Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 1686 – 1697
© TÜBİTAK
doi:10.3906/mat-1710-55

# Extended Laguerre–Appell polynomials via fractional operators and their determinant forms

**Subuhi KHAN*, Shahid Ahmad WANI**
Department of Mathematics, Aligarh Muslim University, Aligarh, India

**Received:** 16.10.2017 • **Accepted/Published Online:** 06.02.2018 • **Final Version:** 24.07.2018

**Abstract:** In this article, the extended form of Laguerre–Appell polynomials is introduced by means of generating function and operational definition. The corresponding results for the extended Laguerre-Bernoulli and Laguerre-Euler polynomials are obtained as applications. Further, the determinant forms of these polynomials are established by using operational techniques.

**Key words:** Appell polynomials, Laguerre polynomials, Laguerre–Appell polynomials, fractional calculus, operational rules, determinant definition

## 1. Introduction and preliminaries

The use of integral transforms to deal with fractional derivatives was originated by Riemann and Liouville [15, 16]. The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional derivatives; see for example [3, 10].

One of the important classes of polynomial sequences is the class of Appell polynomial sequences [2], which arises in numerous problems of applied mathematics, theoretical physics, approximation theory, and several other mathematical branches. The set of Appell sequences is closed under the operation of umbral composition of polynomial sequences. Under this operation the set of Appell sequences forms an abelian group. The Appell sequences are defined by the following generating function:

$$A(x,t) := A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}. \tag{1.1}$$

The power series $A(t)$ is given by

$$A(t) = A_0 + \frac{t}{1!}A_1 + \frac{t^2}{2!}A_2 + \cdots + \frac{t^n}{n!}A_n + \cdots = \sum_{n=0}^{\infty} A_n\frac{t^n}{n!}, \quad A_0 \neq 0, \tag{1.2}$$

with real coefficients $A_i$ $(i = 0, 1, 2, \cdots)$. The function $A(t)$ is an analytic function at $t = 0$. It is easy to see that for any $A(t)$ the derivative of $A_n(x)$ satisfies

$$A_n^{'}(x) = n\ A_{n-1}(x). \tag{1.3}$$

*Correspondence: subuhi2006@gmail.com
2010 *AMS Mathematics Subject Classification:* 26A33, 33B10, 33C45

Sequences of polynomials play an important role in numerous problems of applied mathematics, theoretical physics, approximation theory, and several other mathematical branches. The Bernstein polynomials of order $n$ form a basis for the space of polynomials of degree less than or equal to $n$. Dattoli et al. [9] studied Bernstein polynomials using operational methods. The class of Appell sequences contains a large number of classical polynomial sequences such as the Bernoulli, Euler, Hermite, and Miller–Lee polynomials. Certain new classes of hybrid special polynomials related to the Appell sequences are introduced and studied by Khan et al. [12, 13]. These hybrid polynomials are important due to the fact that they possess important properties such as differential equation, generating function, series definition, and integral representation. The problems arising in different areas of science and engineering are usually expressed in terms of differential equations, which in most cases have special functions as their solutions. The differential equations satisfied by the hybrid special polynomials may be used to express the problems arising in new and emerging areas of sciences.

We recall that the Laguerre–Appell polynomials ${}_LA_n(x,y)$ are introduced as the discrete Appell convolution of the Laguerre polynomials and are defined by means of the following series definition:

$$ {}_LA_n(x,y) = n!\sum_{k=0}^{n} \frac{(-1)^k A_{n-k}(y)x^k}{(n-k)!(k!)^2}. \tag{1.4}$$

These polynomials are connected with Appell polynomials by the following operational rule:

$$\exp\left(-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\ \frac{A_n(-x)}{n!} = {}_LA_n(x,y) \tag{1.5}$$

and specified by the following generating relation:

$$A(t)e^{yt}C_0(xt) = \sum_{n=0}^{\infty} {}_LA_n(x,y)\frac{t^n}{n!}, \tag{1.6}$$

where $C_0(xt)$ denotes the Tricomi function of order zero [1].

