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A study of the tubular surfaces constructed by the spherical indicatrices in Euclidean $3-$space

Fatma ATEŞ[1,2,*], Erdem KOCAKUŞAKLI[2], İsmail GÖK[2], Yusuf YAYLI[2]
[1]Department of Mathematics and Computer Sciences, Necmettin Erbakan University, Konya, Turkey
[2]Department of Mathematics, Faculty of Science, Ankara University, Ankara, Turkey

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Abstract: A basic goal of this paper is to investigate the tubular surface constructed by the spherical indicatrices of any spatial curve in the Euclidean $3-$space. This kind of tubular surface is designed for the alternative moving frame $\{N, C, W\}$ in conjunction with finding a relationship between the tubular surfaces and their special curves, such as geodesic curves, asymptotic curves, and minimal curves. The minimal curve γ on a surface is defined by the property that its fundamental coefficients satisfy Eq. (3.7) along the curve γ. At the end of this article, we exemplify these curves on the tubular surfaces with their figures using the program Mathematica.

Key words: Tubular surface, indicatrix curve, minimal curve

1. Introduction

If the radii of the generating spheres are nonconstant then the surface is called a canal surface. A tubular surface is defined by a canal surface with the constant radius r. These surfaces have many real-life applications in networks of blood vessels and neurons in medicine and tube and hose systems in industrial environments.

Blaga [2] used tubular surfaces as swept surfaces and put forward new approaches for the investigation of tubular and canal surfaces in addition to showing applications of tubular surfaces in scientific visualization. Doğan [3] studied generalized canal surfaces and some special curves on them in his thesis. Karacan and Yaylı [6] investigated the geodesics of tubular surfaces in Minkowski $3-$space. Karacan et al. [7] examined the general and linear Weingarten conditions for tubular surfaces by using the Gaussian and mean curvature of the surface in Euclidean $3-$space. Recently, Yıldız et al. [11] conducted a geometric observation of the ascending colons of some domestic animals such as pigs, horses, ruminants, and dogs; these ascending colons were shown to have a tubular shape along a special curve, which is an application of tubular surfaces in medicine. Tubular and canal surfaces have been also represented as quaternionic by several authors [1, 4].

In the present study, we determine tubular surfaces using their center curves, which are spherical indicatrices of any spatial curve in Euclidean $3-$space. We provide preliminary information on the classical geometry of tubular surfaces and study properties that arise in connection with a computational process of these surfaces. In addition, an example of a tubular surface with its center curve as a slant helix is given and the tubular surfaces around the spherical indicatrices of the slant helix are determined. The theoretical results

*Correspondence: fgokcelik@ankara.edu.tr
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obtained for these curves on the corresponding surfaces will be explained in an example with the corresponding graphs produced by using the programming language Mathematica.

2. Preliminaries

Let $\mathbb{E}^3$ be a 3−dimensional Euclidean space endowed with the standard metric

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

The norm of a vector $x \in \mathbb{E}^3$ is defined by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

and the Euclidean vector product is given by

$$x \times y = (x_2y_3 - y_2x_{3,}\ x_3y_1 - y_3x_{1,}\ x_1y_2 - y_1x_2)\,.$$

The elementary properties of the surface S in $\mathbb{E}^3$ with parameterization $X(s, \theta)$ are explained in [9]. At an arbitrary point $p = p(s, \theta)$ of the surface S, the natural base $\{X_s, X_\theta\}$ spans the tangent space T_pS. Suppose the vector field U is the normal vector of S such that $\{X_s, X_\theta, U\}$ is positively oriented in $\mathbb{E}^3$. The functions $E = \langle X_s, X_s\rangle$, $F = \langle X_s, X_\theta\rangle$, $G = \langle X_\theta, X_\theta\rangle$ are denoted to be the coefficients of the first fundamental form and the functions $L = \langle X_{ss}, U\rangle$, $M = \langle X_{s\theta}, U\rangle$, $N = \langle X_{\theta\theta}, U\rangle$ are defined to form the second fundamental form of S.

The invariant functions of the Gaussian curvature K and the mean curvature H of the surface are defined as follows:

$$K := \frac{LN - M^2}{EG - F^2} \qquad \text{and} \qquad H := \frac{EN - 2FM + GL}{2\,(EG - F^2)}.$$

K and H can be related to each other through the quadratic formula

$$k^2 - 2kH + K = 0$$

with roots

$$k_1 = H + \sqrt{H^2 - K} \quad \text{and} \quad k_2 = H - \sqrt{H^2 - K}$$

where k_1 and k_2 are the principal curvatures of the surface. Next we list the known elementary properties of a space curve in $\mathbb{E}^3$. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed smooth curve with Frenet tools $\{T, N, B, \kappa, \tau\}$ in Euclidean space $\mathbb{E}^3$ where functions κ and τ are called the first and the second curvature functions of the curve α, respectively. Letting N, $C = \dfrac{N'}{\|N'\|}$ and $W = \dfrac{1}{f}\{\tau T + \kappa B\}$ be the unit principal normal vector, the derivative of the principal normal vector, and the darboux vector of the curve α, respectively, a new moving frame $\{N, C, W\}$ along the curve α was introduced by [10] in $\mathbb{E}^3$. In addition, the alternative moving frame derivatives satisfy the equation

$$\begin{bmatrix} N'(s) \\ C'(s) \\ W'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix} \tag{2.1}$$

where $f(s)=\sqrt{\kappa^2(s)+\tau^2(s)}$ and $g(s)=\sigma(s)\ f(s)$, $\sigma(s)\neq 0$ for every $s\in I$, and $\sigma(s)=\left(\dfrac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\dfrac{\tau}{\kappa}\right)'\right)(s)$ is the geodesic curvature of the spherical image of the principal normal indicatrix (N) of α (*see for details* [5]).

Theorem 1 *[5] Let α be a unit speed curve with $\kappa(s)\neq 0$. Then α is a slant helix if and only if $\sigma(s)$ is a constant function.*

A canal surface is the envelope of a one-parameter family of spheres centered at the spine curve $\alpha(s)$ with the radii described by the function $r(s)$. If $r(s)$ is a constant function then the canal surface is called a tubular surface. It is easy to see the connection between the tubular surfaces and a family of spheres, which are great circles lying in the normal plane of the generating curve α, through the equation

$$X(s,\theta)=\alpha(s)+r(\cos\theta N(s)+\sin\theta B(s)),$$

where $N(s)$ is the the principal normal vector and $B(s)$ is the binormal vector of α at point s.

