Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1680 – 1685

doi:10.3906/mat-1704-19

A singularly perturbed differential equation with piecewise constant argument of generalized type

Marat AKHMET[1,*], **Muratkhan DAUYLBAYEV**[2], **Aziza MIRZAKULOVA**[2]
[1]Department of Mathematics, Middle East Technical University, Ankara, Turkey
[2]Department of Mechanics and Mathematics, Kazakh National University, Almaty, Kazakhstan

Received: 06.04.2017 • **Accepted/Published Online:** 06.02.2018 • **Final Version:** 24.07.2018

Abstract: The paper considers the extension of Tikhonov Theorem for singularly perturbed differential equation with piecewise constant argument of generalized type. An approximate solution of the problem has been obtained. A new phenomenon of humping has been observed in the boundary layer area. An illustrative example with simulations is provided.

Key words: Piecewise constant argument of generalized type, small parameter, singular perturbation, approximate solution, humping in the boundary layer area

1. Introduction

Singularly perturbed equations are often used as mathematical models describing processes in physics, chemical kinetics, and mathematical biology. This type of equation often arises during investigation of applied problems of technology and engineering [8, 14, 15, 18, 22]. Prandtl [23] was a pioneer to emphasize the significance of singular problems and the necessity of their appearance as mathematical models. He pointed out the importance of the subject while he was developing the theory of the boundary layer in hydrodynamics in 1904. Several scientists such as Friedrichs, Levinson, and Wazow [12, 16, 30] were then interested in singularly perturbed equations. Systematic investigation of singularly perturbed equations by many mathematicians began only after Tikhonov's proof of fundamental and well-known limit theorems for nonlinear systems of ordinary differential equations [26]. We refer to the books [20, 27, 29] for more information on the recent results on singularly perturbed equations. The initial and boundary value problems considered in the studies [9]-[11] are equivalent to the Cauchy problem with the initial jump for differential and integrodifferential equations in the stable case.

Systematic studies of theoretical and practical problems involving piecewise constant arguments were initiated in the early 1980s. Since then, differential equations with piecewise constant arguments have attracted great attention from researchers in mathematics, biology, engineering, and other fields. A mathematical model including a piecewise constant argument was first considered by Busenberg and Cooke [7] in 1982. They constructed a first-order linear equation to investigate vertically transmitted diseases. Following this work, using the method of reduction to discrete equations, many authors have analyzed various types of differential equations with piecewise constant arguments. A system of differential equation with piecewise constant argument of

*Correspondence: dmk57@mail.ru
2010 *AMS Mathematics Subject Classification:* 34A37, 34D15

generalized type was introduced in [1]–[6].

The contribution of our work relates to a new Tikhonov theorem for singularly perturbed differential equations with piecewise constant arguments.

Tikhonov-type theorems express the limiting behavior of solutions of the singularly perturbed system. It is a powerful instrument for analysis of singular perturbation problems. Due to this fact, it has been studied for many types of differential equations: partial differential equations [17], singularly perturbed differential inclusions [28], and discontinuous differential equations [25]. Although there are some papers that consider singularly perturbed differential equations with piecewise constant arguments [19, 21, 24], there are no articles that discuss the approximation problem, and there is no investigation involving generalized piecewise constant arguments. In the present study, a new interesting phenomenon of humping was observed in the boundary layer area. The hump may be considered a spike and can be applied in neural networks theory [13].

2. Main result

Consider the following system

$$\begin{aligned} \varepsilon\dot{x} &= F(x, y(\beta(t))), \\ \varepsilon\dot{y} &= Q(x, y), \end{aligned} \tag{1}$$

with the initial conditions

$$\begin{aligned} x(0, \varepsilon) &= x_0, \\ y(0, \varepsilon) &= \psi, \end{aligned} \tag{2}$$

where ε is a small positive parameter. System (1) is defined in the domain $G = \{(x, y, t) | \|x\| \leq a,\ \|y\| \leq a,\ 0 \leq t \leq T\}$, where a and T are fixed positive numbers. The functions F and Q are continuously differentiable in the interior of the domain. The piecewise constant argument is determined by the function $\beta(t) = \theta_i$, if $t \in [\theta_i, \theta_{i+1}),\ i = 1, 2, \ldots, p, 0 < \theta_1 < \theta_2 < \ldots \theta_p < T$.

