(t) x^{\Delta}(t) \\ &\quad + P(t) x^{\Delta}(t) x(t) + \mu(t) P(t) x^{\Delta}(t) x^{\Delta}(t) + (P(t)x(t))(x^{\Delta}(t)) \end{aligned} \tag{3.4}$$

$$\begin{aligned} &= (S(t)x(t))(x(t)) + (W(t)x(t))(x(t)) \qquad (3.5) \\ &\leq -Q(t)x(t)(x(t)) \\ &\leq -\beta(t) \|x(t)\|^2. \end{aligned}$$

Then, by Corollary 2.12, Eq. (3.1) is uniformly exponentially stable. □

Theorem 3.2 *Assume there exists an operator $P \in C^1_{rd}(\mathbb{T}, L(X, X^*))$ and a bounded positive differentiable function $h : \mathbb{T} \to \mathbb{R}^+$ that satisfy the following conditions:*

(i) $\eta_1 \leq P(t) \leq \eta_2$, *for some* $\eta_1, \eta_2 \in \mathbb{R}^+$;

(ii) $W(t)h(t) \leq \alpha h^{\Delta}(t)$, *where*

$$\alpha = \begin{cases} \eta_1, & h^{\Delta}(t) \geq 0, \\ \eta_2, & h^{\Delta}(t) < 0; \end{cases}$$

(iii) $S(t) \leq 0$.

Then Eq. (3.1) *is uniformly* h*-stable.*

Proof Let x be a solution of Eq. (3.1). Then in view of equality (3.5) and conditions (ii)–(iii), we have

$$\begin{aligned} V^{\Delta}(t, x(t)) &= (S(t)x(t))(x(t)) + (W(t)x(t))(x(t)) \\ &\leq \alpha \frac{h^{\Delta}(t)\|x(t)\|^2}{h(t)} \\ &\leq \frac{h^{\Delta}(t)}{h(t)} V(t, x(t)). \end{aligned}$$

Then, by Corollary 2.16, Eq. (3.1) is uniformly h-stable. □

Theorem 3.3 *Assume that $\mathbb{T}$ is a time scale with bounded graininess. Let η_1, η_2 be positive constants and β be a positive function with $-\frac{\beta}{\eta_2} \in \mathcal{R}^+ C_{rd}$. Assume there exist $Q \in C_{rd}(\mathbb{T}, L(X, X^*))$ and $P \in C^1_{rd}(\mathbb{T}, L(X, X^*))$ such that the following conditions hold:*

(i) $\eta_1 \leq P(t) \leq \eta_2$,

(ii) $W(t) \leq -Q(t) \leq -\beta(t)$,

(iii) $(S(t)x)(x) + \mu(t)(S_1(t)x)(f(t,x)) \leq 0$, $x \in X$,

(iv) $\|f(t,x)\| \leq \lambda\|x\|$, $x \in X$, *for some* $\lambda > 0$, *which satisfies* $b(t) =: \beta(t) - 2\lambda\|P(t)\|[1 + \|\mu\|_\infty(\|A(t)\| + \frac{\lambda}{2})] > 0$,

Then Eq. (3.2) is uniformly exponentially stable

Proof In view of equality (3.4) and using (ii)–(iv), we obtain

$$\begin{aligned} V^{\Delta}(t, x(t)) &= (P^{\Delta}(t)x(t))(x(t)) + \mu(t)(P^{\Delta}(t)(A(t)x(t) + f(t, x(t)))(x(t)) + \mu(t)(P^{\Delta}(t)x(t))(A(t)x(t) + f(t, x(t))) \\ &\quad + \mu^2(t)(P^{\Delta}(t)(A(t)x(t) + f(t, x(t))))(A(t)x(t) + f(t, x(t))) + (P(t)(A(t)x(t) + f(t, x(t))))(x(t)) \\ &\quad + \mu(t)(P(t)(A(t)x(t) + f(t, x(t))))(A(t)x(t) + f(t, x(t))) + (P(t)x(t))(A(t)x(t) + f(t, x(t))) \end{aligned}$$

$$\begin{aligned} &= (S(t)x(t))(x(t)) + (W(t)x(t))(x(t)) + \mu(t)(S_1(t)x(t))(f(t, x(t))) + (P(t)f(t, x(t)))(x(t)) \\ &\quad + (P(t)x(t))(f(t, x(t))) + \mu(t)(P(t)A(t)x(t))(f(t, x(t))) + \mu(t)(P(t)f(t, x(t)))(A(t)x(t)) \\ &\quad + \mu(t)(P(t)f(t, x(t)))(f(t, x(t))) \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\leq -(Q(t)x(t))(x(t)) + 2\lambda\|P(t)\|[1 + \|\mu\|_\infty(\|A(t)\| + \frac{\lambda}{2})]\|x(t)\|^2 \\ &\leq -[\beta(t) - 2\lambda\|P(t)\|(1 + \|\mu\|_\infty(\|A(t)\| + \frac{\lambda}{2}))]\|x(t)\|^2, \end{aligned}$$

where $\beta(t) > 2\lambda\|P(t)\|(1 + \|\mu\|_\infty(\|A(t)\| + \frac{\lambda}{2}))$. In view of $-\frac{b}{\eta_2} \in \mathcal{R}^+ C_{rd}$, using Corollary 2.12, we conclude that Eq. (3.2) is uniformly exponentially stable. □