Alternatively, the Laguerre–Appell polynomials ${}_LA_n(x,y)$ are also defined by the following generating function:

$$A(t)e^{yt}e^{-D_x^{-1}t} = \sum_{n=0}^{\infty} {}_LA_n(x,y)\frac{t^n}{n!},$$

where $D_x^{-1}$ denotes inverse derivative operator:

$$D_x^{-1} := \int_0^x f(\xi)d\xi.$$

The possibility of using integral transforms in a wider context is discussed by Dattoli et al. [10], where by using Euler's integral

$$a^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-at}t^{\nu-1}dt, \qquad \min\{\mathrm{Re}(\nu),\mathrm{Re}(a)\} > 0, \tag{1.7}$$

it has been shown that [10]

$$\left(\alpha-\frac{\partial}{\partial x}\right)^{-\nu} f(x)= \quad \tfrac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\ e^{t\frac{\partial}{\partial x}}f(x)dt=\tfrac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\ f(x+t)dt, \tag{1.8}$$

whereas for the cases involving second-order derivatives, it is shown that

$$\left(\alpha-\frac{\partial^2}{\partial x^2}\right)^{-\nu} f(x)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\ e^{t\frac{\partial^2}{\partial x^2}}f(x)dt. \tag{1.9}$$

The fractional operators can be treated in an efficient way by combining the properties of exponential operators and suitable integral representations.

In this article, extended Laguerre–Appell polynomials are introduced and studied using fractional operators. In Section 2, extended Laguerre–Appell polynomials are introduced by means of generating function and operational definition using fractional operators. The recurrence relations and summation formulae for the extended Laguerre–Appell polynomials are also established. In Section 3, corresponding results for the Laguerre–Bernoulli and Laguerre–Euler polynomials are obtained as applications. In the last section, the determinant approach to these polynomials is considered.

## 2. Extended Laguerre–Appell polynomials

First we derive the operational rule connecting the Appell and the extended Laguerre–Appell polynomials by proving the following result:

**Theorem 2.1** *For the extended Laguerre–Appell polynomials* ${}_{\nu L}A_n(x,y;\alpha)$, *the following operational connection holds true:*

$$\left(\alpha+\left(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\right)^{-\nu}\frac{A_n(-x)}{n!}={}_{\nu L}A_n(x,y;\alpha). \tag{2.1}$$

**Proof** Replacing $a$ by $\left(\alpha+\left(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\right)$ in integral (1.7) and then operating it on $\frac{A_n(-x)}{n!}$, we find

$$\left(\alpha+\left(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\right)^{-\nu}\frac{A_n(-x)}{n!}=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(-yt\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\frac{A_n(-x)}{n!}dt, \tag{2.2}$$

which on using definition (1.5) on the r.h.s. gives

$$\left(\alpha+\left(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\right)^{-\nu}\frac{A_n(-x)}{n!}=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_LA_n(x,yt)dt. \tag{2.3}$$

The transform on the r.h.s of equation (2.3) defines a new family of special polynomials. Denoting this family of special polynomials by ${}_{\nu L}A_n(x,y;\alpha)$ and naming it extended Laguerre–Appell polynomials, it follows that

$${}_{\nu L}A_n(x,y;\alpha)=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_LA_n(x,yt)dt. \tag{2.4}$$

In view of equations (2.3) and (2.4), assertion (2.1) follows.