The following assumptions are required throughout the paper. There exists a point (φ, ψ) in the domain such that

(C1) The function F satisfies the conditions

$$F(\varphi, \psi) = 0, \tag{3}$$

$$F_x(\varphi, \psi) < 0, \tag{4}$$

$$F_y(\varphi, \psi) < 0. \tag{5}$$

(C2) The function Q satisfies the conditions

$$Q(\varphi, \psi) = 0, \tag{6}$$

$$Q_x(\varphi, \psi) < 0, \tag{7}$$

$$Q_y(\varphi, \psi) < 0. \tag{8}$$

Theorem 2.1 *Assume that the conditions (C1) and (C2) hold. If the initial point* (x_0, ψ) *is in the domain of attraction of the fixed point* (φ, ψ), *then for sufficiently small* ε, *the problem (1)–(2) has a unique solution* $x(t,\varepsilon) = x(t,0,x_0,\psi,\varepsilon),\ y(t,\varepsilon) = y(t,0,x_0,\psi,\varepsilon)$ *such that the following limiting equalities hold:*

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = \varphi, \quad \textit{for} \quad 0 < t \le T, \tag{9}$$

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \psi, \quad \textit{for} \quad 0 < t \le T. \tag{10}$$

Proof. Without loss of generality, we assume that $x_0 < \varphi,\ y_0 = \psi$. For $t \in [0, \theta_1)$ the first equation of system (1) takes the form

$$\varepsilon \frac{dx}{dt} = F(x, \psi). \tag{11}$$

From the assumptions made above, it implies that $F(x,\psi) > 0$. Therefore, the last equation is equivalent to the equation

$$\frac{dt}{dx} = \frac{\varepsilon}{F(x,\psi)}, \tag{12}$$

and the initial condition for equation (12) is

$$t(x_0) = 0. \tag{13}$$

Setting $\varepsilon = 0$ in (12), we obtain a regular problem with the unperturbed equation

$$\frac{dt}{dx} = 0. \tag{14}$$

If ε is sufficiently small, then the problem (12)–(13) has a unique solution in the segment $[x_0, \varphi]$. The solution $t(x, x_0, \psi, \varepsilon)$ can be made arbitrarily close to the horizontal $t = 0$, i.e.

$$\lim_{\varepsilon\to 0} t(x, x_0, y_0, \varepsilon) = 0, \quad x_0 \le x \le \varphi.$$

If we consider $x(t)$ as a function of the independent variable t, then the integral curve starting at the point $(0, x_0)$ is included in the neighborhood of an arbitrary point on the line $x = \varphi$. Next, by using the condition $(C1)$, we apply the stability analysis to obtain that the first coordinate of the solution of (1)–(2) approaches the line $x = \varphi$.

Next we take into account the initial value problem

$$\varepsilon \frac{dy}{dt} = Q(x(t,\varepsilon), y), \quad y(0) = \psi, \tag{15}$$

where $x(t,\varepsilon) < \varphi$ and $x(t,\varepsilon) \to \varphi$ as $\varepsilon \to 0$. We will prove that the solution of equation (15) satisfies $y(t,\varepsilon) > \psi$ and $y(t,\varepsilon) \to \psi$ as $\varepsilon \to 0$ for a fixed $t \in (0, \theta_1]$.

By means of $(C1)$, we have

$$Q(x(t,\varepsilon), y(t,\varepsilon)) > 0 \tag{16}$$

for all t near the initial moment. However, this cannot be true for all $t > 0$. The equality

$$Q(x,y) = Q_x(x_1,y_1)(x-\varphi) + Q_y(x_2,y_2)(y-\psi), \tag{17}$$

is valid, where $x_i, y_i, i = 1, 2,$ are numbers in a neighborhood of (φ, ψ). This is why the condition $x(t) \to \varphi$ implies that $y(t)$ increases only to some moment of time $t = \xi(\varepsilon)$. The moment tends to the zero as $\varepsilon \to 0$, and the function Q is negative for $t > \xi$. Thus, there is a hump near the moment $t = \xi$. A similar discussion can be conducted for the coordinate x.

For $t \in [\theta_1, \theta_2)$ the problem (1)–(2) takes the form

$$\begin{aligned} \varepsilon \dot{x} &= F(x, y(\theta_1)), \\ \varepsilon \dot{y} &= Q(x, y), \end{aligned} \tag{18}$$

with the initial conditions

$$x(\theta_1, \varepsilon) = x(\theta_1), \quad y(\theta_1, \varepsilon) = y(\theta_1). \tag{19}$$

Since the values $x(\theta_1), y(\theta_1)$ approach the point (φ, ψ) as $\varepsilon \to 0$, the proof made above can be repeated for the interval $[\theta_1, \theta_2)$, with uniform convergence.

Proceeding in this way one can prove the limiting equalities of the Theorem in the intervals $t \in [\theta_i, \theta_{i+1}),\ \ i = 2, 3, \ldots, p.$

The theorem is proved.