Theorem 3.4 *Assume there exists* $P \in C^1_{rd}(\mathbb{T}, L(X, X^*))$ *and a bounded positive differentiable function* $h : \mathbb{T} \to \mathbb{R}^+$ *that satisfy the following conditions:*

(i) $\eta_1 \leq P(t) \leq \eta_2$, *for some* $\eta_1, \eta_2 \in \mathbb{R}^+$;

(ii) $W(t)h(t) \leq \alpha h^\Delta(t)$, *where*

$$\alpha = \begin{cases} \eta_1, & h^\Delta(t) \geq 0, \\ \eta_2, & h^\Delta(t) < 0; \end{cases}$$

(iii) $\|f(t,x)\| \leq \lambda \|x\|, x \in X$ *for some* $\lambda > 0$;

(iv) $(S(t)x)(x) + \mu(t)(S_1(t)x)(f(t,x)) \leq -d(t)\|x\|^2$, *for some* $d(t) > 2\lambda\eta_2[1 + \|\mu\|_\infty(\|A(t)\| + \frac{\lambda}{2})]$, $t \in \mathbb{T}$.

Then Eq. (3.2) is uniformly h*-stable.*

Proof In view of equality (3.6) and using (i)–(iv), we obtain

$$\begin{aligned} V^\Delta(t, x(t)) &\leq -d(t)\|x(t)\|^2 + \alpha \frac{h^\Delta(t)}{h(t)}\|x\|^2 + 2\lambda\eta_2[1 + \|\mu\|_\infty(\|A(t)\| + \frac{\lambda}{2})]\|x(t)\|^2 \\ &\leq \alpha \frac{h^\Delta(t)}{h(t)}\|x(t)\|^2 \leq \frac{h^\Delta(t)}{h(t)} V(t, x(t)). \end{aligned}$$

Then, by Corollary 2.16, Eq. (3.2) is uniformly h-stable. □

Now we establish the boundedness, the uniform exponential stability, and the uniform h-stability of the nonhomogeneous dynamic equation

$$x^\Delta(t) = A(t)x(t) + f(t), \; t \in \mathbb{T}^+_\tau, \tag{3.7}$$

under the initial condition $x(\tau) = x_\tau \in X$, where $f \in C_{rd}(\mathbb{T}, X)$.

Theorem 3.5 *Let* η_1, η_2 *be positive constants and* β *be a positive function such that* $\beta > \frac{1}{2}$ *and* $-\frac{\beta}{\eta_2} \in \mathcal{R}^+ C_{rd}$. *Assume there exist* $Q \in C_{rd}(\mathbb{T}, L(X, X^*))$ *and* $P \in C^1_{rd}(\mathbb{T}, L(X, X^*))$ *such that the following conditions hold:*

(i) $\eta_1 \leq P(t) \leq \eta_2$, *for some* $\eta_1, \eta_2 \in \mathbb{R}^+$;

(ii) $W(t) \leq -Q(t) \leq -\beta(t)$;

(iii) $(S(t)x)(x) + \mu(t)(S_1(t)x)(f(t)) \leq 0$, $x \in X$;

(iv) $[\|f(t)\|\|P(t)\|(1 + \|\mu\|_\infty\|A(t)\|)]^2 + \|\mu\|_\infty\|f(t)\|^2\|P(t)\| \leq l(t)$, *for some* $l \in C_{rd}$;

(v) $\int_\tau^t l(u)e_{\ominus(-\omega)}(\sigma(u), t)\Delta u \leq L$, *for some nonnegative constant* L, *where* $\omega =: \inf_{t \in \mathbb{T}} \frac{\beta(t) - \frac{1}{2}}{\eta_2} > 0$.

Then the family of all solutions of Eq. (3.7) *is uniformly bounded with respect to the initial point* τ.