□

**Remark 2.1** Taking $A_n(x) = x^n$ on the l.h.s. of equation (2.1) and denoting the resultant extended Laguerre polynomials on the r.h.s. by ${}_\nu L_n(x, y; \alpha)$, the following operational connection holds true:

$$\left(\alpha + \left(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\right)^{-\nu} \frac{(-x)^n}{n!} = {}_\nu L_n(x, y; \alpha). \tag{2.5}$$

Next we derive the generating function of the extended Laguerre–Appell polynomials ${}_{\nu L}A_n(x, y; \alpha)$ by proving the following result:

**Theorem 2.2** *For the extended Laguerre–Appell polynomials* ${}_{\nu L}A_n(x, y; \alpha)$*, the following generating function holds true:*

$$\frac{A(u)\ C_0(xu)}{(\alpha - yu)^\nu} = \sum_{n=0}^{\infty} {}_{\nu L}A_n(x, y; \alpha)\frac{u^n}{n!}. \tag{2.6}$$

**Proof** Multiplying both sides of equation (2.4) by $\frac{u^n}{n!}$, then summing it over $n$, and making use of equation (1.6) on the r.h.s. of the resultant equation, we find

$$\sum_{n=0}^{\infty} {}_{\nu L}A_n(x, y; \alpha)\frac{u^n}{n!} = \frac{A(u)\ C_0(xu)}{\Gamma(\nu)} \int_0^\infty e^{-\left(\alpha - yu\right)t} t^{\nu-1} dt, \tag{2.7}$$

which in view of integral (1.7) yields assertion (2.6).

□

**Remark 2.2** For $A_n(x) = x^n$, the extended Laguerre–Appell polynomials ${}_{\nu L}A_n(x, y; \alpha)$ reduce to the extended Laguerre polynomials ${}_\nu L_n(x, y; \alpha)$. Therefore, for $A(u) = 1$, the following generating function for the extended Laguerre polynomials holds true:

$$\frac{C_0(xu)}{(\alpha - yu)^\nu} = \sum_{n=0}^{\infty} {}_\nu L_n(x, y; \alpha)\frac{u^n}{n!}. \tag{2.8}$$

**Remark 2.3** From equations (2.6) and (2.8), we find the following series definitions:

$${}_{\nu L}A_n(x, y; \alpha) = \frac{1}{\alpha^\nu\ \Gamma(\nu)} \sum_{m,k=0}^{n} (-1)^{n-m-k} \binom{n-m}{k}\binom{n}{m} \frac{\Gamma(\nu + k)\ y^k\ x^{n-m-k} A_m}{(n-m-k)!\ \alpha^k} \tag{2.9}$$

and

$${}_\nu L_n(x, y; \alpha) = \frac{1}{\alpha^\nu\ \Gamma(\nu)} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{\Gamma(\nu + k)\ y^k\ x^{n-k}}{(n-k)!\ \alpha^k} \tag{2.10}$$

of the extended Laguerre–Appell and the extended Laguerre polynomials, respectively.

Finally, we establish an explicit summation formula for the extended Laguerre–Appell polynomials ${}_{\nu L}A_n(x, y; \alpha)$ by proving the following result:

**Theorem 2.3** *For the extended Laguerre–Appell polynomials* ${}_{\nu L}A_n(x,y;\alpha)$*, the following explicit summation formula in terms of the extended Laguerre polynomials* ${}_{\nu}L_n(x,y;\alpha)$ *and Appell polynomials* $A_n(x)$ *holds true:*

$$
{}_{\nu L}A_n(x,y;\alpha)=\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ A_r(w)\ {}_{\nu}L_{n-k-r}(x,y;\alpha) \tag{2.11}
$$

**Proof** Consider the product of generating functions (1.1) and (2.8) in the following form:

$$
A(t)e^{wt}\,(\alpha-yt)^{-\nu}\,C_0(xt)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}A_r(w)\ {}_{\nu}L_n(x,y;\alpha)\frac{t^{n+r}}{n!\ r!}. \tag{2.12}
$$

Replacing $n$ by $n-r$ on the r.h.s. of equation (2.12), then shifting the first exponential to the r.h.s., and again replacing $n$ by $n-k$ in the resultant equation, it follows that

$$
A(t)\,(\alpha-yt)^{-\nu}\,C_0(xt)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ A_r(w)\ {}_{\nu}L_{n-k-r}(x,y;\alpha)\frac{t^n}{n!}. \tag{2.13}
$$

Finally, using generating function (2.6) on the l.h.s. of equation (2.13) and then equating the coefficients of like powers of $t$ in the resultant equation, assertion (2.11) follows.