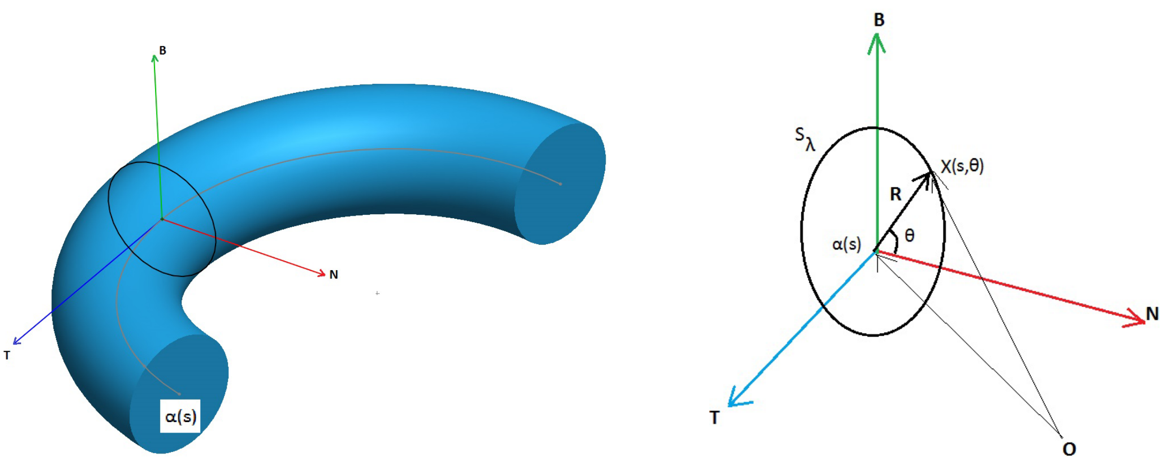


Figure 1. The tubular surface $X(s,\theta)$ and its characteristic circle S_λ.

3. Tubular surfaces constructed by the spherical indicatrices

Let $\alpha: I\subset\mathbb{R}\to\mathbb{E}^3$ be a unit speed smooth curve with arc length parameter s and β_i be a curve that is a spherical indicatrix of the curve α with i considered to be the indices for T, N or B. In this section, we obtain some characterizations of the tubular surfaces whose center curves are the spherical indicatrices of α.

Assume the center curve of the tubular surface is the **tangent indicatrix** of the curve α, that is, $\beta_T(s_T)=\beta_T(\varphi(s))=T(s)$, where $\varphi: I\to I_T$, $s_T=\varphi(s)$ is a regular $C^\infty-$function. Let $\{N(s),C(s),W(s),f(s),g(s)\}$ be the alternative frame apparatus of the curve β_T. The characteristic circles $(S_T)_\lambda$ are lying in the plane $Span\,\{C,W\}$ of the tubular surface. Therefore, the tubular surface can be related to the curve β_T through the formula

$$X_T(s,\theta)=\beta_T(\varphi(s))+r(\cos\theta C(s)+\sin\theta W(s)), \tag{3.1}$$

where $\varphi(s) = \int \kappa(s)ds$ and the angle θ is between the vector C and the position vector of the circles $(S_T)_\lambda$. The normal vector of the tubular surface $X_T(s,\theta)$ is

$$\begin{aligned} U_T &= \frac{(X_T)_s \times (X_T)_\theta}{\|(X_T)_s \times (X_T)_\theta\|} \\ &= \cos\theta C(s) + \sin\theta W(s). \end{aligned} \tag{3.2}$$

Direct computation indicates

$$\begin{gathered} E_T = r^2 g^2(s) + \{\kappa(s) - rf(s)\cos\theta\}^2 \ , \\ F_T = r^2 g(s) \ , \ G_T = r^2, \\ \\ L_T = -rg^2(s) + f(s)\cos\theta\,\{\kappa(s) - rf(s)\cos\theta\} \ , \\ M_T = -rg(s) \ , \ N_T = -r. \end{gathered} \tag{3.3}$$

Proposition 1 *The tubular surface $X_T(s,\theta)$ is a regular surface if and only if the $\kappa(s) - rf(s)\cos\theta$ is not equal to zero.*

Proof Any regular surface has the condition $EG - F^2 \neq 0$. By using Eq. (3.3), we have

$$\left(EG - F^2\right)_T = r^2\left\{\kappa(s) - rf(s)\cos\theta\right\}^2$$

for the surface $X_T(s,\theta)$. Since $r > 0$, it is easy to see

$$\kappa(s) - rf(s)\cos\theta \neq 0.$$

Conversely, if the corresponding assumption holds then we can easily see that the surface is regular. □

Corollary 1 *The Gaussian and mean curvatures of $X_T(s,\theta)$ satisfy*

$$\begin{aligned} K_T &= \frac{-f(s)\cos\theta}{r\left\{\kappa(s) - rf(s)\cos\theta\right\}}, \\ H_T &= -\frac{1}{2}(rK_T + \frac{1}{r}). \end{aligned} \tag{3.4}$$

The principal curvatures of the surface are then given by

$$(k_1)_T = \frac{f(s)\cos\theta}{\kappa(s) - rf(s)\cos\theta} \quad \textit{and} \quad (k_2)_T = -\frac{1}{r}.$$

Proof The curvatures can be easily obtained by the coefficients of the first and the second fundamental forms of $X_T(s,\theta)$. □

Theorem 2 *Let $X_T(s,\theta)$ be a regular tubular surface in $\mathbb{E}^3$. The parameter curves of the tubular surface $X_T(s,\theta)$ have the following properties:*
(i) The $s-$parameter curve of the surface $X_T(s,\theta)$ is an asymptotic curve if and only if

$$\sigma^2(s)f(s) = -\frac{1}{r}\cos\theta\left\{\kappa(s) - rf(s)\cos\theta\right\} \quad \textit{with} \quad |\kappa(s)| \geqslant 2r\,|g(s)|\,. \tag{3.5}$$

(ii) The $\theta-$parameter curve of the surface $X_T(s,\theta)$ cannot be an asymptotic curve.

Proof (i) A space curve γ is an asymptotic curve on a surface in $\mathbb{E}^3$ if and only if the normal vector γ'' of the curve is tangent to the surface at every point, that is $\left\langle U_T, \gamma''\right\rangle = 0$. Thus, we calculate for the $s-$parameter curve

$$\langle U_T, (X_T)_{ss}\rangle = -rg^2(s) + f(s)\cos\theta\,\{\kappa(s) - rf(s)\cos\theta\} = 0.$$

Using this last equation, we obtain Eq. (3.5) for the $s-$parameter curve.