3. An example

Consider the following system with piecewise constant argument

$$\begin{aligned} \varepsilon \dot{x} &= -x - y(\beta(t)) + 5, \\ \varepsilon \dot{y} &= -xy + 2x + y - 1, \end{aligned} \tag{20}$$

where $\theta_i = i, i = 1, 2, 3, 4, 5, T = 6, \varphi = 2, \psi = 3.$ Take $x_0 = 1.$

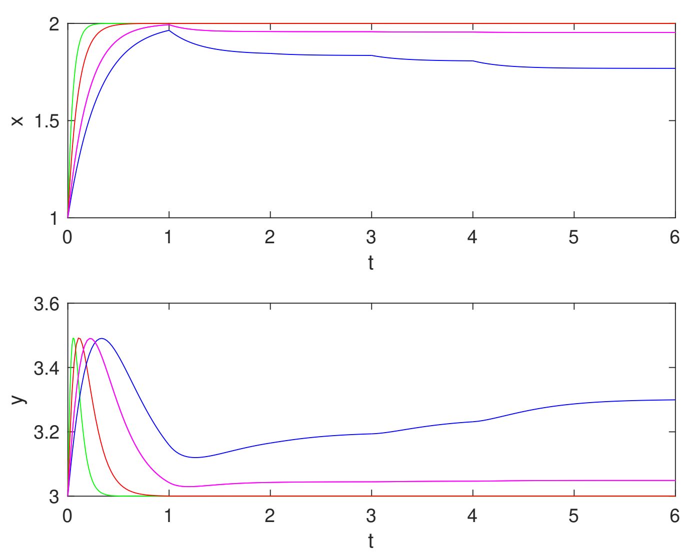

Figure 1. Coordinates $x(t), x(0) = 1,$ and $y(t)$ of the solution are in blue $(\varepsilon = 0.3)$, magenta $(\varepsilon = 0.2)$, red $(\varepsilon = 0.1)$, and green $(\varepsilon = 0.05)$.

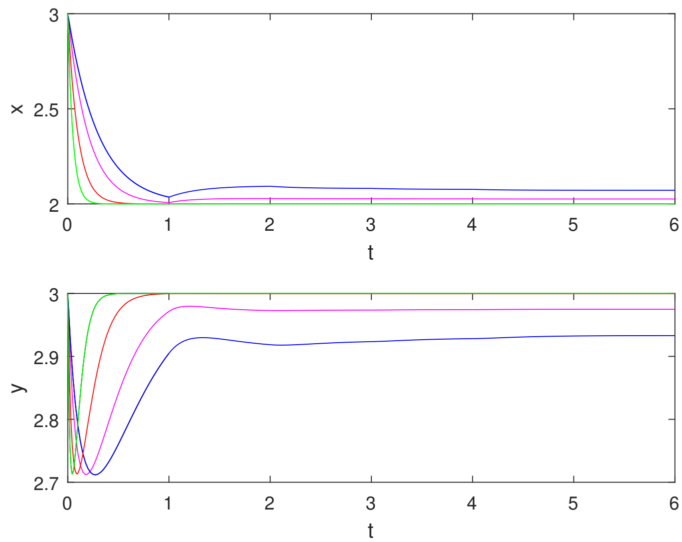

Figure 2. Coordinates $x(t), x(0) = 3$, and $y(t)$ of the solution are in blue ($\varepsilon = 0.3$), magenta ($\varepsilon = 0.2$), red ($\varepsilon = 0.1$), and green ($\varepsilon = 0.05$).

One can easily see that the conditions of Theorem 2.1 are fulfilled. Indeed, system (20) is of the form (1) with $F(x, y) = -x - y + 5, Q(x, y) = -xy + 2x + y - 1$, and $F(2, 3) = 0, Q(2, 3) = 0, F_x(2, 3) = -1 < 0, F_y(2, 3) = -1 < 0, Q_x(2, 3) = -1 < 0, Q_y(2, 3) = -1 < 0$.

We provide results of simulations with two different values of x_0. The first one with $x_0 = 1$ is considered in Figure 1. One can observe the hump in the boundary layer. Another choice with $x_0 = 3$ is simulated in Figure 2, and the hump is observed in the simulations.

Thus, according to the main assertion, the solution of the initial value problem approaches the equilibrium, as the parameter decreases to zero. In both simulations the solutions approach the equilibrium as the parameter decreases.

References

[1] Akhmet M. On the integral manifolds of the differential equations with piecewise constant argument of generalized type. In: Agarval RP, Perera K, editors. Proceedings of the Conference on Differential and Difference Equations at the Florida Institute of Technology, 1–5 August 2005, Melbourne, FL, USA, Hindawi Publishing Corporation, 2006, pp. 11-20.