□

A recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given; each further term of the sequence or array is defined as a function of the preceding terms.

Differentiating generating function (2.6), with respect to $x$, $y$, and $\alpha$ we find the following differential recurrence relations for the extended Laguerre–Appell polynomials ${}_{\nu L}A_n(x,y;\alpha)$:

$$
\begin{aligned}
\frac{\partial}{\partial x}\Big({}_{\nu L}A_n(x,y;\alpha)\Big) &= -n\ {}_{\nu L}A_{n-1}(x,y;\alpha),\\
\frac{\partial}{\partial y}\Big({}_{\nu L}A_n(x,y;\alpha)\Big) &= n\nu\ {}_{\nu+1 L}A_{n-1}(x,y;\alpha),\\
\frac{\partial}{\partial \alpha}\Big({}_{\nu L}A_n(x,y;\alpha)\Big) &= -\nu\ {}_{\nu+1 L}A_{n}(x,y;\alpha).
\end{aligned} \tag{2.14}
$$

Consequently, we have

$$
\frac{\partial}{\partial y}\Big({}_{\nu L}A_n(x,y;\alpha)\Big)=\frac{\partial^2}{\partial x\partial\alpha}\ {}_{\nu L}A_n(x,y;\alpha).
$$

**Note.** It should be noted that for $\alpha=\nu=1$ and $y\to D_y^{-1}$, the extended Laguerre–Appell polynomials ${}_{\nu L}A_n(x,y;\alpha)$ reduce to ${}_{L}A_n(x,y)$. For the same choice of parameters $\alpha$, $\nu$ and variable $y$ the extended Laguerre polynomials ${}_{\nu}L_n(x,y;\alpha)$ reduce to the 2-variable Laguerre polynomials $L_n(x,y)$ [11].

The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional operators [10]. To bolster the contention of using this approach, the extended form of hybrid-type polynomials is introduced. The generating function, summation formula, and recurrence relations for the extended Laguerre–Appell polynomials are derived here. These results may be useful in the investigation of other useful properties of these polynomials and may have applications in physics.

In the next section, we consider the extended forms of the Laguerre–Bernoulli and Laguerre–Euler polynomials as members of the extended Laguerre–Appell family.

## 3. Applications

The functional equations and identities for the Bernoulli and Euler polynomials arise in combinatorial contexts and may lead systematically to well-defined classes of functions. There is a continuous demand of solving problems by means of functional equations, relations, and identities in research fields like classical and quantum optics. The functional equations of hybrid-type special polynomials of more than one variable often appear in applications ranging from electromagnetic processes to combinatorics. The results for the members belonging to the extended Laguerre–Appell family can be obtained from the corresponding results of the members of Appell family. Here we derive certain results for the extended Laguerre–Bernoulli and extended Laguerre–Euler polynomials from the results of Bernoulli and Euler polynomials.

In view of equation (2.1), we find the following operational rules for the extended Laguerre–Bernoulli polynomials ${}_{\nu L}B_n(x,y;\alpha)$ and the extended Laguerre–Euler polynomials ${}_{\nu L}E_n(x,y;\alpha)$:

$$\left(\alpha+\left(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\right)^{-\nu}\left\{\frac{B_n(-x)}{n!}\right\}={}_{\nu L}B_n(x,y;\alpha) \tag{3.1}$$

and

$$\left(\alpha+\left(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\right)^{-\nu}\left\{\frac{E_n(-x)}{n!}\right\}={}_{\nu L}E_n(x,y;\alpha) \tag{3.2}$$