Proof In view of equality (3.4) and using (ii)–(iv), we obtain

$$V^{\Delta}(t,x(t)) = (S(t)x(t))(x(t)) + (W(t)x(t))(x(t)) + \mu(t)(S_1(t)x(t))(f(t)) + (P(t)f(t))(x(t)) + (P(t)x(t))(f(t)) + \mu(t)(P(t)A(t)x(t))(f(t)) + \mu(t)(P(t)f(t))(A(t)x(t)) + \mu(t)(P(t)f(t))(f(t)) \tag{3.8}$$

$$\leq -\beta(t)\|x(t)\|^2 + \frac{1}{2}\|x(t)\|^2 + \frac{1}{2}[2\|f(t)\|\|P(t)\|(1+\|\mu\|_\infty\|A(t)\|)]^2 + \|\mu\|_\infty\|f(t)\|^2\|P(t)\| \tag{3.9}$$

$$\leq -[\beta(t) - \frac{1}{2}]\|x(t)\|^2 + l(t).$$

Here, we used Young's inequality. By Corollary 2.4, the family of all solutions of Eq. (3.7) is uniformly bounded with respect to the initial point τ. □

Theorem 3.6 *Assume that $\mathbb{T}$ is a time scale with bounded graininess. Let η_1, η_2 be positive constants and β a positive function such that $\beta > \frac{1}{2}$ and $-\frac{\beta}{\eta_2} \in \mathcal{R}^+C_{rd}$. Assume there exist $Q \in C_{rd}(\mathbb{T}, L(X,X^*))$ and $P \in C^1_{rd}(\mathbb{T}, L(X,X^*))$ such that the following conditions hold:*

(i) $\eta_1 \leq P(t) \leq \eta_2$;

(ii) $W(t) \leq -Q(t) \leq -\beta(t)$;

(iii) $(S(t)x)(x) + \mu(t)(S_1(t)x)(f(t)) \leq 0$, $x \in X$;

(iv) $[\|f(t)\|\|P(t)\|(1+\|\mu\|_\infty\|A(t)\|)]^2 + \|\mu\|_\infty\|f(t)\|^2\|P(t)\| \leq l(t)$, *for some* $l \in C_{rd}$;

(v) $\displaystyle\int_\tau^t l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u \leq K$, *for some nonnegative constant* K, *where* $\omega =: \inf_{t\in\mathbb{T}} \frac{\beta(t)-\frac{1}{2}}{\eta_2} > 0$.

Then Eq. (3.7) is uniformly exponentially stable.

Proof In view of inequality (3.9) and using (ii)–(iv), we obtain

$$V^{\Delta}(t,x(t)) \leq -[\beta(t) - \frac{1}{2}]\|x(t)\|^2 + l(t).$$

By Corollary 2.11, Eq. (3.7) is uniformly exponentially stable. □

Theorem 3.7 *Assume there exist $P \in C^1_{rd}(\mathbb{T}, L(X,X^*))$ and a bounded positive differentiable function $h : \mathbb{T} \to \mathbb{R}^+$ that satisfy the following conditions:*

(i) $\eta_1 \leq P(t) \leq \eta_2$, *for some* $\eta_1, \eta_2 \in \mathbb{R}^+$;

(ii) $(W(t) + \frac{1}{2})h(t) \leq \alpha h^{\Delta}(t)$, *where*

$$\alpha = \begin{cases} \eta_1, & h^{\Delta}(t) \geq 0, \\ \eta_2, & h^{\Delta}(t) < 0; \end{cases}$$

(iii) $(S(t)x)(x) + \mu(t)(S_1(t)x)(f(t)) \leq 0$;

(iv) $[\|f(t)\|\|P(t)\|(1 + \|\mu\|_\infty \|A(t)\|)]^2 + \|\mu\|_\infty \|f(t)\|^2 \|P(t)\| \leq l(t)$, *for some* $l \in C_{rd}$;

(v) $\displaystyle\int_\tau^t \frac{l(u)h(\tau)}{h(\sigma(u))}\Delta u \leq L$, *for some nonnegative constant* L.

Then Eq. (3.7) is uniformly h*-stable.*

Proof In view of inequality (3.8) and conditions (ii)–(iv), and using Young's inequality, we obtain

$$\begin{aligned}
V^\Delta(t, x(t)) &\leq (W(t)x(t))(x(t)) + \frac{1}{2}\|x(t)\|^2 + \frac{1}{2}[2\|f(t)\|\|P(t)\|(1 + \|\mu\|_\infty \|A(t)\|)]^2 + \|\mu\|_\infty \|f(t)\|^2 \|P(t)\| \\
&\leq \alpha \frac{h^\Delta(t)}{h(t)} \|x(t)\|^2 + l(t) \\
&\leq \frac{h^\Delta(t)}{h(t)} V(t, x(t)) + l(t).
\end{aligned}$$