Under Eq. (3.5), the $s-$parameter curve of $X_T(s,\theta)$ is an asymptotic curve.

(ii) Noting that the inequality $\langle U_T, (X_T)_{\theta\theta}\rangle = -r \neq 0$ holds, the $\theta-$parameter curve cannot be an asymptotic curve. □

Theorem 3 *The s and θ parameter curves of the surface $X_T(s,\theta)$ have the following properties.*
(i) The $s-$parameter curve is a geodesic curve of the surface $X_T(s,\theta)$ if and only if the equation $r^2g^2 + (\kappa - rf\cos\theta)^2$ is a constant.
(ii) The $\theta-$parameter curve of $X_T(s,\theta)$ is always a geodesic curve.

Proof (i) A space curve γ is a geodesic curve on a surface if and only if the vectors γ'' and U_T are linearly dependent, that is, $U \times \gamma'' = 0$. In this case, we obtain the following for the $s-$parameter curve

$$\begin{aligned} U_T \times (X_T)_{ss} \quad &= \quad \left[rg' - f\kappa\sin\theta + rf^2\sin\theta\cos\theta\right] N + \left[\sin\theta(\kappa' - rf'\cos\theta + rfg\sin\theta)\right] C \\ &\quad + \left[-\cos\theta(\kappa' - rf'\cos\theta + rfg\sin\theta)\right] W. \end{aligned}$$

Noting that the set $\{N, C, W\}$ is an orthonormal system and $U_T \times (X_T)_{ss} = 0$ holds, we have the following equalities:

$$\begin{cases} rg' - f\kappa\sin\theta + rf^2\sin\theta\cos\theta = 0, \\ \sin\theta(\kappa' - rf'\cos\theta + rfg\sin\theta) = 0, \\ -\cos\theta(\kappa' - rf'\cos\theta + rfg\sin\theta) = 0. \end{cases} \tag{3.6}$$

By using the last two equations, we obtain $\kappa' - rf'\cos\theta + rfg\sin\theta = 0$; combining this equation with the first equality in Eq. (3.6) results in obtaining the expression $r^2g^2 + (\kappa - rf\cos\theta)^2$, which is a constant.

(ii) We determine $U_T \times (X_T)_{\theta\theta} = 0$ to be satisfied for the $\theta-$parameter curve. Hence, the $\theta-$parameter curve of $X_T(s,\theta)$ is always a geodesic curve. □

The minimal surfaces are characterized by the zero mean curvature. Each surface cannot be a minimal surface, but we can find minimal points on such surfaces. In addition, there exists a curve on the surface called a minimal curve containing the minimal points. The notion of minimal curves was given in the following definition by Şemin (see p. 162, Definition 34-3 in [8] for details).

Definition 1 *On the surface, a hyperbolic point is a minimal point if and only if it satisfies the following equation at this point:*

$$EN + GL - 2MF = 0. \tag{3.7}$$

Theorem 4 *The minimal curves of regular tubular surface $X(s,\theta) = \alpha(s) + r(\cos\theta N(s) + \sin\theta B(s))$ are as follows:*

$$\gamma(s) = \alpha(s) + r\{\cos\theta N(s) + \sin\theta B(s)\},$$

where $\theta = \arccos\left(\dfrac{1}{2r\kappa(s)}\right)$ *with the condition* $\kappa(s) \neq 0$ *satisfied.*

Proof Substitution of the coefficients of the first and the second fundamental forms of the regular tubular surface $X(s,\theta)$ into Eq. (3.7) gives the equality

$$2r^2\kappa^2(s)\cos^2\theta - 3r\kappa(s)\cos\theta + 1 = 0.$$

Direct computations indicate

$$\theta = \arccos\left(\frac{1}{2r\kappa(s)}\right).$$

The minimal curves are obtained by considering the last equation with the tubular surface $X(s,\theta)$. □

Theorem 5 *The minimal curves of the regular tubular surface* $X_T(s,\theta)$ *are as follows:*

$$\gamma_T(s) = \beta_T(\varphi(s)) + r\{\cos\theta C(s) + \sin\theta W(s)\}$$

where $\theta = \arccos\left(\dfrac{\kappa(s)}{2rf(s)}\right)$ *and* $\varphi(s) = \int \kappa(s)ds$*.*

Proof To find the minimal curves of the surface $X_T(s,\theta)$, if the values in the Eq. (3.3) are substituted in Eq. (3.7) then the equation

$$2r^2f^2(s)\cos^2\theta - 3rf(s)\kappa(s)\cos\theta + \kappa^2(s) = 0$$

holds. Since the surface $X_T(s,\theta)$ is regular, it is straightforwardly calculated

$$\theta = \arccos\left(\frac{\kappa(s)}{2rf(s)}\right).$$

The minimal curves of the surface $X_T(s,\theta)$ are obtained using the last equality in Eq. (3.1). □

Let center curve be a **normal indicatrix** of the curve α, that is, $\beta_N(s_N) = \beta_N(\psi(s)) = N(s)$, where $\psi : I \to I_N$, $s_N = \psi(s)$ is a regular $C^\infty-$function. The characteristic circles $(S_N)_\lambda$ lie in the plane $Span\{N, W\}$ of the tubular surface. In this case, we obtain that the tubular surface related to the curve β_N satisfies

$$X_N(s,\theta) = \beta_N(\psi(s)) + r(\cos\theta N(s) + \sin\theta W(s)), \tag{3.8}$$

where $\psi(s) = \int \sqrt{(\kappa^2+\tau^2)(s)}ds$ and θ is the angle between N and the position vector of the circles $(S_N)_\lambda$. The unit normal vector field of the tubular surface $X_N(s,\theta)$ is

$$\begin{aligned} U_N &= \frac{(X_N)_s \times (X_N)_\theta}{\|(X_N)_s \times (X_N)_\theta\|} \\ &= \cos\theta N(s) + \sin\theta W(s). \end{aligned} \tag{3.9}$$

We can easily see that

$$\begin{array}{c} E_N = \{f(s) + rf(s)\cos\theta - rg(s)\sin\theta\}^2 \ , \\ F_N = 0 \ , \ G_N = r^2, \\ \\ L_N = \{f(s) + rf(s)\cos\theta - rg(s)\sin\theta\}\{-f(s)\cos\theta + g(s)\sin\theta\} \ , \\ M_N = 0 \ , \ N_N = -r. \end{array} \tag{3.10}$$