respectively. Again taking $A(u)=\frac{u}{e^u-1}$ (of Bernoulli polynomials) and $A(u)=\frac{2}{e^u+1}$ (of Euler polynomials) in equation (2.6), the generating functions for ${}_{\nu L}B_n(x,y;\alpha)$ and ${}_{\nu L}E_n(x,y;\alpha)$ are obtained as

$$\left(\frac{u}{e^u-1}\right)\frac{\exp(xu)}{(\alpha-(yu))^{\nu}}=\sum_{n=0}^{\infty}{}_{\nu L}B_n(x,y;\alpha)\frac{u^n}{n!} \tag{3.3}$$

and

$$\left(\frac{2}{e^u+1}\right)\frac{\exp(xu)}{(\alpha-(yu))^{\nu}}=\sum_{n=0}^{\infty}{}_{\nu L}E_n(x,y;\alpha)\frac{u^n}{n!}, \tag{3.4}$$

respectively.

Several identities involving Appell polynomials are known. The operational formalism developed in Section 2 can be used to obtain the corresponding identities for the extended Laguerre–Appell polynomials ${}_{\nu L}A_n(x,y;\alpha)$. To achieve this, we perform the following operation:

($\mathcal{O}$) operating $\left(\alpha+(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x})\right)^{-\nu}$ on both sides of a given relation.

We recall the following functional equations involving Bernoulli polynomials $B_n(x)$ [14, p. 26]:

$$B_n(x+1) - B_n(x) = n\ x^{n-1}, \quad n = 0, 1, 2..... \tag{3.5}$$

$$\sum_{m=0}^{n-1} \binom{n}{m} B_m(x) = nx^{n-1}, \quad n = 2, 3, 4..... \tag{3.6}$$

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right), \quad n = 0, 1.2, .....; \quad m = 1, 2, 3.... \tag{3.7}$$

On replacing $x$ by $-x$ in equations (3.5)–(3.7) and then performing the operation $(\mathcal{O})$ on the resultant equation and using operational definitions (3.1) and (2.5), the following identities for the extended Laguerre–Bernoulli polynomials ${}_{\nu L}B_n(x, y; \alpha)$ are obtained:

$${}_{\nu L}B_n(x-1, y; \alpha) - {}_{\nu L}B_n(x, y; \alpha) = {}_{\nu}L_{n-1}(x, y; \alpha), \quad n = 0, 1, 2 \cdots, \tag{3.8}$$

$$\sum_{m=0}^{n-1} \frac{1}{(n-m)!} {}_{\nu L}B_m(x, y; \alpha) = {}_{\nu}L_{n-1}(x, y; \alpha), \quad n = 2, 3, 4 \cdots, \tag{3.9}$$

$${}_{\nu L}B_n(mx, m^2y; \alpha) = m^{n-1} \sum_{k=0}^{m-1} {}_{\nu L}B_{n-1}\left(x - \frac{k}{m}, y; \alpha\right), \quad n = 0, 1, 2, ...; \quad m = 1, 2, 3 \cdots. \tag{3.10}$$

In a similar manner, corresponding to the functional equations involving the Euler polynomials $E_n(x)$ [14, p. 30]:

$$E_n(x+1) + E_n(x) = 2x^n,$$

$$E_n(mx) = m^n \sum_{k=0}^{m-1} (-1)^k E_n\left(x + \frac{k}{m}\right) \quad n = 0, 1, 2...; \quad m \ odd,$$

we find the following identities involving the extended Laguerre–Euler polynomials ${}_{\nu L}E_n(x, y; \alpha)$:

$${}_{\nu L}E_n(x-1, y; \alpha) + {}_{\nu L}E_n(x, y; \alpha) = 2\ {}_{\nu}L_n(x, y; \alpha). \tag{3.11}$$

$${}_{\nu L}E_n(mx, m^2y; \alpha) = m^n \sum_{k=0}^{m-1} (-1)^k {}_{\nu L}E_n\left(x - \frac{k}{m}, y; \alpha\right), \quad n = 0, 1.2, ...; \quad m \ odd. \tag{3.12}$$