Then, by Corollary 2.15, Eq. (3.7) is uniformly h-stable. □

The following example shows the applicability of the theoretical results

Example 3.8 *Consider the dynamic equation*

$$x^\Delta(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}_\tau^+,\ x(\tau) = x_\tau \in \ell_2 = \{(x_n) : \Sigma_{n=1}^\infty |x_n|^2 < \infty\}, \tag{3.10}$$

where $A : \mathbb{T} \to L(l_2)$ *is defined by* $A(t) = (-1 + \frac{1}{2}e_{-2}(t, 0))I$, *and* $f(t) = \sqrt{e_{\ominus 2}(t, 0)}I$. *Here* I *is the identity operator. Let* $\mathbb{T}$ *be a time scale with bounded graininess* $0 \leq \mu(t) \leq \frac{1}{2}$. *Let* $Q(t) = 1 - \frac{3}{8}e_{-2}^2(t, 0)$. *Then* $Q(t) > \frac{5}{8}$. *Therefore, Eq. (3.10) is uniformly exponentially stable.*
Define $P(t)$ *by*

$$P(t) = (1 + \frac{1}{2}e_{-2}(t, 0))I. \tag{3.11}$$

Its derivative is

$$P^\Delta(t) = -e_{-2}(t, 0)I.$$

Take $\eta_1 = 1$, $\eta_2 = \frac{3}{2}$. *One can see that all conditions (i)–(v) of Theorem* 3.6 *hold. Therefore, Eq. (3.10) is uniformly exponentially stable. Indeed, we have*

$$\begin{aligned}
(W(t)x(t))x(t) &\leq -\frac{3}{2}(1 - \frac{1}{4}e_{-2}^2(t, 0))\|x(t)\|^2 \\
&\leq (-1 + \frac{3}{8}e_{-2}^2(t, 0))\|x(t)\|^2 \\
&= -(Q(t)x(t))(x(t)).
\end{aligned}$$

Also, it is clear that $(S(t)x)(x) + \mu(t)(S_1(t)x)(f(t)) \leq 0,\ t \in \mathbb{T},\ x \in X$,

$$\begin{aligned}
[\|f(t)\|\|P(t)\|(1 + \|\mu\|_\infty \|A(t)\|)]^2 + \|\mu\|_\infty \|f(t)\|^2 \|P(t)\| &\leq \frac{3}{2}e_{\ominus 2}(t, 0)[\frac{3}{2}(1 + \frac{1}{2})^2 + \frac{1}{2}] \\
&= \frac{93}{16}e_{\ominus 2}(t, 0) = l(t),
\end{aligned}$$

and

$$\int_{\tau}^{t} l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u = \frac{93}{16}\int_{\tau}^{t} e_{\ominus 2}(u,0)e_{\ominus(-\frac{1}{8})}(\sigma(u),\tau)\Delta u$$
$$= \frac{93}{16}e_{-\frac{1}{8}}(\tau,0)\int_{\tau}^{t}\frac{1}{1-\frac{1}{8}\mu(u)}e_{(\ominus 2)\ominus(-\frac{1}{8})}(u,0)\Delta u \le K.$$

Therefore, by Theorem 3.6, *Eq. (3.10) is uniformly exponentially stable.*
Note that:

(1) If $\mu = 0$, *then*

$$\int_{\tau}^{t} l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u = \frac{93}{16}e_{-\frac{1}{8}}(\tau,0)\int_{\tau}^{t}\frac{1}{1-\frac{1}{8}\mu(u)}e_{(\ominus 2)\ominus(-\frac{1}{8})}(u,0)\Delta u$$
$$= \frac{93}{16}e^{-\frac{1}{8}\tau}\int_{\tau}^{t} e^{-\frac{15}{8}u}du$$
$$= -\frac{93}{16}.\frac{8}{15}e^{-\frac{1}{8}\tau}[e^{-\frac{15}{8}t} - e^{-\frac{15}{8}\tau}]$$
$$\le \frac{31}{10}e^{-2\tau} \le \frac{31}{10}.$$

(2) If $\mu = \frac{1}{2}$, *then*

$$\int_{\tau}^{t} l(u)e_{\ominus(-\omega)}(\sigma(u),\tau)\Delta u = \frac{93}{16}e_{-\frac{1}{8}}(\tau,0)\int_{\tau}^{t}\frac{1}{1-\frac{1}{8}\mu(u)}e_{(\ominus 2)\ominus(-\frac{1}{8})}(u,0)\Delta u$$
$$= \frac{93}{16}e_{-\frac{1}{8}}(\tau,0)\int_{\tau}^{t}\frac{16}{15}e_{-\frac{14}{15}}(u,0)\Delta u$$
$$= -\frac{93}{16}.\frac{16}{14}e_{-\frac{1}{8}}(\tau,0)[e_{-\frac{14}{15}}(t,0) - e_{-\frac{14}{15}}(\tau,0)]$$
$$\le \frac{93}{14}e_{-\frac{7}{8}}(\tau,0) \le \frac{93}{14}.$$

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