Proposition 2 *The tubular surface* $X_N(s,\theta)$ *is a regular surface if and only if* $f(s) + rf(s)\cos\theta - rg(s)\sin\theta$ *is not equal to zero.*

Proof A regular surface has the condition $EG - F^2 \neq 0$. By using Eq. (3.10), we have

$$\left(EG - F^2\right)_N = r^2\{f(s) + rf(s)\cos\theta - rg(s)\sin\theta\}^2.$$

Using $r > 0$ we determine

$$f(s) + rf(s)\cos\theta - rg(s)\sin\theta \neq 0.$$

Conversely, if the corresponding condition holds, then the surface is regular. □

Corollary 2 *The Gaussian, the mean, and the principal curvatures of* $X_N(s,\theta)$ *are*

$$K_N = \frac{f(s)\cos\theta - g(s)\sin\theta}{r(f(s) + rf(s)\cos\theta - rg(s)\sin\theta)} \quad , \quad H_N = -\frac{1}{2}\left(rK_N + \frac{1}{r}\right), \tag{3.11}$$

and

$$(k_1)_N = -\frac{1}{r} \quad \textit{and} \quad (k_2)_N = -\frac{f(s)\cos\theta - g(s)\sin\theta}{f(s) + rf(s)\cos\theta - rg(s)\sin\theta},$$

respectively.

Proof The curvatures can be easily calculated by using Eq. (3.10). □

Theorem 6 *Let* $X_N(s,\theta)$ *be a regular tubular surface in* $\mathbb{E}^3$. *The* s *and* θ *parameter curves of the tubular surface* $X_N(s,\theta)$ *have the following properties:*
(i) *The* $s-$*parameter curve of the surface* $X_N(s,\theta)$ *is an asymptotic curve if and only if the curve* α *is a slant helix.*
(ii) *The* $\theta-$*parameter curve of the surface* $X_N(s,\theta)$ *cannot be an asymptotic curve.*

Proof (i) Assume the $s-$parameter curve is an asymptotic curve. Then

$$\begin{aligned} \langle U_N, (X_N)_{ss}\rangle &= \{g(s)\sin\theta - f(s)\cos\theta\}\{f(s) + rf(s)\cos\theta - rg(s)\sin\theta\} \\ &= 0 \end{aligned}$$

is satisfied. We obtain $\sigma(s) = \cot\theta$ by using the last equation. Noting that $\sigma(s)$ is a constant function, the curve α is a slant helix.

Conversely, we assume that α is a slant helix. Then it can be seen that the $s-$parameter curve of the surface $X_N(s,\theta)$ is an asymptotic curve.

(ii) Using the expression

$$\langle U_{N}, (X_{N})_{\theta\theta}\rangle = -r \neq 0$$

provided for the $\theta-$parameter curve, $X_{N}(s,\theta)$ cannot be an asymptotic curve. □

Theorem 7 *Let $X_{N}(s,\theta)$ be a regular tubular surface determined by the normal indicatrix of the curve α. The parameter curves of this surface in Eq. (3.8) have the following properties.*
(i) The $s-$parameter curve is a geodesic of the surface $X_{N}(s,\theta)$ if and only if the curve α is a slant helix.
(ii) The $\theta-$parameter curve of the surface $X_{N}(s,\theta)$ is a geodesic curve.

Proof (i) Direct computation indicates

$$\begin{aligned} U_{N} \times (X_{N})_{ss} &= [-\sin\theta\,(f' + rf'\cos\theta - rg'\sin\theta)]\,N \\ &\quad + [-(g\cos\theta + f\sin\theta)\,(f + rf\cos\theta - rg\sin\theta)]\,C \\ &\quad + [\cos\theta(f' + rf'\cos\theta - rg'\sin\theta)]\,W. \end{aligned}$$

Since $\{N, C, W\}$ is an orthonormal system and $U_N \times (X_{N})_{ss} = 0$ are satisfied, the equalities

$$\left\{ \begin{array}{c} \sin\theta\,(f' + rf'\cos\theta - rg'\sin\theta) = 0, \\ (g\cos\theta + f\sin\theta)\,(f + rf\cos\theta - rg\sin\theta) = 0, \\ \cos\theta(f' + rf'\cos\theta - rg'\sin\theta) = 0. \end{array} \right. \tag{3.12}$$

are satisfied. By the first and third equations in Eq. (3.12), we have $f' + rf'\cos\theta - rg'\sin\theta = 0$. Integrating this latter equation with respect to s results in

$$f + rf\cos\theta - rg\sin\theta = \text{constant}.$$

Solving this equation and the second equation of Eq. (3.12) together results in obtaining

$$g\cos\theta + f\sin\theta = 0.$$

This means $\sigma(s) = -\tan\theta$ is a constant. Hence, the curve α is a slant helix.

(ii) Using the definition of the geodesic, we get $U_{N} \times (X_{N})_{\theta\theta} = 0$ for the $\theta-$parameter curve. Therefore, the $\theta-$parameter curve on the surface is always a geodesic curve. □

Theorem 8 *The minimal curves of $X_{N}(s,\theta)$ are*

$$\gamma_{N}(s) = \beta_{N}(\psi(s)) + r\left\{\cos\theta N(s) + \sin\theta W(s)\right\}$$

where $\psi(s) = \int f(s)\,ds$ and $\theta = \arccos\left(\left(\frac{-f \mp |g|\sqrt{r^2f^2 + r^2g^2 - 1}}{r(f^2+g^2)}\right)(s)\right)$ with the condition $f(s) \geqslant \dfrac{1}{r}$ satisfied.