Moreover, corresponding to the following relations between the Bernoulli and Euler polynomials [14, pp. 29-30]:

$$B_n(x) = 2^{-n} \sum_{m=0}^{n} \binom{n}{m} B_{n-m} E_m(2x), \quad n = 0, 1, 2..., \tag{3.13}$$

$$E_n(x) = \frac{2^{n+1}}{n+1}\Big[B_{n+1}\Big(\frac{x+1}{2}\Big) - B_{n+1}\Big(\frac{x}{2}\Big)\Big], \quad n = 0, 1, 2..., \tag{3.14}$$

$$E_n(mx) = -\frac{2^{m^n}}{n+1}\sum_{k=0}^{m-1}(-1)^k B_{n+1}\Big(\frac{x+k}{m}\Big), \quad n = 0, 1, 2..., m \ even, \tag{3.15}$$

we obtain the following relations between the extended Laguerre–Bernoulli and extended Laguerre–Euler polynomials:

$${}_{\nu L}B_n(x, y; \alpha) = 2^{-n}\sum_{m=0}^{n}\frac{1}{(n-m)!}B_{n-m} \ {}_{\nu L}E_m(2x, 4y; \alpha), \qquad n = 0, 1, 2..., \tag{3.16}$$

$${}_{\nu L}E_n(x, y; \alpha) = 2^{n+1}\Big[{}_{\nu L}B_{n+1}\Big(\frac{x-1}{2}, \frac{y}{4}; \alpha\Big) - {}_{\nu L}B_{n+1}\Big(\frac{x}{2}, \frac{y}{4}; \alpha\Big)\Big], \quad n = 0, 1, 2..., \tag{3.17}$$

$${}_{\nu L}E_n(mx, m^2y; \alpha) = -2m^n\sum_{k=0}^{m-1}(-1)^k {}_{\nu L}B_{n+1}\left(x - \frac{k}{m}, y; \alpha\right), \quad n = 0, 1.2, ...; \quad m \ even. \tag{3.18}$$

In the next section, we consider the determinant approach to the extended Laguerre–Appell polynomials ${}_{\nu L}A_n(x, y; \alpha)$.

## 4. Determinant approach

In order to establish the determinant form of the extended Laguerre–Appell polynomials, we prove the following result:

**Theorem 4.1** *For the extended Laguerre–Appell polynomials ${}_{\nu L}A_n(x, y; \alpha)$, the following determinant form holds true:*

$${}_{\nu L}A_0(x, y; \alpha) = \frac{1}{\beta_0} \ {}_{\nu L}L_0(x, y; \alpha), \qquad \beta_0 = \frac{1}{A_0}, \tag{4.1}$$

$${}_{\nu L}A_n(x,y;\alpha) = \frac{(-1)^n}{(\beta_0)^{n+1}n!}\begin{vmatrix} {}_{\nu L}L_0(x,y;\alpha) & {}_{\nu L}L_1(x,y;\alpha) & 2! \ {}_{\nu L}L_2(x,y;\alpha) & \cdots & (n-1)! \ {}_{\nu L}L_{n-1}(x,y;\alpha) & n! \ {}_{\nu L}L_n(x,y;\alpha) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{4.2}$$

$$\beta_n = -\frac{1}{A_0}\Big(\sum_{k=1}^{n}\binom{n}{k}A_k\beta_{n-k}\Big), \qquad\qquad n = 1, 2, 3, \cdots,$$

*where* $\beta_0,\ \beta_1,\cdots,\beta_n \in \mathbb{R},\ \beta_0 \neq 0$*, and* ${}_{\nu}L_n(x,y;\alpha)$ $(n=0,1,\cdots)$ *are the extended Laguerre polynomials defined by equation (2.10).*