[2] Akhmet M. Nonlinear Hybrid Continuous/Discrete-Time Models. Paris, France: Atlantis Press, 2011.

[3] Akhmet M. Integral manifolds of differential equations with piecewise constant argument of generalized type. Nonlinear Anal-Theor 2007; 66: 367-383.

[4] Akhmet M. On the reduction principle for differential equations with piecewise constant argument of generalized. J Math Anal Appl 2007; 336: 646-663.

[5] Akhmet M, Fen M, Kirane M. Almost periodic solutions of retarded SICNNs with functional response on piecewise constant argument. Neural Comp and Appl 2015; 27: 2483-2495.

[6] Akhmet M. Exponentially dichotomous linear systems of differential equations with piecewise constant argument. Discontinuity, Nonlinearity, and Complexity 2012; 1: 337-352.

[7] Busenberg S, Cooke KL. Models of vertically transmitted diseases with sequential-continuous dynamics. Nonlinear Phenomena in Mathematical Sciences 1982; New York, NY, USA: Academic Press, pp. 179-187.

[8] Damiano ER, Rabbitt RD. A singular perturbation model of fluid dynamics in the vestibular semicircular canal and ampulla. J Fluid Mech 1996; 307: 333-372.

[9] Dauylbayev MK, Atakhan M. The initial jumps of solutions and integral terms in singular BVP of linear higher order integro-differential equations. Miskolc Mathematical Notes 2015; 16: 747-761.

[10] Dauylbaev MK, Mirzakulova AE. Boundary value problem with initial jumps for singular perturbed integro-differential equations. Nonlinear Oscil 2016; 19: 11-21 (in Russian).

[11] Dauylbaev MK, Mirzakulova AE. Asymptotic behavior of solutions of singular integro-differential equations. Discontinuity, Nonlinearity and Complexity 2016; 5: 147-154.

[12] Friedrichs KO. Kirchhoff's boundary conditions and the edge effect for elastic plates. P S Appl Math 1950; 3: 117-124.

[13] Gerstner M, Kistler WM. Neuron Models, Single Neurons, Populations, Plasticity. Cambridge, UK: Cambridge University Press, 2002.

[14] Gondal I. On the application of singular perturbation techniques to nuclear engineering control problems. IEEE T Nucl Sci 1988; 35: 1080-1085.

[15] Hek G. Geometric singular perturbation theory in biological practice. J Math Biol 2010; 60: 347-386.

[16] Levinson N. Perturbations of discontinuous solutions of nonlinear systems of differential equations. Acta Math 1951; 82: 71-106.

[17] Kadalbajoo MK, Patidar KC. Singularly perturbed problems in partial differential equations: a survey. Appl Math Comput 2003; 134: 371-429.

[18] Kokotovic PV. Applications of singular perturbation techniques to control problems. SIAM Rev 1984; 26: 501-550.

[19] Li X, Wang Z. Global attractivity for a logistic equation with piecewise constant arguments. Fields Inst Commun 2004; 42: 215-222.

[20] O'Malley REJ. Singular Perturbation Methods for Ordinary Differential Equations. Applied Mathematical Sciences. New York, NY, USA: Springer, 1991.

[21] Marconato SAS. The relationship between differential equations with piecewise constant argument and the associated discrete equations, via dichotomous maps. Dynam Cont Dis Ser A 2005; 12: 755-768.

[22] Owen MR, Lewis MA. How predation can slow, stop, or reverse a prey invasion. B Math Biol 2001; 63: 655-684.

[23] Prandtl L. Uber Flussigkeits-bewegung bei kleiner Reibung. Verh Int Math Kongr 1905; 484-491.

[24] Rong Y. The existence of almost periodic solutions of retarded differential equations with piecewise argument. Nonlinear Anal-Theor 2002; 48: 1013-1032.

[25] Simeonov P, Bainov D. Stability of the solutions of singularly perturbed systems with impulse effect. J Math Anal Appl 1988; 136: 575 - 588.

[26] Tikhonov AN. Systems of differential equations containing small parameters in the derivatives. Matematicheskii sbornik 1952; 73: 575-586.

[27] Vasil'eva A, Butuzov V, Kalachev L. The Boundary Function Method for Singular Perturbation Problems. Society for Industrial and Applied Mathematics, 1995.

[28] Veliov V. A generalization of the Tikhonov theorem for singularly perturbed differential inclusions. J Dyn Control Syst 1997; 3: 291-319.

[29] Verhulst F. Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Timescale Dynamics. Texts in Applied Mathematics. New York, NY, USA: Springer, 2005.

[30] Wasow W. On asymptotic solution of boundary value problems for nonlinear differential equations containing a parameter. J Math Phys 1941; 23: 173-183.