Proof Consider the minimal curves of the tubular surface $X_{N}(s,\theta)$ given by Eq. (3.8) satisfy Eq. (3.7). The equation

$$\{rf(s)\cos\theta - rg(s)\sin\theta + 1\}\{rf(s)\cos\theta - rg(s)\sin\theta + f(s)\} = 0.$$

is satisfied. Since the surface $X_{N}(s,\theta)$ is regular, it is straightforwardly obtained

$$\theta = \arccos\left(\left(\frac{-f \mp |g|\sqrt{r^2f^2+r^2g^2-1}}{r\left(f^2+g^2\right)}\right)(s)\right).$$

□

Assume the center curve of the curve α is **binormal indicatrix**, that is, $\beta_{B}(s_{B}) = \beta_{B}(\eta(s) = B(s)$, where $\eta : I \to I_{B}$, $s_{B} = \eta(s)$ is a regular $C^{\infty}-$function. The characteristic circles $(S_{B})_{\lambda}$ are lying in the $Span\{C,W\}$ plane of the center curve β_{B}. Therefore, the tubular surface can be written in the parametric form

$$X_{B}(s,\theta) = \beta_{B}(\eta(s)) + r(\cos\theta C(s) + \sin\theta W(s)),$$

where $\eta(s) = \int \tau(s)ds$ and the angle θ is between the vector C and the position vector of the circles $(S_{B})_{\lambda}$. The unit normal vector field U_{B} is obtained by

$$\begin{aligned} U_{B} &= \frac{(X_{B})_s \times (X_{B})_\theta}{\|(X_{B})_s \times (X_{B})_\theta\|} \\ &= \cos\theta C(s) + \sin\theta W(s). \end{aligned}$$

The components of the curvatures K_{B} and H_{B} are given by the equalities

$$\begin{gathered} E_{B} = \{\tau(s) + rf(s)\cos\theta\}^2 + r^2g^2(s)\ , \\ F_{B} = r^2g(s)\ ,\ G_{B} = r^2\ , \end{gathered}$$

$$\begin{gathered} L_{B} = -f(s)\cos\theta\left\{\tau(s) + rf(s)\cos\theta\right\} - rg^2(s), \\ M_{B} = -rg(s)\ ,\ N_{B} = -r. \end{gathered}$$

The following characterizations of the tubular surfaces related to curve β_{B} will be stated but not proven due to their similar nature to the results proven above.

Proposition 3 *The tubular surface $X_{B}(s,\theta)$ is a regular surface if and only if the inequality $\tau(s) + rf(s)\cos\theta \neq 0$ is provided.*

Corollary 3 *The Gaussian, mean, and principal curvatures of $X_{B}(s,\theta)$ are given by*

$$\begin{aligned} K_{B} &= \frac{f(s)\cos\theta}{r(\tau(s) + rf(s)\cos\theta)}, \\ H_{B} &= -\frac{1}{2}(rK_{B} + \frac{1}{r}). \end{aligned}$$

$$(k_1)_{B} = -\frac{1}{r} \quad and \quad (k_2)_{B} = -\frac{f(s)\cos\theta}{\tau(s) + rf(s)\cos\theta},$$

respectively.

Theorem 9 *Let $X_B(s,\theta)$ be a regular tubular surface in $\mathbb{E}^3$. The parameter curves of $X_B(s,\theta)$ have the following properties.*
(i) The $s-$parameter curve of the surface $X_B(s,\theta)$ is an asymptotic curve if and only if

$$\sigma^2(s)f(s) = -\frac{1}{r}\cos\theta\left\{\tau(s) + rf(s)\cos\theta\right\} \quad \textit{with} \quad |\tau(s)| \geqslant 2r\,|g(s)|\,.$$

(ii) The $\theta-$parameter curve of the surface $X_B(s,\theta)$ cannot be an asymptotic curve.

Theorem 10 *The parameter curves of $X_B(s,\theta)$ have the following properties.*
(i) The $s-$parameter curve is a geodesic curve of $X_B(s,\theta)$ if and only if the equation $r^2g^2 + (\tau + rf\cos\theta)^2$ is a constant.
(ii) The $\theta-$parameter curve of the surface $X_B(s,\theta)$ is always a geodesic curve.

Theorem 11 *The minimal curves of $X_B(s,\theta)$ are given as follows:*

$$\gamma_B(s) = \beta_B(\eta(s)) + r\left\{\cos\theta C(s) + \sin\theta W(s)\right\},$$

where $\eta(s) = \int \tau(s)ds$ and $\theta = \arccos\left(-\dfrac{\tau(s)}{2rf(s)}\right)$ with the condition $\tau(s) \neq 0$ satisfied.

4. Example

In this section, we will give examples of the tubular surfaces generated by the spherical indicatrices of a given curve α. Visualization of several special curves such as minimal, geodesic, and asymptotic curves will be on the corresponding surfaces and will be implemented using the programming language Mathematica.

Example 1 *Let $\alpha = \alpha(s)$ be a slant helix. We choose the curve*

$$\alpha(s) = (\frac{3}{4}\sin s - \frac{1}{12}\sin 3s, -\frac{3}{4}\cos s + \frac{1}{12}\cos 3s, \frac{\sqrt{3}}{2}\sin s).$$

Alternative moving frame apparatus $\{N, C, W, f, g\}$ and Frenet curvatures κ, τ of the curve α can be calculated as

$$\begin{aligned} N(s) &= (\frac{\sqrt{3}}{2}\cos 2s, \frac{\sqrt{3}}{2}\sin 2s, -\frac{1}{2}), \\ C(s) &= (-\sin 2s, \cos 2s, 0), \\ W(s) &= (\frac{1}{2}\cos 2s, \frac{1}{2}\sin 2s, \frac{\sqrt{3}}{2}), \\ \kappa(s) &= \sqrt{3}\sin s,\ \tau(s) = \sqrt{3}\cos s,\ f(s) = \sqrt{3} \textit{ and } g(s) = -1. \end{aligned}$$

The equation of the tubular surface around the curve α is given by the equation

$$X(s,\theta) = \alpha(s) + r(\cos\theta N(s) + \sin\theta B(s))$$

indicating

$$\begin{aligned} X(s,\theta) &= (\frac{3}{4}\sin s - \frac{1}{12}\sin 3s + \frac{\sqrt{3}}{2}\cos\theta\cos 2s - \frac{3}{4}\sin\theta\sin s - \frac{1}{4}\sin\theta\sin 3s, \\ & -\frac{3}{4}\cos s + \frac{1}{12}\cos 3s + \frac{\sqrt{3}}{2}\cos\theta\sin 2s + \frac{3}{4}\sin\theta\cos s + \frac{1}{4}\sin\theta\cos 3s, \\ & \frac{\sqrt{3}}{2}\sin s - \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\sin s). \end{aligned}$$

On this surface, if $\theta = \arccos\left(\frac{1}{2\sqrt{3}\sin s}\right)$ *is assumed to hold then we obtain minimal curves formed by hyperbolic points on the surface. The graphs of this surface and its minimal curves are displayed in Figure* 2 *below.*