**Proof** Since the Appell polynomials possess the following determinant definition [6, p. 1533]:

$$A_0(x) = \frac{1}{\beta_0},\quad \beta_0 = \frac{1}{A_0}, \tag{4.3}$$

$$A_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{4.4}$$

$$\beta_n = -\frac{1}{A_0}\Big(\sum_{k=1}^{n}\binom{n}{k}A_k\ \beta_{n-k}\Big), n=1,2,3,\cdots,$$

where $\beta_0,\ \beta_1,\cdots,\beta_n \in \mathbb{R},\ \ \beta_0 \neq 0$.

Taking $n=0$ in equation (2.11) and then using equation (4.3) in the resultant equation, assertion (4.1) follows.

Next, replacing $x$ by $-x$ in equation (4.4) and then expanding the determinant on the r.h.s. of the resultant equation w.r.t. the first row, it follows that

$$A_n(-x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}$$

$$-\frac{(-1)^n(-x)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} + \frac{(-1)^n(-x)^2}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} + \cdots$$

$$+\frac{(-1)^{2n-1}(-x)^{n-1}}{(\beta_0)^{n+1}}\begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\beta_1 \end{vmatrix} + \frac{(-x)^n}{(\beta_0)^{n+1}}\begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \beta_0 \end{vmatrix}. \tag{4.5}$$

Since each minor in equation (4.5) is independent of $x$, operating $\left(\alpha+\left(y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\right)^{-\nu}$ on both sides of equation (4.5), then using equations (2.1) and (2.5), and combining the terms on the r.h.s. of the resultant equation, we are led to assertion (4.2). □

**Remark 4.1** For $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}$ $(i = 1, 2, ....., n)$ the determinant definition of the Appell polynomials $A_n(x)$ given by equations (4.3) and (4.4) gives the determinant definition of the Bernoulli polynomials $B_n(x)$ [5]. Therefore, taking $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}$ $(i = 1, 2, ....., n)$ in equations (4.1) and (4.2), the following determinant definition of the extended Laguerre–Bernoulli polynomials ${}_{\nu L}B_n(x, y; \alpha)$ is obtained:

$${}_{\nu L}B_0(x, y; \alpha) = {}_{\nu}L_0(x, y; \alpha),$$

$${}_{\nu L}B_n(x, y; \alpha)$$

$$= \frac{(-1)^n}{(\beta_0)^{n+1} n!}\begin{vmatrix} {}_{\nu}L_0(x,y;\alpha) & {}_{\nu}L_1(x,y;\alpha) & 2!\ {}_{\nu}L_2(x,y;\alpha) & \cdots & (n-1)!\ {}_{\nu}L_{n-1}(x,y;\alpha) & n!\ {}_{\nu}L_n(x,y;\alpha) \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & \frac{n-1}{2} & \frac{n}{2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \frac{n}{2} \end{vmatrix}. \tag{4.6}$$

**Remark 4.2** In view of the fact that for $\beta_0 = 1$ and $\beta_i = \frac{1}{2}$ $(i = 1, 2, ....., n)$ the determinant definition of the Appell polynomials $A_n(x)$ given by equations (4.3) and (4.4) reduces to the determinant definition of the Euler polynomials $E_n(x)$ [6], taking $\beta_0 = 1$ and $\beta_i = \frac{1}{2}$ $(i = 1, 2, ....., n)$ in equations (4.1) and (4.2), the following determinant definition for the generalized Laguerre–Euler polynomials ${}_{\nu L}E_n(x, y; \alpha)$ is obtained:

$${}_{\nu L}E_0(x, y; \alpha) = {}_{\nu}L_0(x, y; \alpha),$$

$$
{}_{\nu L}E_n(x,y;\alpha)
$$

$$
= \frac{(-1)^n}{(\beta_0)^{n+1} n!} \begin{vmatrix} {}_{\nu}L_0(x,y;\alpha) & {}_{\nu}L_1(x,y;\alpha) & 2!\ {}_{\nu}L_2(x,y;\alpha) & \cdots & (n-1)!\ {}_{\nu}L_{n-1}(x,y;\alpha) & n!\ {}_{\nu}L_n(x,y;\alpha) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{2} & \binom{n}{1}\frac{1}{2} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2} & \binom{n}{2}\frac{1}{2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}. \tag{4.7}
$$