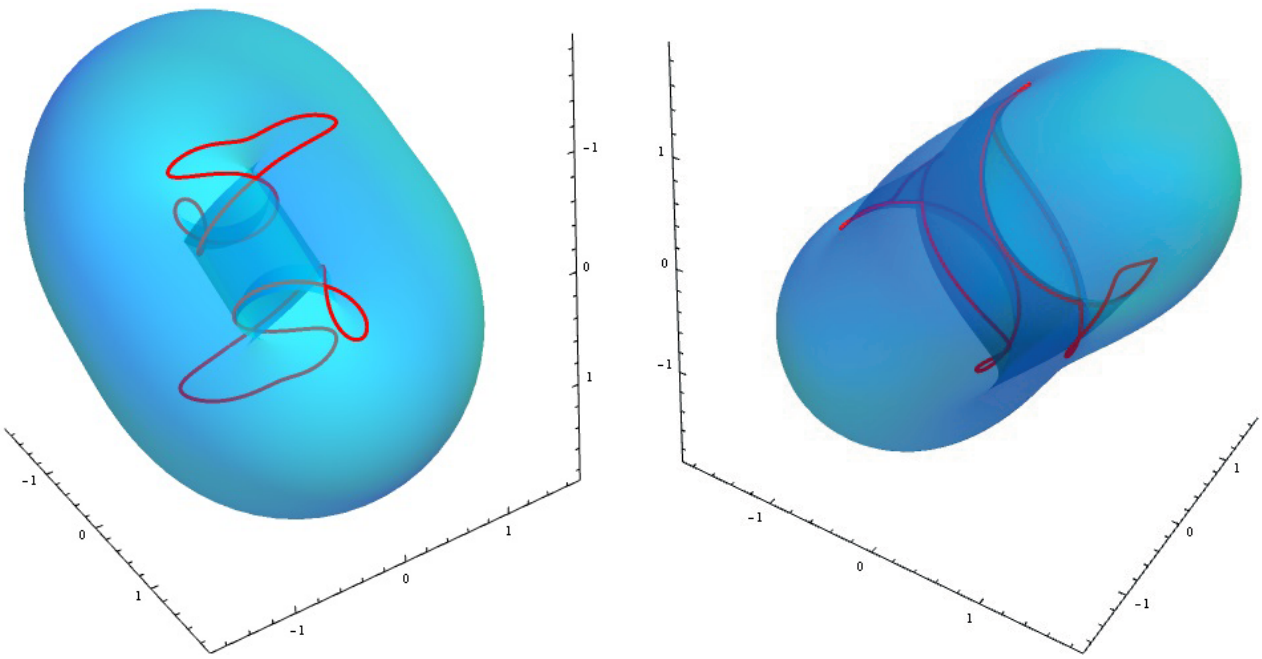


Figure 2. The tubular surface $X(s,\theta)$ and its minimal curves.

The formula of the tubular surface related to the curve β_T *is*

$$X_T(s,\theta) = \beta_T(\varphi(s)) + r(\cos\theta C(s) + \sin\theta W(s)). \tag{4.1}$$

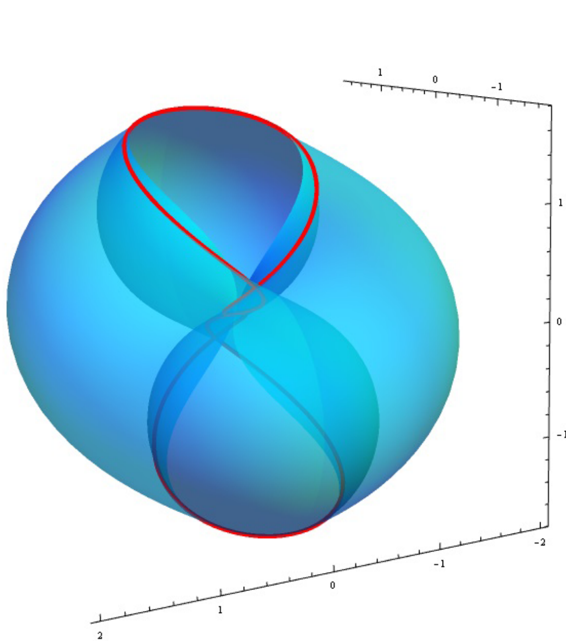
$$\begin{aligned} X_T(s,\theta) &= (\frac{3}{4}\cos s - \frac{1}{4}\cos 3s - \cos\theta\sin 2s + \frac{1}{2}\sin\theta\cos 2s, \\ & \frac{3}{4}\sin s - \frac{1}{4}\sin 3s + \cos\theta\cos 2s + \frac{1}{2}\sin\theta\sin 2s, \\ & \frac{\sqrt{3}}{2}\cos s + \frac{\sqrt{3}}{2}\sin\theta). \end{aligned}$$

Since the conditions in Eq. (3.5) *and Theorem* 3 *are not provided for this example, the* $s-$ *parameter curve of the surface* $X_T(s,\theta)$ *is not an asymptotic curve and a geodesic curve, respectively.*

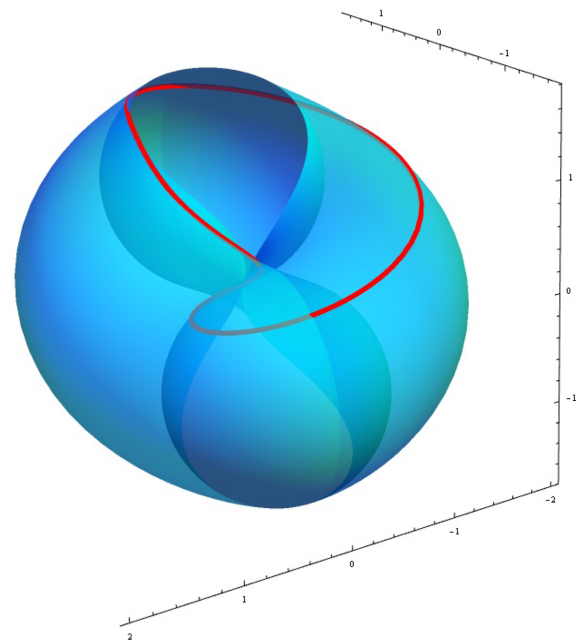
To find the minimal curves γ_T *of the surface* $X_T(s,\theta)$, *we substitute the value* $\cos\theta = \frac{1}{2}\sin s$ *obtained from Theorem* 5 *in Eq.* (4.1). *Since the Gaussian curvature* $K_T = -1 < 0$ *is satisfied at the points on the* *minimal curves γ_T, the minimal curves of the surface $X_T(s,\theta)$ consist of the hyperbolic points on the surface $X_T(s,\theta)$. The graphs of this surface and its special curves are displayed in Figure 3.*