Costabile [4, 5, 8] has given several approaches to Bernoulli polynomials. An important approach based on a determinant definition was given in [5]. This approach is further extended to provide determinant definitions of the Appell and Sheffer polynomial sequences by Costabile and Longo in [6] and [7], respectively. The equivalence of the determinant approach with other existing approaches is also established. The simplicity of the algebraic approach to the Appell and Sheffer polynomials established in [6, 7] allows several applications. The abovementioned research works by Costabile and Longo and the importance of operational methods in the theory of special functions motivated the authors to establish the determinant forms of the extended Laguerre–Appell polynomials.

Operational methods can be exploited to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of special functions. The use of operational techniques in the study of special functions provides explicit solutions for the families of partial differential equations including those of heat and D′Alembert type. The method proposed in this article can be used in combination with the monomiality principle as a useful tool in analyzing the solutions of a wide class of partial differential equations often encountered in physical problems.

## References


[1] Andrews LC. Special Functions for Engineers and Applied Mathematicians. New York, NY, USA: Macmillan Publishing Company, 1985.

[2] Appell P. Sur une classe de polynômes. Ann Sci École Norm Sup 1880; 9: 119-144 (in French).

[3] Assante D, Cesarano C, Fornaro C, Vazquez L. Higher order and fractional diffusive equations. Journal of Engineering Science and Technology Review 2015; 8: 202-204.

[4] Costabile FA. On expansion of a real function in Bernoulli polynomials and applications. Conferenze del sem Matem Uni Bari (IT) $n^{273}$ 1999.

[5] Costabile FA, Dell′Accio F, Gualtieri MI. A new approach to Bernoulli polynomials. Rend Mat Appl 2006; 26: 1-12.

[6] Costabile FA, Longo E. A determinantal approach to Appell polynomials. J Comput Appl Math 2010; 234: 1528-1542.

[7] Costabile FA, Longo E. An algebraic approach to Sheffer polynomial sequences. Integral Transforms Spec Funct 2014; 25: 295-311.

[8] Costabile FA, Longo E. An algebraic exposition of umbral calculus with application to general interpolation problem - a survey. Publications de L′institut Mathématique Nouvelle série, tome 2014; 96: 67-83.

[9] Dattotli G, Lorenzutta S, Cesarano C. Bernstein polynomials and operational methods. J Comput Anal Appl 2006; 8: 369-377.

[10] Dattoli G, Ricci PE, Cesarano C, Vázquez L. Special polynomials and fractional calculus. Math Comput Modelling 2003; 37: 729-733.

[11] Dattoli G, Torre A. Operational methods and two variable Laguerre polynomials. Atti Acad Sci Torino CL Sci Fis Mat Natur 1998; 132: 1-7.

[12] Khan S, Al-Saad MWM, Khan R. Laguerre-based Appell polynomials: properties and applications. Math Comput Modelling 2010; 52: 247-259.

[13] Khan S, Yasmin G, Khan R, Hassan NAM. Hermite-based Appell polynomials: properties and applications. J Math Anal Appl 2009; 351;: 756-764.

[14] Magnus W, Oberhettinger F, Soni RP. Formulas and Theorems for Special Functions of Mathematical Physics. New York, NY, USA: Springer-Verlag, 1956.

[15] Oldham H, Spanier N. The Fractional Calculus. San Diego, CA, USA: Academic Press, 1974.

[16] Widder DV. An Introduction to Transform Theory. New York, NY, USA: Academic Press, 1971.