The formula of the tubular surface related to the curve β_N is given as follows:

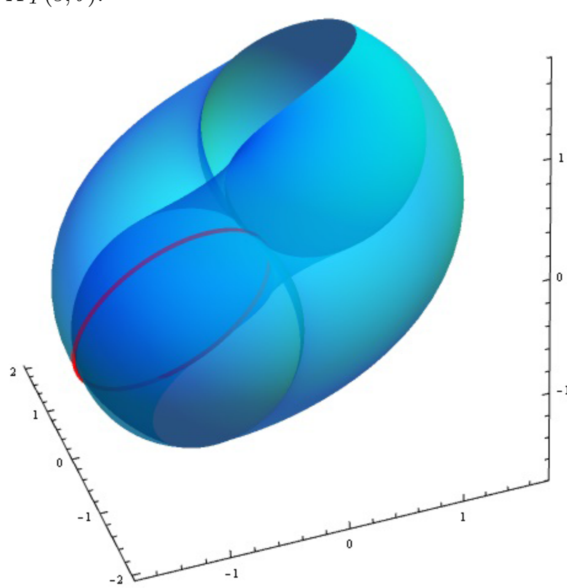
$$X_N(s,\theta) = \beta_N(\psi(s)) + r\left(\cos\theta N(s) + \sin\theta W(s)\right).$$



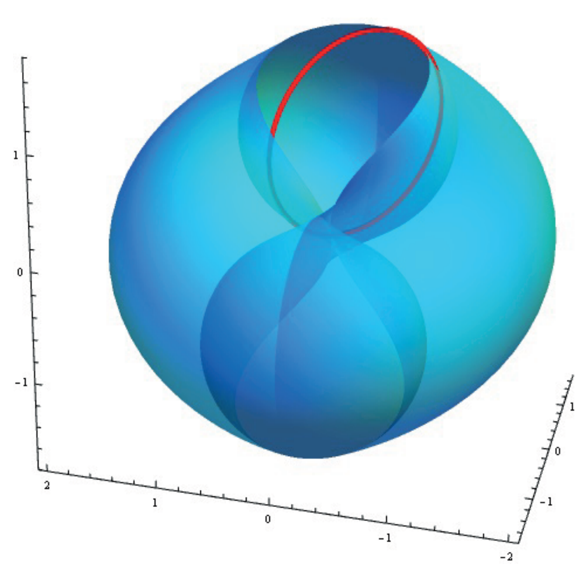
(a) The minimal curves γ_T of the surface $X_T(s,\theta)$.



(b) The nongeodesic and nonasymptotic s-parameter curve for $\theta = 1$.



(c) The geodesic θ-parameter curve for $s = 5/2$.



(d) The geodesic θ-parameter curve of $X_T(s,\theta)$ for $s = 6$.

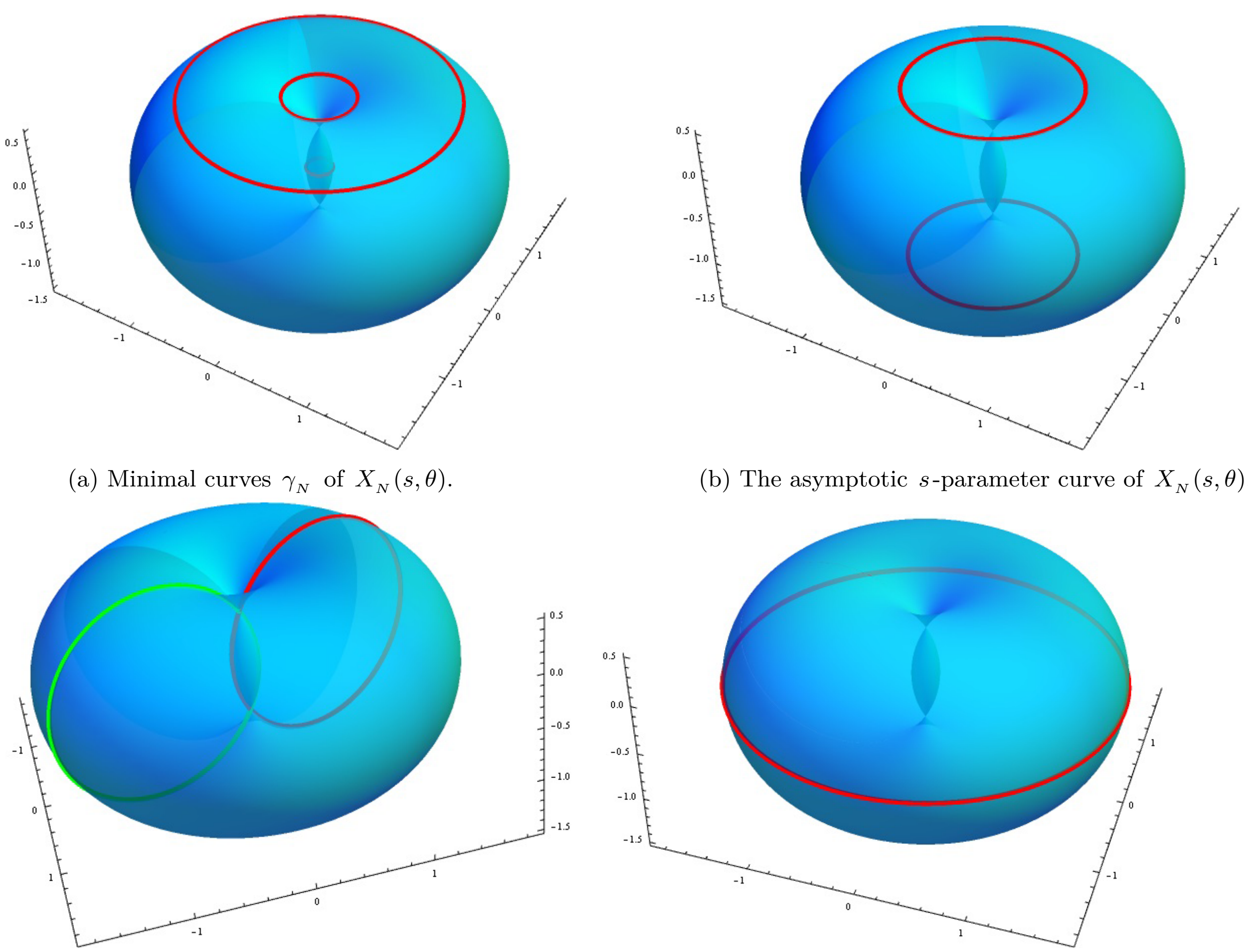
Figure 3. The tubular surface $X_T(s,\theta)$ and its special curves.

$$
\begin{aligned}
X_{_N}(s,\theta) \quad &= \quad \left(\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta\right)\cos 2s,\right.\\
&\quad\quad \left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta\right)\sin 2s,\\
&\quad\quad \left.-\frac{1}{2}-\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right).
\end{aligned}
$$

The graphs of this surface and its special curves are included in Figure 4*.*

The tubular surface generated by β_B *is given by the equation*

$$X_{_B}(s,\theta)=\beta_B(\eta(s))+r(\cos\theta C(s)+\sin\theta W(s)).$$



(a) Minimal curves $\gamma_{_N}$ of $X_{_N}(s,\theta)$.

(b) The asymptotic s-parameter curve of $X_{_N}(s,\theta)$.

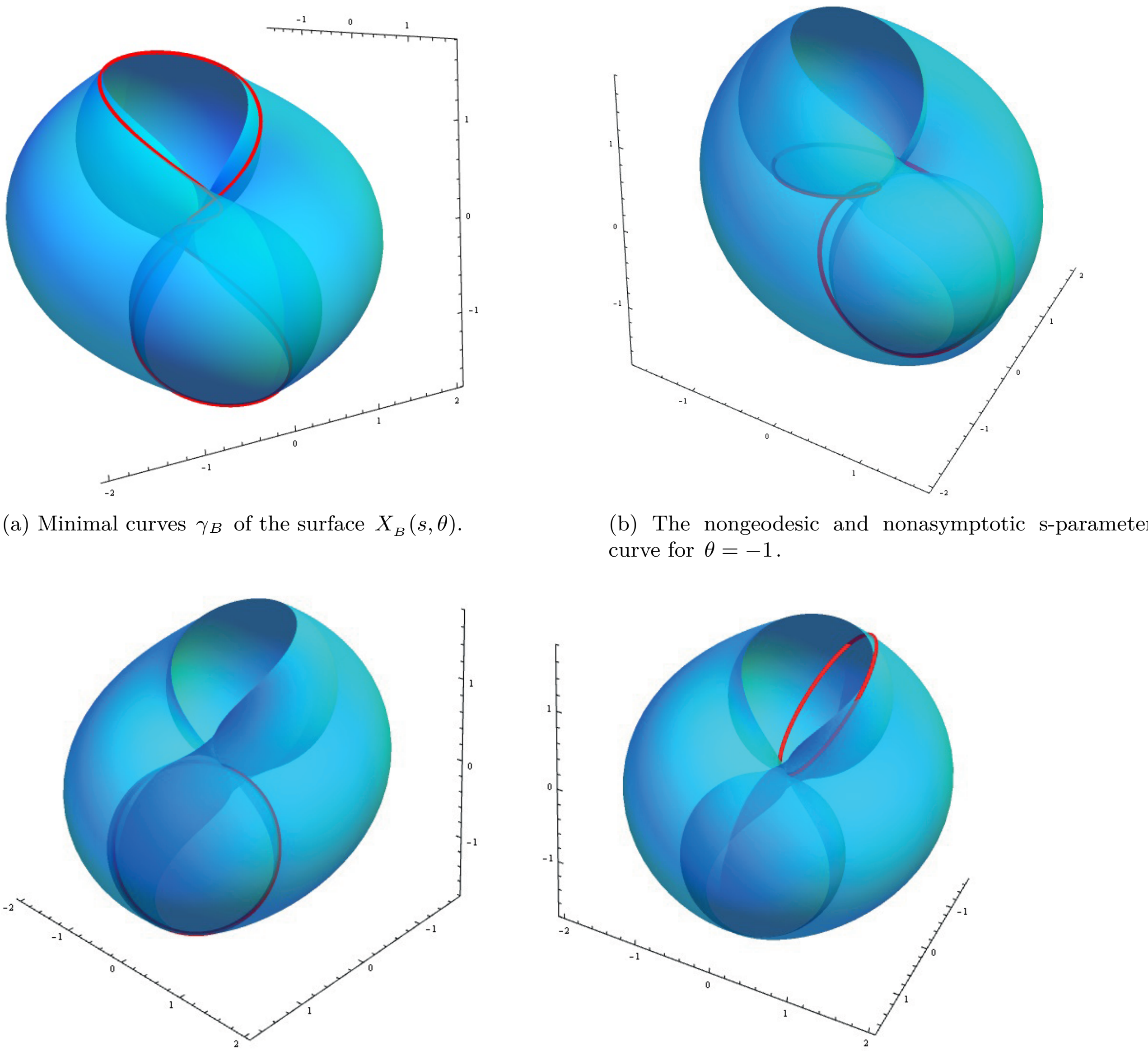
(c) The geodesic θ-parameter curves for $s=1$ (red) and $s=5/2$ (green).

(d) The geodesic s-parameter curve of $X_{_N}(s,\theta)$.

Figure 4. The tubular surface $X_{_N}(s,\theta)$ and its special curves.

$$
\begin{aligned}
X_B(s,\theta) \quad = \quad & (-\frac{3}{4}\sin s - \frac{1}{4}\sin 3s - \cos\theta \sin 2s + \frac{1}{2}\sin\theta\cos 2s, \\
& \frac{3}{4}\cos s + \frac{1}{4}\cos 3s + \cos\theta\cos 2s + \frac{1}{2}\sin\theta\sin 2s, \\
& \frac{\sqrt{3}}{2}\sin s + \frac{\sqrt{3}}{2}\sin\theta).
\end{aligned}
$$

The graphs of this surface and its special curves can be seen in Figure 5*.*



(a) Minimal curves γ_B of the surface $X_B(s,\theta)$.

(b) The nongeodesic and nonasymptotic s-parameter curve for $\theta = -1$.

(c) The geodesic θ-parameter curve for $s = 5$.

(d) The geodesic θ-parameter curve for $s = 1$.

Figure 5. The tubular surface $X_B(s,\theta)$ and its special curves.

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